New Shell Models for Rayleigh-Bénard Turbulence

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Abstract

Two new shell models are introduced to investigate a turbulence driven by a thermal instability (Rayleigh-Bénard turbulence). One is the nearest neighbour real component model (I) and the other is the nearest and the second neighbour complex component model (II). Using these models, quantities such as a mean value of total fluctuation energy, its standard deviation, time averaged wave spectrum, frequency spectrum, the maximum instantaneous Lyapunov exponent are evaluated and the characteristics of turbulence are analyzed. Intermittency nature in the models are also investigated and are compared with results obtained by the model which describes Navier-Stokes turbulence.

Keywords:

shell model, Rayleigh-Bénard turbulence, chaos, intermittency

1. Introduction

The analysis of nonlinear dynamics such as chaos or complexity is a hot topic in physics. Problems related to convective instability are seen in many fields. For instance, the appearance of chaos due to the transition from laminar to turbulent flow via convective nonlinearity has been investigated extensively in the field of fluids turbulence. To challenge such a problem, we need to clarify the following points; what kinds of quantities are relevant for characterizing nonlinear instability and turbulent transports, and to what extent they can be calculated as meaningful parameters in the turbulent state [1].

Various kind of models have been used to investigate the turbulent characteristics and chaotic features in fluid turbulence. Here, we categorize hierarchical model equations, i.e., (I) a model with a few degrees of freedom, e.g., the Lorenz model [2,3], (II) a model with an intermediate degrees of freedom like a shell model [4-7] and (III) a model with many degrees

of freedom such as a Navier-Stokes equation [8].

The model (I) is a very simple one and reveals chaotic nature for some Rayleigh number and Prandtl number. Recently, the extended Lorenz model is proposed as the model describing the dynamics of Scrape Off Layer (SOL) in tokamak plasmas [9-13]. Lorenz model and the model with five Fourier components are compared with each other numerically, which shows that the validity of these models are limited in a low Rayleigh number [14]. Therefore model with intermediate or large number of degrees of freedom should be needed to investigate the turbulence in high Rayleigh number region which is relevant to the plasma edge region.

The model (II) such as a shell model still has a number of properties in common with the Navier-Stokes equation and has an advantage in numerical simulations, although it does not take a real spatial structure in turbulence into account. Ohkitani and Yamada have numerically shown that Gledzer-Ohkitani-Yamada

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(GOY) model (a complex version of a shell model) has a chaotic characteristic [7]. The GOY model is reduced from Navier-Stokes equation, retaining only one complex mode as a representative of all the Fourier modes in the shell of wave numbers between an octave, and only nonlinear couplings to the next nearest shell are kept. In this model, the turbulence is sustained by the external pumping. Jensen *et al.* have introduced a shell model for the small-scale statistics of a scalar field passively carried by a three dimensional turbulence fluid; this model consists of GOY model and passive scalar field [15]. They calculated the structure functions and the probability distribution functions of increments of the passive scalar which show an intermittent nature. The application of a shell model for plasma turbulence is not yet considered.

The model (III) is realistic, however, it is very complicated and is time-consuming to solve it numerically. Yagi *et al.* numerically showed that the self-sustainment of plasma turbulence is possible via the nonlinear instability through the normal cascade process based on the Reduced Magneto Hydro Dynamics (RMHD) [16]. It is found that both the normal cascade and inverse cascade are essential in establishing the stationary turbulent state. However, the numerical simulation was limited in a low Reynolds number due to the computer capability.

In this paper, we adopt a shell model which is relevant to studying a turbulence driven by the thermal instability (Rayleigh-Bénard turbulence) [17,18]. Here our interest is a turbulence driven by the instability, not by the external force, which is quite poorly understood at the moment. A new shell model has been proposed and the basic feature has been analyzed [19]. Based on this shell model, we present an example of numerical analysis for a fluid in high Rayleigh number regime, which covers not only energy containing regime but also inertial range. This model enables us to treat a fully developed turbulence which contains enough positive Lyapunov exponents and to elucidate the characteristics of turbulence such as spectrum cascade, intermittency and so on. Not only coarse grained quantities but also micro quantities, (i) a mean value of total energy, (ii) it's standard deviation, (iii) time averaged wave spectrum, (iv) a probability distribution function, (v) frequency spectrum, (vi) an instantaneous maximum Lyapunov exponent, (vii) scaling exponents of the higher order structure functions are evaluated. Based on these quantities, the characteristics of the turbulence driven by the thermal instability is analyzed.

The organization of this paper is as follows. In Sec. 2, we introduce a new shell model extended to the one which is relevant to studying the turbulence driven by the thermal instability (Rayleigh-Bénard convection). In Sec. 3, we examine the stationary solutions and the linear stability analysis of the model. In Sec. 4, we analyze our numerical results, such as a mean value of total energy, it's standard deviation, the time averaged wave spectrum, a probability distribution function, the frequency spectrum, an instantaneous maximum Lyapunov exponent and the scaling exponents of the higher order structure functions. In Sec. 5, the scale exponents of higher order structure functions are compared between a new shell model and GOY shell model to investigate the intermittency nature of turbulence.

Summary and discussions are given in Sec. 6.

2. Derivation of Model Equation

Here we develop a simplified model which describes Rayleigh-Bénard turbulence, heuristically. More quantitative discussion of model equation is given in ref. [17].

We assume that the fluid layer of height h under the gravity acceleration g is submitted to a temperature gradient $\beta = \Delta T/h > 0$ where $\Delta T = T_b - T_t (> 0)$ is the temperature difference between the bottom (subscript 'b') and top (subscript 't') plates [18]. The temperature field is written as

$$T = T_0(x_3) + \theta, \quad (1)$$

where $T_0(z) = T_b - \beta x_3$ is a purely conducting temperature, x_3 , the distance from bottom plate and θ , a fluctuating temperature. The density distribution is governed by

$$\rho_0 = \rho(T_0(x_3)) = \rho_b(1 + \alpha\beta x_3), \quad (2)$$

where α is the thermal expansion coefficient and ρ_b is the reference density at the bottom. Then the differential buoyancy is given as $-g(\rho - \rho_0) = -g(\rho(T) - \rho(T_0)) = \rho_b\alpha g\theta$. Navier-Stokes equation for incompressible fluid and heat equation are written as

$$\nabla \cdot \mathbf{v} = 0, \quad (3)$$

$$\frac{\partial v_i}{\partial t} + (\mathbf{v} \cdot \nabla) v_i = -\frac{\partial p}{\partial x_i} + \nu \nabla^2 v_i + \alpha g \theta \delta_{i3}, \quad (4)$$

$$\frac{\partial v_i}{\partial t} + (\mathbf{v} \cdot \nabla) \theta = \kappa \nabla^2 \theta + \beta \delta_{i3} v_i, \quad (5)$$

where ν the kinematic viscosity, κ is the heat diffusivity

and $\delta_{\alpha\beta}$ is Kronecker delta and $i = 3$ represents the direction of buoyancy. Fourier transforming eqs. (3)–(5) using the relation

$$f(\mathbf{x}) = \sum_{\mathbf{k}} f(\mathbf{k}) \exp(i\mathbf{k} \cdot \mathbf{x}) = \sum_{\mathbf{k}} f(\mathbf{k}) \exp(i\mathbf{k}_i x_i), \quad (6)$$

and eliminating pressure in eq. (4) using incompressibility, we obtain

$$\begin{aligned} \frac{\partial v_i(\mathbf{k})}{\partial t} = & -\frac{i}{2} (k_j D_{ij}(\mathbf{k}) + k_i D_{ij}(\mathbf{k})) \\ & \sum_{\mathbf{p}, \mathbf{q}} \delta_{k, \mathbf{p}+\mathbf{q}} v_j(\mathbf{p}) v_i(\mathbf{q}) - \nu k^2 v_i(\mathbf{k}) \\ & + \alpha g \delta_{i3}(\mathbf{k}) \theta(\mathbf{k}), \end{aligned} \quad (7)$$

and

$$\begin{aligned} \frac{\partial \theta(\mathbf{k})}{\partial t} = & -ik_i \sum_{\mathbf{p}, \mathbf{q}} \delta_{k, \mathbf{p}+\mathbf{q}} v_i(\mathbf{p}) \theta(\mathbf{q}) \\ & - \kappa k^2 \theta(\mathbf{k}) + \beta \delta_{i3} v_i(\mathbf{k}), \end{aligned} \quad (8)$$

where

$$D_{ij}(\mathbf{k}) = \delta_{ij} - \frac{k_i k_j}{k^2}. \quad (9)$$

Extracting the component of $i = 3$ direction which couples with thermal buoyancy, we obtain the equations of the velocity and temperature fluctuations

$$\begin{aligned} \frac{du(k)}{dt} = & -i \sum_{k'} k' u(k - k') u(k') \\ & + P_r p(k) - P_r k^2 u(k), \end{aligned} \quad (10)$$

$$\begin{aligned} \frac{dp(k)}{dt} = & -i \sum_{k'} k' u(k - k') p(k') \\ & + R_a u(k) - k^2 p(k), \end{aligned} \quad (11)$$

where $P_r = \nu/\kappa$ is the Prandtl number and $R_a = \alpha g \beta h^4 / (\nu \kappa)$, the Rayleigh number. To derive these equations, $k \simeq k_3$ is assumed and the normalizations: $kh \rightarrow k$, $t\kappa h^2 \rightarrow t$, $v_3 h/\kappa \rightarrow u$, $\theta \alpha g h^3 / (\kappa \nu) \rightarrow p$, are used. From (10) and (11), we now derive two new shell models in the following. One is the nearest neighbour real component model which is the extension of the Desnyansky and Novikov model [4]. The other is the nearest and the second nearest neighbour complex component model, which is the extension of the original GOY model [7] supplemented by the passive scalar component [15].

The nearest neighbour and real component model is written as

$$\begin{aligned} \frac{du_n}{dt} = & k_n (u_{n-1}^2 - 2u_n u_{n+1}) \\ & - P_r k_n^2 u_n + P_r p_n, \end{aligned} \quad (12)$$

$$\begin{aligned} \frac{dp_n}{dt} = & k_n (u_{n-1} p_{n-1} - 2u_n p_{n+1}) \\ & - k_n^2 p_n + R_a u_n. \end{aligned} \quad (13)$$

If we neglect the third term in the right hand side (RHS) of (12) and add the forcing term to it, the Desnyansky and Novikov model is recovered.

The nearest and the second nearest neighbour complex component model is given by

$$\begin{aligned} \frac{du_n}{dt} = & i(k_n u_{n+1} u_{n+2} - k_{n-2} u_{n-1} u_{n+1} \\ & - k_{n-3} u_{n-1} u_{n-2})^* - P_r k_n^2 u_n + P_r p_n, \end{aligned} \quad (14)$$

$$\begin{aligned} \frac{dp_n}{dt} = & i[k_{n-1} (u_{n-1} p_{n+1} - u_{n+1} p_{n-1}) \\ & - k_{n-2} (u_{n-2} p_{n-1} + u_{n-1} p_{n-2}) \\ & + k_n (u_{n+1} p_{n+2} + u_{n+2} p_{n+1})]^* \\ & - k_n^2 p_n + R_a u_n, \end{aligned} \quad (15)$$

where $k_n = C \cdot 2^{n-1}$ ($n = 1, 2, \dots, N$), N is the number of shells, C is a numerical constant and the boundary conditions are given as

$$u_{-1} = u_0 = u_{N+1} = u_{N+2} = 0, \quad (16)$$

$$p_{-1} = p_0 = p_{N+1} = p_{N+2} = 0, \quad (17)$$

which ensures no interaction between N -shells and the vacuum outside of N -shells. If we neglect the third terms in RHS of (14) and (15) and add the force terms to them, respectively, then (14) corresponds to the GOY model and the (15) corresponds to passive scalar field which involves with GOY model.

In this paper, we introduce these new models above which are appropriate to describe Rayleigh-Bénard turbulence and are the extended versions of Desnyansky and Novikov model and of GOY model. In the following sections, we analyze these model equations.

3. Theoretical Analysis of Model Equation

In this section we firstly discuss the linear stability analysis of our models. Secondary, stationary solutions are investigated. In the shell models which describe the

fully developed turbulence of neutral fluid, the variable, $u_n(k)$ obeys the scaling law, $u_n(k) \sim k_n^{-1/3}$ in the stationary state. For comparison, we investigate the stationary similarity mode of our model equation.

3.1 Linear stability analysis

By linearizing (12) and (13), or (14) and (15) we obtain

$$\frac{du_n}{dt} = -P_r k_n^2 u_n + P_r p_n, \quad (18)$$

$$\frac{dp_n}{dt} = -R_a u_n - k_n^2 p_n, \quad (19)$$

for both two models. Assuming an exponential time dependence involving an unknown growth rate s : $\{u_n, p_n\} \rightarrow \{\hat{u}_n, \hat{p}_n\} \exp(s_n t)$, we obtain

$$(s_n + P_r k_n^2) \hat{u}_n - P_r \hat{p}_n = 0, \quad (20)$$

$$-R_a \hat{u}_n + (s_n + k_n^2) \hat{p}_n = 0. \quad (21)$$

A perturbation will exist if this system has a nontrivial solution, which can be the case only if its determinant vanishes:

$$0 = (s_n + P_r k_n^2)(s_n + k_n^2) - P_r R_a \\ = s_n^2 + k_n^2(1 + P_r) s_n + P_r(k_n^4 - R_a). \quad (22)$$

This equation is quadratic in s_n . It can have two roots, $s_n^\pm$, with

$$s_n^+ + s_n^- = -k_n^2(1 + P_r) \equiv \Sigma, \quad (23)$$

$$s_n^+ s_n^- = P_r(k_n^4 - R_a) \equiv \Pi. \quad (24)$$

s_n^+ and s_n^- can be real or complex according to the sign of the determinant

$$\Delta \equiv k_n^4(1 + P_r)^2 - 4P_r(k_n^4 - R_a) \quad (25)$$

$$= k_n^4(P_r - 1)^2 + 4P_r R_a. \quad (26)$$

Δ can be negative only when R_a is negative, i.e., when the layer is heated from above. When R_a is negative and sufficiently large, the two roots are complex conjugate but their real part $\Sigma/2$ remains negative so that the corresponding perturbation is damped; there cannot be unstable oscillatory modes.

When R_a is positive, Δ remains positive and the roots are real but their Σ stays negative. In order to

detect the instability, we need only to discuss the sign of the product Π . When it is positive both roots are negative and we have two stable modes. This occurs when R_a is small, especially when $R_a \rightarrow 0$, i.e., close to thermal equilibrium. When Π is negative, root s_n^- remains negative but s_n^+ becomes positive; one of the modes grows exponentially and the layer is unstable. This happens when R_a is large so that the release of gravitational energy by buoyancy can overcome dissipation losses.

Marginal stability corresponds to $s_n^+ = 0$, i.e.,

$$R_a = k_n^4. \quad (27)$$

The condition in which the system is unstable is given by

$$R_a > k_n^4, \quad (28)$$

and that in which the system is stable is given by

$$R_a < k_n^4. \quad (29)$$

3.2 Stationary similarity mode

(a) nearest neighbor real component model

In the stationary state, since $du_n/dt = 0$ and $dp_n/dt = 0$, the model equations (12) and (13) are in the forms of

$$k_n(u_{n-1}^2 - 2u_n u_{n+1}) - P_r u_n + P_r p_n = 0 \quad (30)$$

$$k_n(u_{n-1}^2 p_{n-1} - 2u_n p_{n+1}) - k_n^2 p_n + R_a u_n = 0. \quad (31)$$

Here, we assume the wavenumber dependences of u_n and p_n such that, $u_n = A k_n^a$ and $p_n = B k_n^b$, respectively, and find the stationary similarity mode. Substituting these equations into (30) and (31), we obtain the following equations

$$0 = (k_n^{2a+1} 2^{-2a} - k_n^{2a+1} 2^{a+1}) A^2 \\ - P_r A k_n^{a+2} + P_r B k_n^b, \quad (32)$$

$$0 = (k_n^{a+b+1} 2^{-a-b} - k_n^{a+b+1} 2^{b+1}) A B \\ - B k_n^{b+2} + R_a A k_n^a. \quad (33)$$

When $a = -1/3$ and $b = -1/3$, the nonlinear terms vanish in (32) and (33). Then, we obtain

$$A k_n^{5/3} - B k_n^{-1/3} = 0, \quad (34)$$

$$-R_a A k_n^{-1/3} - B k_n^{5/3} = 0. \quad (35)$$

Substituting (35) into (34) and eliminating k_n , we have the relation equation between A and B as

$$-R_a A^2 + B^2 = 0, \quad (36)$$

i.e.,

$$B = R_a^{1/2} A. \quad (37)$$

(b) nearest and second nearest neighbor complex component model

Before analyzing the stationary similarity mode of the model equations (14) and (15), we rewrite the model equations in terms of the real and imaginary parts of u_n and p_n . Then, the following equations are obtained:

$$\begin{aligned} \frac{du_n^r}{dt} = & k_n (u_{n+1}^r u_{n+2}^i + u_{n+1}^i u_{n+2}^r) \\ & - k_{n-2} (u_{n-1}^r u_{n+1}^i + u_{n-1}^i u_{n+1}^r) \\ & - k_{n-3} (u_{n-1}^r u_{n-2}^i + u_{n-1}^i u_{n-2}^r) \\ & - P_r k_n^2 u_n^r + P_r p_n^i, \end{aligned} \quad (38)$$

$$\begin{aligned} \frac{du_n^i}{dt} = & k_n (u_{n+1}^r u_{n+2}^r - u_{n+1}^i u_{n+2}^i) \\ & - k_{n-2} (u_{n-1}^i u_{n+1}^i - u_{n-1}^r u_{n+1}^r) \\ & - k_{n-3} (u_{n-1}^i u_{n-2}^i - u_{n-1}^r u_{n-2}^r) \\ & - P_r k_n^2 u_n^i + P_r p_n^r, \end{aligned} \quad (39)$$

$$\begin{aligned} \frac{dp_n^r}{dt} = & k_{n-1} (u_{n-1}^r p_{n+1}^i + u_{n-1}^i p_{n+1}^r) \\ & - k_{n-1} (u_{n+1}^r p_{n-1}^i + u_{n+1}^i p_{n-1}^r) \\ & - k_{n-2} (u_{n-2}^r p_{n-1}^i + u_{n-2}^i p_{n-1}^r) \\ & - k_{n-2} (u_{n-1}^r p_{n-2}^i + u_{n-1}^i p_{n-2}^r) \\ & + k_n (u_{n+1}^r p_{n+2}^i + u_{n+1}^i p_{n+2}^r) \\ & + k_n (u_{n+2}^r p_{n+1}^i + u_{n+2}^i p_{n+1}^r) \\ & - k_n^2 p_n^r + R_a u_n^r, \end{aligned} \quad (40)$$

and

$$\begin{aligned} \frac{dp_n^i}{dt} = & k_{n-1} (u_{n-1}^r p_{n+1}^r - u_{n-1}^i p_{n+1}^i) \\ & - k_{n-1} (u_{n+1}^r p_{n-1}^r - u_{n+1}^i p_{n-1}^i) \\ & - k_{n-2} (u_{n-2}^r p_{n-1}^r - u_{n-2}^i p_{n-1}^i) \\ & - k_{n-2} (u_{n-1}^r p_{n-2}^r - u_{n-1}^i p_{n-2}^i) \\ & + k_n (u_{n+1}^r p_{n+2}^r - u_{n+1}^i p_{n+2}^i) \\ & + k_n (u_{n+2}^r p_{n+1}^r - u_{n+2}^i p_{n+1}^i) \\ & - k_n^2 p_n^i + R_a u_n^i, \end{aligned} \quad (41)$$

where superscripts 'r' and 'i' represent the real part and the imaginary part of the variables, respectively.

Here, we assume the wavenumber dependences of the variables given by $u_n^r = A k_n^a$, $u_n^i = B k_n^b$, $p_n^r = C k_n^c$ and $p_n^i = D k_n^d$, respectively, to obtain the stationary similarity mode. Substituting these relation equations into (38)–(41) and taking a similar process as in the case of (a), we obtain $a = b = c = d = -1/3$. The relations between the coefficients of u_n^r , u_n^i , p_n^r and p_n^i are given by

$$C = R_a^{1/2} A, \quad (42)$$

$$D = R_a^{1/2} B. \quad (43)$$

In both cases (a) and (b), the energy spectrum $E(k_n)$, which is defined as $E(k_n) = \Sigma_n (|u_n|^2 + |p_n|^2) / (2k_n)$, scales like $k^{-5/3}$, i.e., the Kolmogorov's type scaling laws [20].

4. Numerical Analysis

Numerical simulations of the new two shell models are carried out and their results are shown. Two models are given by (I); (12) with (13) and (II); (14) with (15), respectively.

4.1 Linear analysis

Figures 1 and 2 show the wavenumber dependence of the linear growth rate of the model equations.

Figure 1 is the case for various values of Rayleigh number with the fixed Prandtl number, $P_r = 1.0$. The solid line corresponds to the case of $R_a = 2 \times 10^4$. The dotted-dashed line shows the case of $R_a = 10^6$ and the dotted line shows the case of $R_a = 10^7$. The growth rate increases according to the increase of the Rayleigh number. The system is linearly stable in the region of $n > 8$ ($k_n > 2^8$) for $R_a = 2 \times 10^4$, in the region of $n > 9$ ($k_n > 2^9$) for $R_a = 10^6$ and in the region of $n > 10$ ($k_n > 2^{10}$) for $R_a = 10^7$, respectively.

Figure 2 is the case for various values of Prandtl number with the fixed Rayleigh number, $R_a = 2 \times 10^4$. The solid line is the case of $P_r = 2.0$. The dotted-dashed line shows the case $P_r = 5.0$ and the dotted line corresponds to the case of $P_r = 8.0$. The growth rate increases according to the increase of the Prandtl number. The maximum linear growth rate is determined by the longest wavelength mode. It is found that the normalized profile of the growth rate versus the wave number shows the similar profile for different Rayleigh numbers.

Fig. 1 The wavenumber dependence of the linear growth rate for various values of Rayleigh number with the fixed Prandtl number, $P_r = 1.0$. The solid line: $R_a = 2 \times 10^4$, the dotted-dashed line: $R_a = 10^6$ and dotted line: $R_a = 10^7$.

Fig. 2 The wavenumber dependence of the linear growth rate for various values of Prandtl number with the fixed Rayleigh number, $R_a = 2 \times 10^4$. The solid line: $P_r = 2.0$, the dotted-dashed line: $P_r = 5.0$ and the dotted line: $P_r = 8.0$.

4.2 Nonlinear analysis

In this section, the following parameters are used for numerical simulations; $C = 1.0$ and the numbers of shells $N = 13$ for $R_a = 2 \times 10^4$, $N = 14$ for $R_a = 10^6$ and $N = 14$ for $R_a = 10^7$.

(a) Time averaged wavenumber spectrum

Figure 3 shows the log-log plot of the time averaged wave number spectra of the variables, $u(k)$ and $p(k)$ by the nearest neighbour real component model (I) in the case of $R_a = 2 \times 10^4$ and $P_r = 1.0$. The circles correspond to the wavenumber spectrum of $u(k)$, the triangles to the wavenumber spectrum of $p(k)$, respectively. Time averaging is done from $t = 0.2$ to $t = 0.5$. The macro scale length in this system is given by $k_1 \sim 10^0$, and the dissipation scale length is given $k_d \sim 5 \times$

Fig. 3 The log-log plot of the time averaged wavenumber spectra of the variables $u(k)$ and $p(k)$ by the nearest neighbour real component model (I) for $R_a = 2 \times 10^4$ and $P_r = 1.0$. The circles correspond to the wavenumber spectrum of $u(k)$, the triangles, to the wavenumber spectrum of $p(k)$, respectively.

Fig. 4 The log-log plot of the time averaged wavenumber spectra of $u(k)$ and $p(k)$ by the nearest and the second nearest neighbour complex component model (II) for $R_a = 2 \times 10^4$ and $P_r = 1.0$. The circle corresponds to the spectrum of $u(k)$, the triangles, to the spectrum of $p(k)$, respectively.

10^2 . The linear instability exists in the region $k_1 \leq k \leq k_m^{\text{lin}} \sim 10^1$. Therefore, we may regard the region $10^1 \leq k \leq 5 \times 10^2$ as the inertia range. In the wavenumber region, $2 \leq k \leq 64$ (mode number $2 \leq n \leq 7$), both variables obey the scaling law, $u(k) \sim k^{-1/3}$ and $p(k) \sim k^{-1/3}$, these are plotted by dotted lines. The Kolmogorov's type scaling law is obtained in this region.

Figure 4 shows the log-log plot of the time averaged wavenumber spectra of $u(k)$ and $p(k)$ by the nearest and the second nearest neighbour complex

component model (II) in the case of $R_a = 2 \times 10^4$ and $P_r = 1.0$. The circles corresponds to the spectrum of $u(k)$, the triangles correspond to the spectrum of $p(k)$, respectively. Time averaging is performed from $t = 0.2$ to $t = 10.0$. In the wave number region, $2 \leq k \leq 16$ (mode number $2 \leq n \leq 5$), both variables seem to obey the scaling law, $u(k) \sim k^{-1/3}$ and $p(k) \sim k^{-1/3}$ as are shown by the dotted lines, although they fluctuate around the line proportional to $k^{-1/3}$.

(b) Total energy

Firstly, we investigate the Rayleigh number dependence of the saturation level of total energy. Figure 5 shows the time evolution of total energy $E(t)$ by the nearest neighbour real component model (I) in

Fig. 5 The time evolution of the total energy $E(t)$ of the nearest neighbor real component model (I) in the case of $P_r = 1.0$. The solid line corresponds to the case of $R_a = 2 \times 10^4$. The dotted-dashed line shows the case of $R_a = 10^6$ and the dotted line shows the case of $R_a = 10^7$.

Fig. 6 The time evolution of the total energy of the nearest and the second nearest neighbor complex component model (II) in the case of $P_r = 1.0$. The solid line corresponds to the case of $R_a = 2 \times 10^4$. The dotted-dashed line shows the case of $R_a = 10^6$ and the dotted line shows the case of $R_a = 10^7$.

the case of $P_r = 1.0$. The solid line corresponds to the case of $R_a = 2 \times 10^4$. The dotted-dashed line shows the case of $R_a = 10^6$ and the dotted line shows the case of $R_a = 10^7$. Here, the total energy is defined by

$$E(t) = \frac{1}{2} \sum_n (|u_n|^2 + |p_n|^2). \quad (44)$$

For each case, the total energy converges at a constant value in the nonlinear phase. In other words, the attractor of this model converges to the fixed point. The larger Rayleigh number is, the higher the saturation level of total energy becomes.

Figure 6 shows the time evolution of total energy $E(t)$ by the nearest and second nearest neighbour complex component model (II) for the same parameters in Fig. 5. The solid line corresponds to the case of $R_a = 2 \times 10^4$, the dotted-dashed line to the case of $R_a = 10^6$ and the dotted line to the case of $R_a = 10^7$. The total energy tends to saturate in the nonlinear phase, however the fluctuation around the mean value of the total energy is observed, which shows the chaotic nature of the system. Both the mean value and the fluctuation amplitude increase with the Rayleigh number.

The saturation level of the total energy is roughly determined by the maximum linear growth rate and the nonlinear damping rate and the higher saturation level is obtained for the model of the nearest and the second nearest neighbour complex component model (II). This is because the two different models do not show the same results even if the parameters of R_a and P_r are given. Model (I) consists of real variables and model (II) of complex variables. The scheme of shell model may cause the differences in the nonlinear interaction rate (the nonlinear damping rate) and/or in the nonlinear interaction regime in k space.

Next, we investigate the Prandtl number dependence of the saturation level of total energy. Figure 7 shows the time evolution of the total energy by the nearest neighbor real component model (I) in the case of $R_a = 2 \times 10^4$. Cases with various Prandtl numbers are plotted. The solid line represents the case of $P_r = 2.0$, the dotted-dashed line, the case of $P_r = 5.0$ and the dotted line, the case of $P_r = 8.0$. The total energy converges to a constant value in nonlinear phase. The total energy becomes higher with the increase of Prandtl number as is expected from the linear analysis.

Figure 8 shows the time evolution of the total energy by the nearest and the second nearest neighbor complex component model (II) with $R_a = 2 \times 10^4$. The solid line corresponds to the case of $P_r = 2.0$, the dotted-

dashed line is the case of $P_r = 5.0$ and the dotted line shows the case of $P_r = 8.0$. The total energy also fluctuates as is observed in figure 6. In contrast with the Rayleigh number dependence shown in Fig. 6, the fluctuation amplitude relative to the mean value of the total energy is found to decrease as the Prandtl number increases. The increase of mean value of saturation level with the increase of Prandtl number is correlated with the maximum linear growth rate, as is the case of the Rayleigh number dependence.

It is found that the nearest neighbour real component model (I) does not have a chaotic attractor. Therefore the following analysis, (c), (d), (e) and (f) are done by employing the nearest and the second nearest neighbour complex component model (II). In order to

Fig. 7 The time evolution of the total energy by the nearest neighbour real component model (I) in the case of $R_a = 2 \times 10^4$. The solid line represents the case of $P_r = 2.0$, the dotted-dashed line, the case of $P_r = 5.0$ and the dotted line, the case of $P_r = 8.0$.

Fig. 8 The time evolution of the total energy of the nearest and the second nearest neighbour complex model (II) in the case of $R_a = 2 \times 10^4$. The solid line represents the case of $P_r = 2.0$, the dotted-dashed line, the case of $P_r = 5.0$ and the dotted line, the case of $P_r = 8.0$.

investigate the fluctuation amplitude of the total energy by the nearest and the second nearest neighbour complex component model (II), we study the mean value of the total energy and its variance in the next section.

(c) Mean value and its variance of total energy

Figure 9 shows the log-log plot of the mean value of the total energy and its variance versus Rayleigh number by the nearest and the second nearest neighbour complex component model (II).

Fig. 9 The log-log plot of the mean value of total energy and its variance versus Rayleigh number by the nearest and second nearest neighbor complex component model (II) in the case of $P_r = 1.0$. The square symbols correspond to the mean value of the total energy and the circle symbols, to the variance.

Fig. 10 The log-log plot of the mean value of total energy versus Prandtl number by the nearest and second nearest neighbor complex component model (II) in the case of $R_a = 2 \times 10^4$. The square symbols correspond to the mean value of the total energy and the circle symbols, to the variance.

The mean value of the total energy is defined by

$$\langle E \rangle = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} E(t) dt, \quad (45)$$

where $t_1 = 0.2$ and $t_2 = 10.0$ are chosen. Its variance is defined by

$$\sigma^2 = \langle E^2 \rangle - \langle E \rangle^2. \quad (46)$$

The square symbols correspond to the mean value of the total energy and the circle symbols to the variance in Fig. 9. The mean value increases with the increase of the Rayleigh number. This corresponds to the rise of the saturation level of the total energy in figure 6. The variance of the total energy also increases with the increase of the Rayleigh number. The dependence of the variance on Rayleigh number is $\sim R_a^{1.53}$. Figure 10 shows the log-log plot of the mean value of the total energy and its variance versus Prandtl number. In this figure Rayleigh number is fixed to $R_a = 2 \times 10^4$. The square symbols correspond to the mean value of the total energy and the circle symbols to the variance. The mean value of the total energy increases with the increase of the Prandtl number. It is found that the variance increases in the region $0.1 < P_r < 7.5$, on the other hand, it decreases in the region $P_r > 7.5$.

In order to study the ratio between the variance and the mean value, we show the log-log plot of the ratio of the variance to the mean value of the total energy versus Rayleigh number in Fig. 11. Prandtl number is fixed to the value of $P_r = 1.0$ in this case. The circle symbols show the ratio. It is found that the ratio is not a monotonic function of the Rayleigh number. The ratio increases with the increase of the Rayleigh number in the region $10^4 < R_a < 10^5$, then it decreases in the region $10^5 < R_a < 2 \times 10^6$. For $R_a > 2 \times 10^6$, it again increases with the Rayleigh number.

Figure 12 shows the log-log plot of the ratio of the variance to the mean value of the total energy versus Prandtl number. Rayleigh number is fixed to the value of $R_a = 2 \times 10^4$ in this case. The circles show the ratio. The value of the ratio decreases with the increase of the Prandtl number in the region of $P_r < 3.0$. The value of the ratio stays almost constant in the region of $3.0 < P_r < 7.5$. The value of the ratio decreases rapidly in the region $P_r > 7.5$. These results indicate that nonlinear instability in phase space is stabilized for $P_r > 7.5$ and the attractor of this system falls into the limit cycle from chaotic attractor.

Figure 13 shows the log-log plot of the maximum

Fig. 11 The log-log plot of the ratio of the variance to the mean value of the total energy versus Rayleigh number in the case of $P_r = 1.0$. The circle symbols show the ratio.

Fig. 12 The log-log plot of the ratio of the variance to the mean value of the total energy versus Rayleigh number in the case of $R_a = 2 \times 10^4$. The circle symbols show the ratio.

Lyapunov exponent versus the Rayleigh number for the fixed Prandtl number, $P_r = 1.0$. The procedure to obtain Lyapunov exponent is explained in Appendix A. For these Rayleigh numbers the maximum Lyapunov exponent is found to be positive. This means that the system shows a chaotic nature, which is also related with an intermittency. The dependence of the maximum Lyapunov exponent of the Rayleigh number is $\sim R_a^{0.53}$, which shows a little stronger dependence than that of the linear growth rate in the inviscid limit ($\sim R_a^{1/2}$).

Figure 14 shows the plot of the maximum Lyapunov exponent versus the Prandtl number for the fixed Rayleigh number, $R_a = 2 \times 10^4$. The values of maximum Lyapunov exponent are positive in the region, $0.5 \leq P_r \leq 7.5$. It is found that the values of the maximum Lyapunov exponent are almost zero in the

Fig. 13 The log-log plot of the maximum Lyapunov exponent versus Rayleigh number in the case of $P_r = 1.0$. The dotted line corresponds to the line proportional to $Ra^{0.53}$.

Fig. 14 The log-log plot of the maximum Lyapunov exponent versus Prandtl number in the case of $Ra = 2 \times 10^4$.

region, $8.0 \leq P_r \leq 10.0$. We conjecture that both parameters, Ra and P_r , have the destabilizing effect on the linear mode, which reflects to the mean value of saturation amplitude. On the other hand, the Rayleigh number has an effect of destabilizing on the nonlinear mode, while, the Prandtl number has an effects of both stabilizing and destabilizing on the nonlinear mode in these parameter regime.

(d) Probability density function

Figure 15 shows the probability density function of $P_{n=3}(s)$ versus the normalized temperature component in the case of $Ra = 2 \times 10^4$ and $P_r = 1.0$. The normalized temperature component, s is defined by

$$s = \frac{|p_n|}{\langle |p_n|^2 \rangle^{1/2}}. \quad (47)$$

Fig. 15 The probability density function of $P_{n=3}(s)$ versus normalized temperature component, s in the case of $Ra = 2 \times 10^4$ and $P_r = 1.0$.

Fig. 16 The plot of the probability density function of $P_{n=11}(s)$ versus normalized temperature, s in the case of $Ra = 2 \times 10^4$ and $P_r = 1.0$.

We took the mode, $n = 3$ which is located in the inertial range. The circles are the data obtained from the model equation and the dotted line corresponds to the Gaussian distribution. The probability density function does not deviate much from the Gaussian form.

Figure 16 shows the plot of the probability density function of $P_{n=11}(s)$ versus normalized temperature component, the definition of s is the same as the case of Fig. 15. We took the mode, $n = 11$ which is located in the dissipation range. The circles show the data from the model equation and the dotted line is the Gaussian distribution. In this case the probability density function has a tail of the temperature component. The probability density function of the shell model with a passive-scalar has been reported in [15], which is close to the Gaussian form in the inertial range and has a tail in the dissipation

range. Our shell model shows the similar tendency with respect to the probability density function.

(e) Power spectrum of total energy

The log-log plot of the power spectrum of the total energy with $Pr = 1.0$ is shown for various Rayleigh numbers in Fig. 17. The frequency f is defined as $f = (j - 1)\Delta f$ ($j = 1, 2, \dots$) where $\Delta f = 2.5$. The solid line shows the power spectrum for $R_a = 2 \times 10^4$, the dotted-dashed line is the case of $R_a = 10^6$ and the dotted line corresponds to the case of $R_a = 10^7$. Continuous spectra are obtained in these cases. In the high frequency range $1,000 < f < 2,500$, the power spectra are fitted as $E(f) \sim f^{-3.88}$, $E(f) \sim f^{-2.45}$ and $E(f) \sim f^{-2.25}$ for $R_a = 2 \times 10^4$, $R_a = 10^6$ and $R_a = 10^7$, respectively. Especially, if we pay

Fig. 17 The log-log plot of power spectra of the total energy in the case of $Pr = 1.0$. The solid line shows the power spectrum for $R_a = 2 \times 10^4$, the dotted-dashed line is the case of $R_a = 10^6$ and the dotted line corresponds to the case of $R_a = 10^7$.

Fig. 18 The log-log plot of power spectra of the total energy in the case of $R_a = 2.0 \times 10^4$. The solid line shows the power spectrum for $Pr = 2.0$, the dotted-dashed line is the case of $Pr = 5.0$ and the dotted line corresponds to the case of $Pr = 8.0$.

attention to the middle frequency range $2 \times 10^2 < f < 10^3$ for $R_a = 10^7$, we found that the power spectrum is fitted as $E(f) = f^{-0.28}$ which is somewhat weaker dependence than $1/f$.

The power spectrum of the total energy with $R_a = 2 \times 10^4$ is shown for various Prandtl numbers in Fig. 18. The solid line corresponds to the case of $Pr = 2.0$, the dotted-dashed line to the case of $Pr = 5.0$ and the dotted line to the case of $Pr = 8.0$. For $Pr = 2.0$ and $Pr = 5.0$ the power spectra are fitted as $(f) \sim f^{-3.15}$ and $(f) \sim f^{-2.20}$, respectively, in the range of $1,000 < f < 2,500$. On the other hand, the period doubling cascade such as $f/16, f/8, f/4, f/2, \dots$, where f is initial frequency, is observed in the case of $Pr = 8.0$ [21]. This corresponds the fact that the fluctuation amplitude is very small and converges to the limit cycle.

Fig. 19 The R_a dependence on the mean value of heat flux $\langle q \rangle$ in the case with $Pr = 1$. The time average is taken in the interval between $t_1 = 0.2$ and $t_2 = 10.0$.

Fig. 20 The Pr dependence on the mean value of heat flux $\langle q \rangle$ in the case with $R_a = 2 \times 10^4$.

(f) Heat flux

The heat flux of this model is defined by

$$q = \sum_{n=1}^N u_n p_n^* \quad (48)$$

Figure 19 shows the R_a dependence on the mean value of heat flux $\langle q \rangle$ in the case with $P_r = 1$. The time average is taken in the interval between $t_1 = 0.2$ and $t_2 = 10.0$. It is found that the mean value of heat flux is fitted as $\langle q \rangle \sim R_a^{0.49}$ in the range of $2 \times 10^4 < R_a < 10^5$ and as $\langle q \rangle \sim R_a^{1.21}$ in the range of $2 \times 10^5 < R_a < 4 \times 10^6$, respectively. For $6 \times 10^6 < R_a$, the R_a dependence is somewhat weaker than $R_a^{1.21}$.

Figure 20 shows the P_r dependence on the mean value of heat flux $\langle q \rangle$ in the case with $R_a = 2 \times 10^4$. It is found that the mean value of heat flux is fitted as $\langle q \rangle \sim P_r^{1.65}$ in the wide range of P_r . In this case, the drastic change of P_r dependence across the threshold $P_r \sim 8$ does not observed compared to the results in Figs. 12 and 14.

5. Analysis of Intermittency by Shell Models for Rayleigh-Bénard Turbulence and by GOY Shell Model

In this section, we analyze an intermittency of the model equations with respect to the scaling exponent of higher order structure function. The number of shell is fixed for $N = 20$ and a constant C is fixed for $C = 10^{-2}$ in these calculations. In our model the p th order structure functions of u_n and p_n are defined by

$$S_p(n) = \langle |u_n|^p \rangle, \quad (49)$$

and

$$S'_p(n) = \langle |p_n|^p \rangle, \quad (50)$$

respectively. In the inertial range, the scaling exponents, ζ_p and ζ'_p are defined by

$$S_p(n) \propto k_n^{-\zeta_p}, \quad (51)$$

and

$$S'_p(n) \propto k_n^{-\zeta'_p}. \quad (52)$$

Firstly, the Rayleigh number dependence of the intermittency is investigated. Figure 21 shows the scaling exponents of $S_p(n)$ by the nearest neighbour real component model (I) for the fixed Prandtl number, $P_r = 1.0$. Cases with various Rayleigh numbers are plotted. The circles correspond to the exponents for $R_a = 2 \times 10^4$, the squares to the exponents for $R_a = 2 \times 10^5$ and the solid line to the K41 prediction ($\zeta_p = p/3$) [20]. The exponents obtained by this model are in a good

agreement with the K41 prediction. The deviation from the K41 prediction indicates an intermittency. Therefore, this model does not contain an intermittent nature, in other words, does not have a chaotic attractor.

Figure 22 is the plot of the scaling exponents of $S_p(n)$ obtained by the nearest and second nearest neighbour complex component model (II). The solid line corresponds to the K41 prediction. Circles and squares show the exponents with $R_a = 2 \times 10^4$ and $R_a = 2 \times 10^5$, respectively. The dotted line is the result from the GOY model [22]. The scaling exponents are found to be

Fig. 21 The scaling exponent of $S_p(n)$ by the nearest neighbour real component model (I) in the case of $P_r = 1.0$. The circles correspond to the exponents for $R_a = 2 \times 10^4$, the squares, to the exponents for $R_a = 2 \times 10^5$ and the solid line, to the K41 prediction.

Fig. 22 The plot of the scaling exponent obtained of $S_p(n)$ by the nearest and the second nearest neighbour complex component model (II). The circles correspond to the exponents for $R_a = 2 \times 10^4$, the squares, to the exponents for $R_a = 2 \times 10^5$ and the solid line, to the K41 prediction.

different from the K41 prediction. The strength of intermittency is a little bit weaker than that of GOY model for $p > 7$.

Figure 23 shows the plot of the scaling exponents of $S'_p(n)$ by the nearest and the second nearest complex component model (II) for the fixed Prandtl number, $P_r = 1.0$. The circles correspond to the exponents for $R_a = 2 \times 10^4$, the squares to the exponents for $R_a = 2 \times 10^5$ and the dotted line to the exponents calculated by the passive scalar field [15]. The difference of the scaling exponents between our model and the passive scalar field clearly seen in $S'_p(n)$, however, the Rayleigh number dependence of the scaling exponents is not clearly observed.

Next, the Prandtl number dependence of intermittency is investigated. Figure 24 shows the plot of the scaling exponents of $S_p(n)$ obtained by the nearest and the second nearest neighbour complex component model (II) for the fixed Rayleigh number, $R_a = 2 \times 10^4$. Cases of various Prandtl numbers are plotted. The circles represent the exponents for $P_r = 1.0$, the squares the exponents for $P_r = 2.0$, the triangles the exponents for $P_r = 5.0$ and the black circles correspond to the exponents for $P_r = 8.0$. The dotted line shows the results from the GOY model. The intermittency is observed for cases with $P_r = 1.0, 2.0$ and 5.0 . However for $P_r = 8.0$, it tends to approach the K41 line. The disappearance of intermittency is also confirmed by the maximum Lyapunov exponent as is shown in Fig. 14. In this case, the dissipation by the viscosity is stronger than that by the thermal diffusivity and the effect of thermal fluctuation is weak. In addition to it, the maximum linear growth rate increases according to the increase of P_r as is shown in Fig. 2, which indicates that the long wave length modes dominate the dynamics. The change of the strength of intermittency could be related with these facts. It is also found that the strength of intermittency tends to be weak in the small P_r limit ($P_r \ll 1$), which is consistent with the decrease of maximum Lyapunov exponent in the small P_r region shown in Fig. 14. In this case, the absolute values of linear growth rate are also small, so that nonlinear stabilization mechanism could act effectively on the short wave length region, which could be related with the strength of intermittency in this limit.

Figure 25 shows the exponents of $S'_p(n)$ obtained by the nearest and the second nearest neighbour complex component model (II) for the fixed Rayleigh number $R_a = 2 \times 10^4$. The circles correspond to the exponents for $P_r = 1.0$, the squares to the exponents for $P_r = 2.0$, the

triangles to the exponents for $P_r = 5.0$, the black circles to the exponents for $P_r = 8.0$ and the solid line to the K41 prediction. The dotted line shows the exponents in the case with a passive scalar field [15]. The fluctuation amplitude relative to the mean value of the total energy shown in Fig. 8 may be related with the strength of the intermittency. The disappearance of intermittency for $P_r = 8.0$ is also observed as in Fig. 24.

Fig. 23 The plot of the scaling exponent of $S'_p(n)$ by the nearest and the second nearest neighbour complex component model (II) in the case of $P_r = 1.0$. The circles correspond to the exponents for $R_a = 2 \times 10^4$, the squares, to the exponents for $R_a = 2 \times 10^5$, the solid line, to the K41 prediction and the dotted line, to the exponents calculated by the passive scalar field.

Fig. 24 The plot of the scaling exponent of $S_p(n)$ by the nearest and the second nearest neighbour complex component model (II) in the case of $R_a = 2 \times 10^4$. The circles represent the exponents for $P_r = 1.0$, the squares, the exponents for $P_r = 2.0$, the triangles, the exponents for $P_r = 5.0$ and the black circles correspond to the exponents for $P_r = 8.0$. The dotted line shows the results from GOY model.

Fig. 25 The exponent of $S'_p(n)$ by the nearest and the second nearest neighbour complex component model (II) in the case of $R_a = 2.0 \times 10^4$. The circles represent the exponents for $P_r = 1.0$, the squares, the exponents for $P_r = 2.0$, the triangles, the exponents for $P_r = 5.0$ and the black circles correspond to the exponents for $P_r = 8.0$. The dotted line shows the exponents calculated by a passive scalar field.

From these calculations, it is confirmed that the nearest neighbour real component model (I) does not show an intermittency, and the nearest and the second nearest neighbour complex component model (II) shows it. The freedom of the phase difference, i.e. a complex variable is an essential factor to produce the chaotic nature or intermittency. We also construct the nearest neighbour complex component model and confirm the nature of intermittency of it. Similarly, the original GOY model (complex component model) [6] shows the intermittency nature, on the other hand Gledzer model (real component model) [5] does not show it. The Prandtl number dependence of the intermittency is a newly obtained by our model. The original GOY model does not contain such a nature.

6. Summary and Discussions

In this paper, two new shell models, which are extended to describe a turbulence driven by the thermal instability are introduced and analyzed. One is the nearest neighbour real component model (I) and other is the nearest and the second nearest neighbour complex component model (II). Numerical simulations of two new shell models are performed. Parameter surveys are done with respect to the Rayleigh number, R_a and the Prandtl number, P_r .

For the nearest neighbour real component model (I), the time averaged wavenumber spectra of variables, $u_n(k)$ and $p_n(k)$ show the scaling laws, $u_n(k) \propto k_n^{-1/3}$ and

$p_n(k) \propto k_n^{-1/3}$, respectively in the inertial range. The Kolmogorov type scaling law is obtained. The total energy converges to a constant value in the nonlinear phase for arbitrary values of the Rayleigh number and the Prandtl number, the fact indicates that this model does not have a chaotic attractor.

For the nearest and the second nearest neighbour complex component model (II), the time averaged wavenumber spectra of variables, $u_n(k)$ and $p_n(k)$ are examined. They show the scaling like the Kolmogorov type scaling law as is obtained in the nearest neighbour real component model (I). The mean value of the total energy and its variance increase with the increase of the Rayleigh number. The ratio of the fluctuation amplitude to the mean value of the total energy is 20% ~ 90%, which shows a chaotic nature contained in this model. The maximum Lyapunov exponents are positive and the Rayleigh number dependence is found to be $-R_a^{0.53}$. This result indicates that a chaotic nature exists in the system. The power spectrum of the total energy is found to be a smooth function and has a fractional power law. This is another evidence that the system contains the chaos. The ratio of the variance to the mean value of total energy decreases with the increase of the Prandtl number. This indicates that the system moves from the chaotic attractor to the limit cycle. The maximum Lyapunov exponent becomes nearly 0 for $P_r = 8.0$ and period doubling cascade such as $f/16, f/8, f/4, f/2$ is observed. The nearest and the second nearest neighbour complex component model (II) shows both a chaotic attractor and a limit cycle, the appearance depends on the Prandtl number. This result is firstly pointed out by this paper. The dependences of R_a and P_r on the mean value of heat flux are examined. For R_a , the mean value of heat flux is fitted as $\langle q \rangle \sim R_a^{0.49}$ in the range of $2 \times 10^4 < R_a < 10^5$ and as $\langle q \rangle \sim R_a^{1.21}$ in the range of $2 \times 10^5 < R_a < 4 \times 10^6$, respectively. For P_r , the mean value of heat flux is fitted as $\langle q \rangle \sim P_r^{1.65}$ in the wide range of P_r . In this case, the drastic change of P_r dependence across the threshold $P_r \sim 8$ does not observed compared to the results in Figs. 12 and 14.

We also investigated the property of an intermittency characteristic. In the nearest neighbour real component model (I), the scaling exponents of both variables, $u_n(k)$ and $p_n(k)$ show the good agreement with the K41 prediction. This fact indicates that the nearest neighbour real component model (I) does not contain the characteristic of an intermittency. In the nearest and the second nearest neighbour complex component model (II) the scaling exponent is different from the K41

prediction. The clear dependence on the Rayleigh number is not observed from the calculation of scaling exponents. The dependence on the Prandtl number is newly obtained, which is the decreasing function of the Prandtl number. When $P_r \rightarrow 8.0$, the scaling exponent approaches the K41 prediction. This may correspond to the disappearance of an intermittency. The scaling exponents of $S_p(n)$ and $S_p'(n)$ are found to be smaller than those in the GOY model with the passive scalar field. From our calculations, we confirm the phase difference between the shell (i.e., complex variables) are important for existing a chaos and an intermittency.

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Appendix A: The Maximum Lyapunov Exponent

In this section, we show the procedure to calculate a Lyapunov exponent. Lyapunov exponent by the nearest and the second nearest neighbour complex component model is evaluated as follows.

Linearizing (14) and (15) we obtain the equations:

$$\begin{aligned} \frac{d\delta u_n}{dt} = & i(k_n \delta u_{n+1} u_{n+2} + k_n u_{n+1} \delta u_{n+2} \\ & - k_{n-2} \delta u_{n-1} u_{n+1} - k_{n-2} u_{n-1} \delta u_{n+1} \\ & - k_{n-3} \delta u_{n-1} u_{n-2} - k_{n-3} u_{n-1} \delta u_{n-2})^* \\ & + P_r \delta p_n - P_r k_n^2 \delta u_n, \end{aligned} \quad (A1)$$

and

$$\begin{aligned} \frac{d\delta p_n}{dt} = & i[k_{n-1}(\delta u_{n-1} p_{n+1} + u_{n-1} \delta p_{n+1} \\ & - \delta u_{n+1} p_{n-1} + u_{n+1} \delta p_{n-1}) \\ & - k_{n-3}(\delta u_{n-2} p_{n-1} + u_{n-2} \delta p_{n-1} \\ & + \delta u_{n-1} p_{n-2} + u_{n-1} \delta p_{n-2}) \\ & + k_{n+2}(\delta u_{n+1} p_{n+2} + u_{n+1} \delta p_{n+2} \\ & + \delta u_{n+2} p_{n+1} + u_{n+2} \delta p_{n+1})]^* \\ & + R_a \delta u_n - k_n^2 \delta p_n. \end{aligned} \quad (A2)$$

This system has $4N$ Lyapunov exponents. We define the k th eigenvector ($k = 1, 2, \dots, 4N$) given by

$$\mathbf{v}^{(k)} = \begin{pmatrix} \delta u_1^r \\ \delta u_1^i \\ \vdots \\ \delta u_N^r \\ \delta u_N^i \\ \delta p_1^r \\ \delta p_1^i \\ \vdots \\ \delta p_N^r \\ \delta p_N^i \end{pmatrix} \quad (A3)$$

where superscript 'r' and 'i' represent the real part and the imaginary part, respectively. We integrate (A1) and (A2) with the $4N$ initial conditions given by

$$\begin{aligned} \mathbf{v}_0^{(1)} &= (1, 0, \dots, 0), \mathbf{v}_0^{(2)} = (0, 1, \dots, 0), \dots, \\ \mathbf{v}_0^{(4N)} &= (0, 0, \dots, 1). \end{aligned} \quad (A4)$$

By the Gram-Schmidt method we normalize the $4N$ linear solutions, $\mathbf{v}^{(k)}$, $k = 1, \dots, 4N$ under the constant intervals;

$$\begin{aligned} \mathbf{v}^{(1)} &= \frac{\mathbf{v}^{(1)}}{\|\mathbf{v}^{(1)}\|}, \\ \mathbf{v}^{(2)} &= \frac{\mathbf{v}^{(2)} - (\mathbf{v}^{(2)} \cdot \mathbf{v}^{(1)}) \mathbf{v}^{(1)}}{\|\mathbf{v}^{(2)} - (\mathbf{v}^{(2)} \cdot \mathbf{v}^{(1)}) \mathbf{v}^{(1)}\|}, \\ \mathbf{v}^{(4N)} &= \frac{\mathbf{v}^{(4N)} - (\mathbf{v}^{(4N)} \cdot \mathbf{v}^{(4N-1)}) \mathbf{v}^{(4N-1)} - \dots - (\mathbf{v}^{(4N)} \cdot \mathbf{v}^{(1)}) \mathbf{v}^{(1)}}{\|\mathbf{v}^{(4N)} - (\mathbf{v}^{(4N)} \cdot \mathbf{v}^{(4N-1)}) \mathbf{v}^{(4N-1)} - \dots - (\mathbf{v}^{(4N)} \cdot \mathbf{v}^{(1)}) \mathbf{v}^{(1)}\|}. \end{aligned}$$

where $\|\dots\|$ represents the norm. By these normalizations, we obtain the increments of the solutions, $\varepsilon_k = \|\mathbf{v}^{(k)}(t)\|$ in the orthogonal direction in the phase space. A local Lyapunov exponent, λ_i is defined as the local growth rate of an error, ε_k measured over the finite time interval, $[t_0, t]$:

$$\lambda_i(t) = \frac{\log \{\varepsilon_i(t) / \varepsilon_i(t_0)\}}{t - t_0}, \quad (A5)$$

where $i = 1, \dots, 4N$. In the limit of $t \rightarrow \infty$, we obtain the Lyapunov exponent, λ_i . The maximum value of λ_i is defined as the maximum Lyapunov exponent.

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