Steady Plasma Electric Field as a Function of
Ionization Degree

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Preface

An electric field is necessary to maintain steady plasma. This field is determined as a function of the ionization degree from the transport equations for three components system. A maximum appears at a value of the ionization degree. The presence of the maximum electric field E_m means that if the applied electric field is less than E_m fully ionized plasma can not be attained.

1. Introduction

Fully ionized plasma is an idealized form which is free from complicated collision processes associated with neutral particles. The author has been concerned if the fully ionized plasma is really attainable under the ordinary laboratory conditions or what is the necessary condition for the maintenance of the fully ionized plasma. Due to poor reliability of the estimated value of the plasma density, no one has had confidence for the fact that the fully

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ionized plasma is really obtained. The author will point out in this paper that the attainment of the fully ionized plasma is limited by various collision processes resulting in energy loss, which are dependent on the velocity and density of the particles associated with the process.

To establish self-sustained plasma, the energy loss should be compensated by an externally applied energy. The external energy is usually given by an applied electric field. Thus the solution to the present problem will be provided by finding the necessary electric field as a function of the average velocity and density of the particles, i.e., the temperature and ionization degree. Whilst, the temperature of the constituent may be a function of the ionization degree. As a consequence, the necessary electric field will be described in terms of the ionization degree only.

There may take place many collision processes between the constituents, i.e., the electron, ion and neutral particle, in partially ionized plasma, which appears before reaching the fully ionized plasma. Also the interaction between the particles and radiation may occur. The rate of these collision processes is expressed by the collision frequency, which depends on the density of the target particle and the relative velocity of the testing particle, to that of the target particle in other words, the ionization degree and the temperature of the particle.

II. Basic Equations

We start with the transport equations for three components system, i.e., those describing the continuity, the momentum conservation and the energy balance for the electron, ion and neutral particle,

respectively. We restrict ourselves to the case of uniform plasma for simplicity. Also we assume that negative ion and cumulative process are not present.

The continuity equations for three components are given as

$$\frac{\partial n_e}{\partial t} = n_e \nu_i - \beta n_e n_i \quad (1)$$

$$\frac{\partial n_i}{\partial t} = n_e \nu_i - \beta n_e n_i \quad (2)$$

$$\frac{\partial n_g}{\partial t} = -n_e \nu_i + \beta n_e n_i \quad (3)$$

where n is the density, t the time, ν_i the ionizing collision frequency, β the recombination coefficient. The subscripts e , i and g symbolize the electron, ion and neutral particle, respectively.

The equations describing the conservation of momentum for three components are given by

$$\frac{\partial v_e}{\partial t} + \frac{eE}{m_e} = -\nu_{eg} (v_e - v_g) - \nu_{ei} (v_e - v_i) \quad (4)$$

$$\frac{\partial v_i}{\partial t} - \frac{eE}{m_i} = -\nu_{ig} (v_i - v_g) - \frac{m_e}{m_i} \nu_{ei} (v_i - v_e) \quad (5)$$

$$\frac{\partial v_g}{\partial t} = -\frac{m_e}{m_g} \frac{n_e}{n_g} \nu_{eg} (v_g - v_e) - \frac{n_i}{n_g} \nu_{ig} (v_g - v_i) \quad (6)$$

where v is the velocity, e the electron charge, m the mass, ν_{mn} the collision frequency of the m th particle against the n th particle.

The energy balance equations for three components are

$$\begin{aligned} \frac{\partial W_e}{\partial t} + 2eE v_e = & -\nu_{eg} \left(\frac{2m_e}{m_g} \right) (W_e - W_g) - \nu_{ei} \left(\frac{2m_e}{m_i} \right) (W_e - W_i) - \nu_i e V_i \\ & - \sum e V_* \nu_* - \Gamma_{br} - W_w + \sum e V_m \nu_{em} + A \end{aligned} \quad (7)$$

$$\frac{\partial W_i}{\partial t} - 2eE v_i = -\nu_{ig} (W_i - W_g) - \left(\frac{2m_e}{m_i} \right) \nu_{ei} (W_i - W_e) \quad (8)$$

$$\frac{\partial W_g}{\partial t} = -\left(\frac{2m_e}{m_g} \right) \left(\frac{n_e}{n_g} \right) (W_g - W_e) - \left(\frac{n_i}{n_g} \right) \nu_{ig} (W_g - W_i) + \frac{n_e n_i}{n_g} \beta e V_i \quad (9)$$

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where W is the energy, V_i the ionization potential, V_* the excitation potential, Γ_{br} the loss due to the Bremsstrahlung, W_w the loss to the surrounding, β' the radiation-free recombination coefficient, V_m the metastable potential, ν_{em} the super elastic collision frequency and A the energy increment due to the self-absorption of radiation, including the bound-free and free-free transitions.

The right hand side of the above equations represent the energy losses through the elastic and inelastic collisions, in which the positive sign indicates restoration of energy.

III. Steady State Solution

Here, we are concerned only with a steady state. Setting $\partial/\partial t = 0$ in all the equations above described, and combining eqs.(4), (5) and (6) to eliminate v_g , we obtain the following equation :

$$v_e - v_i = - \frac{\left(\frac{\nu_i g}{m_e} + \frac{\nu_e g}{m_i} \right) e E}{\nu_i g \nu_{eg} + \nu_i g \nu_{ei} + \left(\frac{m_e}{m_i} \right) \nu_{ei} \nu_{eg}} \quad (10)$$

On the other hand, combining eqs.(7), (8) and (9) leads to

$$0 = 2 e E (v_e - v_i) + L \quad (11)$$

Where

$$L = \nu_i e V_i + \sum e V_* \nu_* + \Gamma_{br} + W_w - \sum e V_m \nu_{em} - n_i \beta' e V_i - A \quad (12)$$

By substituting eq.(12) into eq.(11), we get the following expression for the steady electric field in terms of the collision frequencies and energy losses :

$$E^2 = \frac{L}{2e^2} \frac{\nu_i g \nu_{eg} + \nu_i g \nu_{ei} + \frac{m_e}{m_i} \nu_{ei} \nu_{eg}}{\frac{\nu_i g}{m_e} + \frac{\nu_e g}{m_i}} \quad (13)$$

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Now, it should be noted that the quantities appearing in the right side of eq.(13) are complicated functions of the relative velocity of the test particle with respect to the target particle and the density of the target particle. As a result, E is a complicated function of the temperature of particle and ionization degree.

The ionization degree is defined as

$$x = \frac{n_e}{n_{go}} = \frac{n_i}{n_{go}} \quad (14)$$

where n_{go} is the density of the neutral particle when no ionization occurs. From eq.(14), it follows :

$$n_g = n_{go} (1 - x) \quad (15)$$

For illustration of general behavior, it is sufficient to use approximate expressions for the collision frequencies and loss terms. According to the earlier investigations, ^{(1),(2)} the elastic collision frequencies, $\nu_{eg} = \sigma_{eg} n_g \bar{v}_e$ for He, H₂, A and Ne have the following forms, if the energy is less than about 10 eV :

$$\left. \begin{aligned} \nu_{eg} &= C_{eg} n_{go} (1 - x) T_e^{1/2} & \text{for He} \\ &= C_{eg} n_{go} (1 - x) & \text{for H}_2 \\ &= C_{eg} n_{go} (1 - x) T_e^{3/2} & \text{for A} \\ &= C_{eg} n_{go} (1 - x) T_e & \text{for Ne} \end{aligned} \right\} \quad (16)$$

However, if the energy exceeds about 10 eV, the expressions for four gases have the same type as follows :

$$\nu_{eg} = C_{eg} n_{go} (1 - x) \quad (17)$$

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because the cross section is a decreasing function, i.e., it is
inversely proportional to $T_e^{\frac{1}{2}}$.

The collision frequencies ν_{ei} and ν_{ig} and the ionizing frequency
 ν_i are given by ⁽¹⁾⁽²⁾⁽³⁾

$$\nu_{ei} = C_{ei} x n_{go} T_e^{-3/2} \quad (18)$$

$$\nu_{ig} = C_{ig} n_{go} (1-x) T_i^{1/2} \quad (19)$$

$$\nu_i = C_i n_{go} (1-x) T_e^{1/2} e^{-eV_i/kT_e} \quad (20)$$

The energy loss due to the Bremsstrahlung Γ_{br} is

$$\Gamma_{br} = C_{br} n_{go} x T_e^{1/2} \quad (21)$$

Finally, concerning the recombination coefficient β , one has
proved experimentally to be

$$\beta = C_r T_e^{-3/2} \quad (22)$$

In the subsequent discussion, other quantities such as ν_* , ν_{em} ,
 β' and A will be neglected for simplicity, so that the approximate
forms for these quantities are not given here.

By use of these approximate expressions, E is described in terms
of T_e and x only, provided that T_i is regarded to be constant.
Further, we obtain a relation between x and T_e from any one of
eqs.(1) to (3) by setting $\partial/\partial t = 0$.

Taking the case of He as an example, it is

$$(1-x) C_i e^{-eV_i/kT_e} = x C_r T_e^{-2} \quad (23)$$

Thus, the resulting expression for E after introducing eqs.(16) to
(23) turns out to be a unique function of x . To proceed the
analysis, it is convenient to make further assumption that the

electron temperature is either extremely high or low.

For the low temperature case, we may use the following approximation :

$$e^{-eV_i/kT_e} \approx b \left(\frac{T_e}{a} \right)^s, \quad a = \frac{eV_i}{k} \quad (24)$$

where s is taken to be approximately 6 and b to be $1.37 \cdot 10^2$.

On the contrary, for the high temperature case, it is assumed that

$$e^{-eV_i/kT_e} \approx 1 \quad (25)$$

V. Numerical Examples

In this section, we shall illustrate the dependence of E on x for several cases. The gas considered here are He, H_2 , A and Ne. Various coefficients characterizing collision processes are tabulated in Table 1 for 1 mm Hg. With these numerical values, the

Table 1

	He	H_2	A	Ne
C_{eg} (High Temp)	$8.1 \cdot 10^{-6}$	$1.8 \cdot 10^{-5}$	$4.1 \cdot 10^{-5}$	$1.2 \cdot 10^{-5}$
C_{eg} (Low Temp)	$6.0 \cdot 10^{-10}$	$1.8 \cdot 10^{-5}$	$9.3 \cdot 10^{-15}$	6.10^{-13}
C_{ig}	$1.2 \cdot 10^{-10}$	$7.8 \cdot 10^{-10}$	$3.3 \cdot 10^{-10}$	$2.4 \cdot 10^{-11}$
C_i	$2.1 \cdot 10^{-11}$	$5.9 \cdot 10^{-11}$	$2.0 \cdot 10^{-10}$	$2.3 \cdot 10^{-11}$
C_{ci}	$2.6 \cdot 10^1$	$2.6 \cdot 10^1$	$2.6 \cdot 10^1$	$2.6 \cdot 10$
C_r	$5.0 \cdot 10^{-9}$	$5.0 \cdot 10^{-9}$	$5.0 \cdot 10^{-9}$	$5.0 \cdot 10^{-9}$
C_{br}	$3.1 \cdot 10^{-27}$	$1.6 \cdot 10^{-27}$	$2.8 \cdot 10^{-26}$	$1.6 \cdot 10^{-26}$

(units; sec, cm, V, °K)

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numerical value of E as a function of the ionization degree x can be calculated. Three cases are considered as typical examples.

(a) Low temperature and ionization loss. When the temperature is low and the ionization loss is dominant, introducing eq.(24) into eq.(20) and using the relation $L = \nu_i eV_i$ yield the following final expressions for E :

$$E^2 = K \frac{C_{eg} C_{ig} (1-x)^{\frac{19}{16}} x^{\frac{1}{8}} T_i^{\frac{1}{2}} B^{\frac{1}{16}} + C_{ig} C_{ei} (1-x)^{\frac{1}{16}} x^{\frac{13}{8}} B^{-\frac{3}{16}} T_i^{\frac{1}{2}} + \frac{m_e}{m_i} C_{ei} C_{eg} (1-x)^{\frac{5}{16}} x^{\frac{27}{16}} B^{-\frac{1}{8}}}{(1-x)^{\frac{1}{16}} \frac{C_{ig} T_i^{\frac{1}{2}}}{m_e} + \frac{C_{eg}}{m_i} x^{\frac{1}{16}} B^{\frac{1}{16}}} \quad \text{for He} \quad (26)$$

$$E^2 = K \frac{C_{eg} C_{ig} (1-x)^{\frac{19}{16}} x^{\frac{13}{16}} T_i^{\frac{1}{2}} + C_{ig} C_{ei} (1-x)^{\frac{3}{8}} x^{\frac{13}{8}} B^{-\frac{3}{16}} T_i^{\frac{1}{2}} + \frac{m_e}{m_i} C_{ei} C_{eg} (1-x)^{\frac{3}{8}} x^{\frac{13}{8}} B^{-\frac{3}{16}}}{\frac{C_{ig} T_i^{\frac{1}{2}}}{m_e} + \frac{C_{eg}}{m_i}} \quad \text{for H}_2 \quad (27)$$

$$E^2 = K \frac{C_{eg} C_{ig} (1-x)^{\frac{19}{16}} x B^{\frac{3}{16}} T_i^{\frac{1}{2}} + C_{ig} C_{ei} (1-x)^{\frac{9}{16}} x^{\frac{13}{8}} B^{-\frac{3}{16}} T_i^{\frac{1}{2}} + \frac{m_e}{m_i} C_{ei} C_{eg} (1-x)^{\frac{3}{8}} x^{\frac{29}{16}}}{\frac{C_{ig} T_i^{\frac{1}{2}}}{m_e} (1-x)^{\frac{3}{16}} + \frac{C_{eg}}{m_i} x^{\frac{3}{16}} B^{\frac{3}{16}}} \quad \text{for A} \quad (28)$$

$$E^2 = K \frac{C_{eg} C_{ig} (1-x)^{19/16} x^{15/16} B^{1/8} T_i^{1/2} + C_{ig} C_{ci} (1-x)^{1/2} x^{13/8} B^{-3/16} T_i^{1/2} + \frac{m_e}{m_i} C_{ci} C_{eg} (1-x)^{3/8} x^{7/4} B^{-5/16}}{C_{ig} T_i^{1/2} (1-x)^{1/8} + \frac{C_{eg}}{m_i} x^{1/8} B^{1/8}} \quad \text{for Ne} \quad (29)$$

$$K = \frac{n g_0^2 C_i^{3/16} V_i b^{3/16} C_r^{13/16}}{2 e a^{9/8}} \quad B = \frac{a^6 C_r}{C_i b}$$

Typical plot of E as a function of x is shown in Fig. 1, showing that E is zero at $x=0$ and $x=1$ and between them a maximum appears. The maximum values of the electric field E_m and the values of x corresponding to E_m for four gases are tabulated in Table 2.

From the table, it is seen that the maximum electric fields E_m for different gases appear at the same value of x but with different magnitudes.

The numerators of the above equations consist of three terms. The first, the second and the third terms arise from such collision that the neutral, ion and electron acts as the test particle, respectively. Since each term has a different dependence on the ionization degree, the dominant term is altered from one to another as the ionization is varied. When the ionization degree is low, the first term is dominant, but as the ionization degree is increased the second term becomes larger than the first. The ionization degree at which both terms are equal is different with the gas, as shown in Table 3. Also, the equality between the second and the third occurs close to $x=1$.

(b) High energy and ionization loss. When the temperature is high and the ionization loss is dominant, the expression for E in terms of x is given by use of eq.(25) and the relation $L = \nu_i e V_i$ as follows :

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Fig. 1

case	x_m	E_m (V/cm)			
		He	H ₂	A	Ne
a	0.81	$6.2 \cdot 10^{-1}$	$1.2 \cdot 10^{-1}$	$2.0 \cdot 10^{-1}$	$2.0 \cdot 10^{-1}$
b	0.25	$5.1 \cdot 10^3$	$9.9 \cdot 10^3$	$3.1 \cdot 10^3$	$5.1 \cdot 10^3$
c	0.75	$2.0 \cdot 10^1$	5.7	$1.0 \cdot 10^2$	$4.5 \cdot 10^1$

Table 2

case	He	H ₂	A	Ne
a	$6 \cdot 4 \cdot 10^{-4}$	$3 \cdot 10^{-1}$	$7 \cdot 10^{-7}$	$1 \cdot 10^{-5}$
b	$6 \cdot 4 \cdot 10^{-18}$	$3 \cdot 9 \cdot 10^{-18}$	$9 \cdot 1 \cdot 10^{-19}$	$1 \cdot 2 \cdot 10^{-17}$
c	$3 \cdot 3 \cdot 10^{-20}$	$4 \cdot 5 \cdot 10^{-20}$	$5 \cdot 7 \cdot 10^{-20}$	$5 \cdot 5 \cdot 10^{-20}$

Table 3

$$E^2 = \frac{C_i V_i n_{go}^2 C_{eg} C_{ig} (1-x)^{1/4} x^{1/4} C_r^{1/4} C_2^{-1/4} T_i^{1/2} + (C_{ig} C_{ei} T_i^{1/2} + \frac{m_e}{m_i} C_{ei} C_{eg}) (1-x)^{3/2} x^{1/2} C_r^{-3/4} C_i^{3/4}}{2e \frac{C_{ig} T_i^{1/2}}{m_e} \frac{C_{eg}}{m_i}}$$

(30)

for all gases.

The shape of the plot of E versus x is similar to that for the case (a). The maximum electric field E_m and the corresponding x_m are tabulated in Table 2.

The second of the numerator of eq.(30) is always comparable to the third term and the equal contribution of the first and the second terms is expected to occur at a certain value of x , as shown in Table 3.

(c) High temperature and Bremsstrahlung loss. When the temperature is high and the loss due to the Bremsstrahlung is dominant, the expression for E is given by use of eq.(25) and relation

$I = \Gamma_{br}$ as

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$$E^2 = \frac{G_{br} n_{go}^2 C_{eg} C_{ig} (1-x)^{3/4} x^{5/4} C_r^{1/4} C_i^{-1/4} T_i^{1/2} + (C_{ig} C_{ei} C_t^{-1/2} C_i T_i^{1/2} + \frac{m_e}{m_i} C_{ei} C_{eg} C_r^{-1/2} C_i^{1/2} (1-x)^{1/2} x^{3/2}}{2 e^2} \quad (31)$$

for all gases.

Also we obtain a similar plot of E versus x as the formerly described cases. The numerical values of E_m and x_m are given in Table 2.

The second term of the numerator of eq.(31) is comparable to the third, regardless of x . The equal contribution from the first and second terms is attained at an extremely small x , as shown in Table 3.

VI. Conclusion

Even though we treated the problem only for extreme cases, the result thus obtained makes us to understand what happens as the ionization degree is increased from zero to 1. For brevity, it is said that a maximum electric field is needed before reaching completely ionized plasma, in other words, the applied electric field must be greater than the maximum electric field E_m to realize the fully ionized plasma.

Also the present work suggests that it would be better to raise the ionization degree when the energy is kept at a low value. Thus the preionization process necessary to achieve hot plasma should be such that only the ionization degree is enhanced but not the temperature of the charged particles.

Furthermore, the collision in which the charged particle takes part of the test particle plays substantial role even when ionization is very weak.

References

- (1) E.W.McDaniel, Collision Phenomena in Ionized Gas, 1964.
- (2) S.C.Brown, Basic Data of Plasma Physics, 1959.
- (3) L.Spitzer, Physics of Fully Ionized Gases, 1955.
- (4) S.Glasstone and R.H.Lovberg, Controlled Thermonuclear Reactions, 1960.

Figure Caption

Fig.1. Steady electric field as a function of ionization degree for the case (a).

Table Captions

- Table 1. Various coefficients characterizing collision processes for 1 mm Hg.
- Table 2. Maximum electric field E_m and corresponding ionization degree x_m .
- Table 3. Critical ionization degree at which the first and second terms have equal contribution.