A radio Frequency Method for Determining Plasma Density

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Abstract

A method for determining plasma density by use of a solenoid is proposed. The calculation of the distributions of the electric and magnetic field in a homogeneous plasma placed in the solenoid is made and the current flowing in the solenoid or the impedance looking into the solenoid is expressed in terms of the plasma density. The method is to apply a radio frequency voltage across the solenoid and to measure the impedance by either a Lissajous' figure on a X-Y scope or an impedance bridge.

I. Introduction

As one of the convenient means for determining the electrical conductivity or the electron density of a plasma, the radio frequency method by using a solenoid has been extensively developed.

Lary and Olson have made use of a single-layered solenoid enclosed within an insulating tube in conjunction with a radio frequency oscillator detector to measure the ohmic dissipation of a small amount of r.f. power within the plasma.⁽¹⁾ The existence of the plasma leads to an apparent increase of the resistance and an apparent decrease of the inductance. The change of the resistance as a function of the conductivity and frequency was found with a grid dip meter connected to the solenoid. The

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frequency of 10 to 50 Mc/s was chosen in order to obtain a detectable change of the grid current of the oscillator. The solenoid samples only the plasma within its immediate vicinity and could therefore be used to resolve the profile of the conductivity or plasma density, if the size of the solenoid is small.

Marshall, Hill and Crapo developed a similar method in which a small solenoid was mounted in ferrite supports.⁽²⁾ The voltage across the solenoid is decreased by the presence of plasma within it, because of a decrease of apparent coil impedance. Their treatment is based on the theory of ideal transformer in the electrical engineering. The secondary impedance, especially resistance, arising from an interaction between the electromagnetic field and plasma is reduced to the primary impedance according to the turn ratio of the transformer.

Also, Person has used an external solenoid to measure the plasma conductivity.⁽³⁾ He obtained an expression for the voltage change across the solenoid by solving a time-dependent circuit equation, but ignoring the distortion of the electromagnetic field due to an azimuthal current in the plasma. His measurement was made with a pulse voltage of the duration of 1μ sec., instead of the sinusoidal voltage.

In the present paper, we derive a formula for the impedance looking into the solenoid in which a homogeneous plasma is present, and propose a new method based upon the impedance measurement.

This problem is similar to those appeared in the treatment of induction type high frequency discharge and induction heating⁽⁴⁾⁽⁵⁾

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of conductive materials.

II. Basic Equations

Let us consider a cylindrical plasma with an uniform density distribution bound by a solenoid, across which a radio frequency voltage is applied. Neglecting the end effect, the electric and magnetic field within the solenoid can be evaluated with the help of Maxwell's equation. The impedance formula can be derived from the result thus obtained together with the circuit equation. Considering the above mentioned assumptions and taking the permittivity equal to unity, Maxwell's equation and the current density equation are written as

$$\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{H}}{\partial t} \quad (1)$$

$$\nabla \times \mathbf{H} = \frac{4\pi}{c} \mathbf{J} + \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} \quad (2)$$

$$\nabla \cdot \mathbf{H} = 0 \quad (3)$$

$$\mathbf{J} = \sigma \mathbf{E} = (\sigma_r + i\sigma_i) \mathbf{E} \quad (4)$$

where $\mathbf{E}$ is the electric field, $\mathbf{H}$ the magnetic field, c the velocity of the light, $\mathbf{J}$ the current density, σ_r the real part of the conductivity, σ_i the imaginary part of the conductivity.

The above equations (1) to (4) yield the following equations describing the components of the fields in cylindrical coordinates when the fields harmonically vary as $e^{i\omega t}$:

$$\frac{1}{r} \frac{\partial E_z}{\partial \theta} - \frac{\partial E_\theta}{\partial z} = -\frac{1}{c} \frac{\partial H_r}{\partial t} = -\frac{i \omega}{c} H_r \quad (5)$$

$$\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r} = -\frac{1}{c} \frac{\partial H_\theta}{\partial t} = -\frac{i \omega}{c} H_\theta \quad (6)$$

$$\frac{1}{r} \left\{ \frac{\partial}{\partial r} (r E_\theta) - \frac{\partial E_r}{\partial \theta} \right\} = -\frac{1}{c} \frac{\partial H_z}{\partial t} = -\frac{i \omega}{c} H_z \quad (7)$$

$$\frac{1}{r} \frac{\partial H_z}{\partial \theta} - \frac{\partial H_\theta}{\partial z} = \left(\frac{4\pi\sigma}{c} + \frac{i\omega}{c} \right) E_r \quad (8)$$

$$\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r} = \left(\frac{4\pi\sigma}{c} + \frac{i\omega}{c} \right) E_\theta \quad (9)$$

$$\frac{1}{r} \left\{ \frac{\partial}{\partial r} (r H_\theta) - \frac{\partial H_r}{\partial \theta} \right\} = \left(\frac{4\pi\sigma}{c} + \frac{i\omega}{c} \right) E_z \quad (10)$$

III. The Field Distributions and Azimuthal Current

We may now assume that the derivatives with respect to θ and z are equal to zero, and consequently it follows that

$$E_z = 0, \quad H_r = 0 \quad (11)$$

Hence, eq. (7) reduces to

$$\frac{1}{r} \frac{\partial}{\partial r} (r E_\theta) = -\frac{i \omega}{c} H_z \quad \text{or} \quad \frac{dE}{dr} + \frac{E}{r} = -\frac{i \omega}{c} H \quad (12)$$

Furthermore, from eq. (9) we obtain a similar equation for the magnetic field as

$$\frac{dH}{dr} = -\left(\frac{4\pi\sigma}{c} + \frac{i\omega}{c} \right) E \quad (13)$$

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After eliminating E or H from eqs. (12) and (13), we have

$$\frac{d^2 E}{dr^2} + \frac{1}{r} \frac{dE}{dr} - \left(\frac{1}{r^2} + \frac{i 4 \pi \omega \sigma}{c^2} - \frac{\omega^2}{c^2} \right) E = 0 \quad (14)$$

$$\frac{d^2 H}{dr^2} + \frac{1}{r} \frac{dH}{dr} - \left(\frac{i 4 \pi \omega \sigma}{c^2} - \frac{\omega^2}{c^2} \right) H = 0 \quad (15)$$

Now, we introduce a new variable x

$$x = i n r \quad (16)$$

where

$$n^2 = \frac{i 4 \pi \omega \sigma}{c^2} - \frac{\omega^2}{c^2} \quad (17)$$

Making use of the new variable, the following transformations are obtained :

$$\frac{dH}{dr} = \frac{dH}{dx} \frac{dx}{dr} = \frac{dH}{dx} i n \quad (18)$$

$$\frac{d^2 H}{dr^2} = \frac{d}{dr} \left(\frac{dH}{dr} \right) = \frac{d}{dr} \left(\frac{dH}{dx} i n \right) = \frac{d}{dx} \left(\frac{dH}{dx} i n \right) \frac{dx}{dr} = - \frac{d^2 H}{dx^2} n^2 \quad (19)$$

Substituting the above relations into eq. (15) yields

$$\frac{d^2 H}{dx^2} + \frac{1}{x} \frac{dH}{dx} + H = 0 \quad (20)$$

The solution to eq. (20) is found to be expressed by the zeroth Bessel function as

$$H = A J_0(x) = A J_0(i n r) \quad (21)$$

where A is a constant.

Returning to the electric field E, we see that eq.(14) is the modified Bessel equation. The solution to eq.(14) is then given by

$$E = B I_1(nr) = -i B J_1(inr) \quad (22)$$

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Setting $r=R$ in eqs. (21) and (22), we obtain the expressions for the magnetic and electric field respectively

$$H = H_R \frac{J_0(inr)}{J_0(inR)} \quad (23)$$

$$E = E_R \frac{J_1(inr)}{J_1(inR)} \quad (24)$$

where H_R and E_R are the magnetic and electric field at $r=R$, respectively.

From eq.(13), the electric field is written by

$$E = -\frac{c}{4\pi\sigma} \frac{dH}{dr} - \frac{i\omega E}{4\pi\sigma} \quad (25)$$

Substituting eq.(23) into eq. (25), we have

$$E = \frac{cH_R}{4\pi\sigma} \frac{in J_1(inr)}{J_0(inR)} - \frac{i\omega E}{4\pi\sigma} \quad (26)$$

Hence, the current density is given by

$$J = \frac{cH_R}{4\pi\sigma} \frac{in J_1(inr)}{J_0(inR)} - \frac{i\omega E}{4\pi} \quad (27)$$

The relation between E_R and H_R is found from eq. (26) by setting $r=R$. It is

$$\frac{E_R}{H_R} = \frac{icn J_1(inR)}{4\pi\sigma + i\omega} \quad (28)$$

The azimuthal current per unit length I is then calculated from eq. (27) as

$$I = \int_0^R J dr = \left\{ -\frac{cH_R}{4\pi J_0(inR)} + \frac{\omega E_R}{4\pi n J_1(inR)} \right\} \{ J_0(inR) - J_0(in0) \} \quad (29)$$

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IV. Impedance formula

In order to evaluate the impedance looking into the solenoid, a circuit analysis is needed. The electric circuit for the present case is shown in Fig. 1. R_1 and L_1 represent the primary resistance and inductance respectively, while R_2 and L_2 those of the secondary circuit or the plasma in which the azimuthal current flows.

Fig. 1. Measuring Circuit.

The Kirchhoff's law for the circuit shown in Fig. 1 is described as

$$i \omega L_1 I_1 + R_1 I_1 + i \omega M I_2 = V \quad (30)$$

$$i \omega L_2 I_2 + R_2 I_2 + i \omega M I_1 = 0 \quad (31)$$

where M is the mutual inductance, I_1 and I_2 are the primary and secondary current respectively. Also, I_2 is

$$I_2 = I \ell \quad (32)$$

where ℓ is the length of the solenoid.

Combining eqs.(29), (30) and (32) and putting a relation $H_R = 4 \pi N I_1 / c$, where N is the number of the turns per unit

length, we obtain the following equation:

$$i\omega L_1 I_1 + R_1 I_1 - i\omega M \ell \left\{ \frac{N I_1}{J_0(inR)} - \frac{\omega E_R}{4\pi n J_1(inR)} \right\} \{ J_0(inR) - J_0(ino) \} = V \quad (33)$$

where allowance is made that the sign of I_2 is opposite.

Rearranging the above equation, we finally have

$$I_1 \left\{ i\omega L_1 + R_1 + i\omega M N \ell \frac{J_0(inR) - J_0(ino)}{J_0(inR)} \right\} = V + \frac{i\omega^2 M \ell E_R}{4\pi n J_1(inR)} \times \{ J_0(inR) - J_0(ino) \} \quad (34)$$

The second term on the right-hand side represents the component due to a capacitive current.

A particular attention is directed to the case of such a low frequency that the capacitive current can be neglected. In this case, the formula for the impedance looking into the solenoid is easily obtained from eq.(34). It is

$$Z_1 = \frac{V}{I_1} = R_1 + i\omega L_1 + i\omega M N \ell \frac{J_0(inR) - J_0(ino)}{J_0(inR)} \quad (35)$$

where $n^2 = i4\pi\sigma\omega/c^2$ and $inR = i(4\pi\sigma\omega/c^2)^{1/2} R = (R/c)(4\pi\sigma\omega)^{1/2} i^{3/2}$.

We shall discuss the above result for two extreme cases, i.e., pure resistive ($\sigma = \sigma_r$) and pure inductive ($\sigma = i\sigma_i$). The former corresponds to the condition that the angular frequency of the field is lower than the collision frequency, i.e., $\omega \ll \nu_c$, while the latter to the condition that $\omega \gg \nu_c$, since the following relation holds between σ and ν_c :

$$\sigma = \sigma_r + i\sigma_i = \frac{ne^2}{m} \left(\frac{\nu_c}{\nu_c^2 + \omega^2} - i \frac{\omega_c}{\nu_c^2 + \omega^2} \right) \quad (36)$$

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For the resistive case, it follows that

$$\begin{aligned}
 J_0(i n R) &= J_0 \left\{ \frac{R}{c} (4 \pi \sigma_r \omega)^{1/2} i^{3/2} \right\} = \text{ber} \left\{ \frac{R}{c} (4 \pi \sigma_r \omega)^{1/2} \right\} \\
 &+ i \text{bei} \left\{ \frac{R}{c} (4 \pi \sigma_r \omega)^{1/2} \right\} = \text{ber} \left\{ \frac{R}{c} \left(\frac{\omega_p^2 \omega}{\nu_c} \right)^{1/2} \right\} \\
 &+ i \text{bei} \left\{ \frac{R}{c} \left(\frac{\omega_p^2 \omega}{\nu_c} \right)^{1/2} \right\}
 \end{aligned} \tag{37}$$

where ω_p is the plasma frequency defined as $\omega_p^2 = 4 \pi n e^2 / m$

and $\sigma_r = \omega_p^2 / 4 \pi \nu_c$.

As a consequence, eq. (35) becomes

$$Z_1 = R_1 + i \omega L_1 + i \omega M N \ell \left\{ \frac{\text{ber} \left\{ \frac{R}{c} \left(\frac{\omega_p^2 \omega}{\nu_c} \right)^{1/2} \right\} + i \text{bei} \left\{ \frac{R}{c} \left(\frac{\omega_p^2 \omega}{\nu_c} \right)^{1/2} \right\} - 1}{\text{ber} \left\{ \frac{R}{c} \left(\frac{\omega_p^2 \omega}{\nu_c} \right)^{1/2} \right\} + i \text{bei} \left\{ \frac{R}{c} \left(\frac{\omega_p^2 \omega}{\nu_c} \right)^{1/2} \right\}} \right\} \tag{38}$$

The above equation states that the presence of plasma affects both the resistive and inductive component of the impedance looking into the solenoid. Of course, it is apparent that the effect becomes appreciable as the plasma density increases.

On the other hand, for the inductive case, we know that $J_0(inR) = J_0 \left\{ - \left(\frac{R}{c} \right) (4 \pi \sigma_i \omega)^{1/2} \right\} = J_0(-i R \omega_p / c) = J_0(R \omega_p / c)$,

where the relation $\sigma_i = -n e^2 / m \omega = -\omega_p^2 / 4 \pi \omega$.

The impedance formula for this case is thus given by

$$Z_1 = R_1 + i \omega L_1 + i \omega M N \ell \left\{ \frac{J_0 \left(\frac{R \omega_p}{c} \right) - 1}{J_0 \left(\frac{R \omega_p}{c} \right)} \right\} \tag{39}$$

We may say from the above result that the interaction between plasma and electromagnetic field leads to a decrease of the apparent inductance.

V. Discussion and Conclusion

In the preceding paragraphs we solved the simplified problem by neglecting the capacitive current and assuming the uniformity of density. The formula for the impedance looking into the solenoid was derived for two extreme cases, i.e. $\omega \ll \nu_c$ and $\omega \gg \nu_c$.

The interaction term arising from the existence of plasma inside the solenoid was described as a function of the plasma frequency or plasma density. Thus the measurement of the impedance allows us to estimate the plasma density, provided that the mutual inductance M is known. However, the theoretical evaluation of M is not simple, so that we must be satisfied with a calibrating method by use of a conducting material of a known conductivity or density. In practice, the calibrating method has been extensively used in other work.⁽¹⁾⁽²⁾

The measurement of the impedance may be carried out on a X-Y scope or by an impedance bridge.

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A Lissajous' figure displayed on the X-Y scope may provide an information of the phase angle between the voltage and current, from which the inductive or resistive component is evaluated. Especially, for the inductive case, the evaluation of plasma density by means of the Lissajous' figure is easy because of the simplicity of the impedance formula. In our method another precaution must be paid for eliminating the capacitive current due to the voltage across the solenoid. This is accomplished by a shielding metallic cylinder placed just inside the solenoid.

The choice of the frequency is important for the purpose of detection with high sensibility and reliability. The first key point is to avoid the electron cyclotron resonance. The second is to avoid the resonance of a cable, which is of necessity connected to the remaining circuit and has a resonant frequency of the order of 10 Mc/s.

References

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