A Negative V" System by Use of Helical Tube

by Tsuneo Amano

Dep. of Physics Kyoto Univ., Kyoto*

(Received 25th May, 1965)

Since the pioneering work of Furth and Rosenbluth (1), stable confinements of plasma in closed (troidal) systems, where the integral $\oint dl/B$ has a local maximum, have been considered by many authors (2) ~ (6). In this note we present a new such system which makes use of the curvature in a helix. We consider a helical tube whose axis is a helix of radius a and pitch $2\pi/h$ twisted around some axis, say the z -axis of a cartesian coordinates system (x,y,z) . When a solenoidal winding is applied on the vacuum vessel of this tube, the magnetic field lines have stabilizing or destabilizing curvatures according as they lie on the inside edge of the tube (the side of z -axis) or the outside edge. Then stability can be obtained by using magnetic surfaces in which the lines of force spend a longer time at the inner edge than at the outer edge. Such magnetic surfaces can be obtained by means of $l=2$ and $l=3$ helical windings on the vacuum vessel of the tube in addition to the solenoidal winding.

* Now at Plasma Physics Laboratory, Princeton Univ., Princeton, New Jersey

The orthogonal coordinates system (ξ, η, ζ) referred to the axis of the helix is obtained from the transformation,

$$\begin{aligned} x &= a \cos r \zeta + \xi \cos u \cos r \zeta - (1 + a^2 h^2)^{-1/2} \xi \sin u \sin r \zeta \\ y &= a \sin r \zeta + \xi \cos u \sin r \zeta + (1 + a^2 h^2)^{-1/2} \xi \sin u \cos r \zeta \\ z &= -a r \xi \sin u + (1 + a^2 h^2)^{-1/2} \zeta \end{aligned} \quad (1)$$

with $r = h(1 + a^2 h^2)^{-1/2}$ and $u = \eta - h \zeta (1 + a^2 h^2)^{-1}$. The line element, ds , in this system is defined by

$$ds^2 = g_1^2 (d\xi)^2 + g_2^2 (\xi d\eta)^2 + g_3^2 (d\zeta)^2, \quad (2)$$

with

$$g_1 = 1, g_2 = \xi, g_3 = 1 + \epsilon \xi \cos u, \epsilon = a^2 h^2 (1 + a^2 h^2)^{-1}.$$

We consider the magnetic field given by the scalar potential

$$\phi = \zeta + \delta_2 r \xi^2 \sin 2u + \delta_3 r^2 \xi^3 \sin 3u \quad (3)$$

This satisfies Laplace equation up to terms of order $(a r)^2$ which we assume to be small.

The magnetic surfaces must satisfy

$$\nabla \phi \cdot \nabla \phi = 0 \quad (4)$$

This can be solved as a power series in $r \xi$ to give

$$\begin{aligned} \phi &= \xi^2 - 2 \delta_2 \xi^2 \cos 2u - 4 \epsilon \delta_2 (1 - 2 \delta_2) \xi^3 \cos u - \{ 2 \delta_2 r^2 \\ &\quad + 4 \epsilon \delta_2 / 3 (1 - 2 \delta_2) \} \xi^3 \cos 3u + \dots \end{aligned} \quad (5)$$

We can evaluate $V''(\phi)$ by calculating

$$W = \frac{1}{L} \int_0^L d\ell / B = \int_0^{2\pi} \frac{g_3}{B_0} \frac{d\zeta}{du} du / \int_0^{2\pi} g_3 \frac{d\zeta}{du} du, \quad (6)$$

with the integral taken over a length such that a given field line goes through a complete period. By expanding in powers of $r \phi^{1/2}$, we obtain

$$V''(\phi) = \frac{dW}{d\phi} = \frac{1}{2\pi a^2} \left\{ \frac{(1 + 2 \delta_2) (1 + 2 \delta_2 + 2 \delta_2^2)}{(1 - 4 \delta_2^2)^{3/2} (1 - 2 \delta_2)} \right\} \epsilon^2$$

A Negative V'' System by Use of Helical Tube

$$+ \frac{6 \epsilon \delta_2 (1 + 2 \delta_2)}{(1 - 4 \delta_2^2)} (2 \delta + \frac{\epsilon \delta_2}{3 r a (1 - 2 \delta_2)}) a r + \frac{8 \delta_2^2 a^2 r^2}{(1 - 4 \delta_2^2)^{3/2}} \} \\ + \frac{18 r^4 \phi \delta_3^2 (1 + 2 \delta_2) (1 - 2 \delta_2 + 16 \delta_2^2 + 3 \delta_2^3)}{\pi^2 (1 - 4 \delta_2^2)^2} + \dots \quad (7)$$

Here we have retained only the significant terms in the ϕ order. We also have replaced ψ by ϕ , the magnetic flux contained in the magnetic surface ϕ , using the relation

$$\phi = \frac{1}{2} \int_0^{2\pi} \xi^2 d\eta = \frac{\pi \psi}{(1 - 4 \delta_2^2)^{3/2}} + \dots \quad (8)$$

It is clear that we can achieve negative V'' by making $\delta_2 \delta_3 < 0$ and $|\delta_3|$ sufficiently large.

Instead of helical tube, we can equivalently use a straight tube by applying an $l = 1$ field on it. We consider the magnetic field given by the scalar potential

$$\phi = z + \delta_1 r (1 + \frac{h^2 r^2}{8}) \sin(\theta - h z) + \delta_2 h r^2 \sin 2(\theta - h z) + \delta_3 h^2 r^3 \sin 3(\theta - h z), \quad (9)$$

which is similar to Johnson's system⁽⁴⁾, but the field pitches are different. The magnetic surfaces are given by

$$\psi = r^2 - \frac{2 \delta_1 r}{h} (1 + \frac{3}{8} h^2 r^2) \cos(\theta - h z) - 2 \delta_2 r^2 \cos 2(\theta - h z) - 2 \delta_3 h r^3 \cos 3(\theta - h z), \quad (10)$$

the axis of which is a helix of pitch $2\pi/h$ and radius

$$a = \frac{1}{(\frac{9}{4} \delta_1 + \delta_3) h} \left\{ 1 - 2 \delta_2 - \sqrt{1 - 2 \delta_2 - 2 \delta_1 (6 \delta_3 + \frac{9}{4} \delta_1)} \right\} \quad (11)$$

If we use Eq.(1) to transform to the coordinates system centered on this magnetic axis, the potential (9) becomes

$$\phi = \frac{\zeta}{(1+a^2 h^2)} + \frac{\delta_1 h^2}{8} \xi^3 \sin u + h \left\{ \frac{\zeta a h \delta_1}{8} + \delta_2 + 3 a h \delta_3 \right\} \xi^2 \sin 2u \\ + \delta_3 h^2 \xi^3 \sin 3u + \dots \quad (12)$$

This potential is clearly identical to the $l = 2$ and 3 system in a helical tube, except for the second term which is small.

Furthermore, it can be shown that the second term always contributes favorably to the stability.

We can also obtain the stability by using the curvature of a torus instead of a helical tube⁽⁵⁾. Here we propose $l = 2$ and 3 fields in place of Taylor's $l = 3$ and transverse fields. We consider the magnetic field,

$$B = \nabla \left\{ \xi + \delta_2 h \xi^2 \sin(2\eta - h\zeta) + \delta_3 h^2 \xi^3 \sin(3\eta - h\zeta) \right\} \quad (13)$$

and the metric

$$g_3 = 1 - K \xi \cos \eta, \quad (14)$$

where we denoted the curvature of the torus by K . Then on the axis

$$V'' = \frac{1}{\pi} \left(4 \delta_2^2 h^2 + \frac{6 h K \delta_3}{\delta_2} + \frac{K^2}{2} \right), \quad (15)$$

where we assumed that δ_2 and δ_3 are comparable order of magnitude.

This system is superior to Taylor's, because the $l = 2$ field can provide large rotational transform so that the distance between good and bad regions of toroidal curvature can be made short and the critical β against the so-called "balloning instability"⁽⁶⁾ is much improved.

The author wishes to express his sincerest gratitude to Dr. J. L. Johnson for valuable discussions and criticism.

A Negative V" System by Use of Helical Tube

He also wishes to express his thanks to Drs. L. Spitzer, Jr., R. M. Kulsrud and his colleagues in the theoretical group at the Princeton Plasma Physics Laboratory for helpful discussions.

References

- (1) H. P. Furth and M. N. Rosenbluth, Phys. Fluids 7, 764 (1964).
- (2) D. J. Bendaniel and H. Hurivitz, Jr., Phys. Fluids 7, 1974 (1964).
- (3) A. Lenard, Phys. Fluids 7, 1875 (1964).
- (4) J. L. Johnson, Phys. Fluids 7, 2015 (1964).
- (5) J. B. Taylor,
- (6) F. Shure and R. M. Kulsrud, Bull. Am. Phys. Soc. II 10, 199 (1965).