Thermal Instability of a Thermonuclear Plasma in a DD Fusion Reactor. III. Inhomogeneous Temperature

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Abstract

A thermal thermonuclear instability of a deuterium plasma with an inhomogeneous temperature in a strong magnetic field is studied theoretically. The growth rate of the instability is obtained. The plasmas with an inhomogeneous temperature in a DD fusion reactor and in a catalyzed DD fusion reactor are unstable for the thermal instability.

§ 1. Introduction

In Paper I, 1) a thermal thermonuclear instability of a homogeneous deuterium plasma in a strong magnetic field was studied. In Paper II, 2) the instability in a catalyzed DD fusion reactor was discussed.

In this paper, we present an analysis of the instability of a deuterium plasma with an inhomogeneous temperature. The thermal instability of an inhomogeneous plasma in DT fusion reactors has been discussed by some authors. 3-4)

Kolesnichenko, Reznik and Yavorskij 3) have developed the method for investigating the stability of a steady-state thermonuclear reaction in an inhomogeneous deuterium-tritium plasma. They have analyzed the differential equation of the second order in partial derivatives which describes the instabilities of the energy balance in a thermonuclear DT Plasma.

We want to know the stability of a DD fusion reactor. In this paper, we will use the method developed in ref. 3) for analyzing the stability of a thermonuclear deuterium plasma.

In § 2, the instability in a DD fusion reactor is studied. The instability in a catalyzed DD fusion reactor is discussed in § 3.

§ 2. DD fusion reactor

First, we discuss the instability in a DD fusion reactor. We assume that (1) the
deuterium plasma is isothermal \( (T_e = T_i = T) \), (2) the plasma energy losses are due only to heat conduction across a magnetic field and bremsstrahlung, (3) the number density \( n \) and the Coulomb logarithm \( \lambda \) are constant in space and time, and (4) the magnetic field \( B \) is uniform in space and constant in time.

For simplicity, we discuss the plasma of a plane layer. Then, we have the following equation for the energy balance: ¹)

\[
3n \frac{\partial T(x,t)}{\partial t} = \frac{\partial}{\partial x} \left( \frac{An^2}{T_{1/2}(x,t)} \frac{\partial T(x,t)}{\partial x} \right) + P(T),
\]

where

\[
P(T) = Cn^2 T^{1.37} - Dn^2 T^{1/2},
\]

\[
A = 4.48 \times 10^{-4} \frac{(\lambda/10)}{B^2},
\]

\[
C = 1.21 \times 10^{-16},
\]

\[
D = 3.34 \times 10^{-15},
\]

\[
\lambda = 33.48 + \frac{1}{0.8686} \log_{10} \frac{T^3}{n},
\]

\( (T(\text{keV}), n(\text{cm}^{-3}), B(\text{G}), t(\text{sec}), x(\text{cm})) \).

The \( x \)-axis is perpendicular to \( B \) and to the surface of the layer.

We assume that the temperature of the 0-th order \( T_0 \) depends only on \( x \). Then, we obtain the equation of the 0-th order from eq. (1):

\[
\frac{d}{dx} \left( \frac{An^2}{T_{1/2}^2(x)} \frac{dT_0(x)}{dx} \right) + P(T_0) = 0.
\]

Putting

\( T(x,t) = T_0(x) + T_1(x,t); \ T_0 \gg T_1 \),

and using eq. (7), we have the first order equation from eq. (1):

\[
3n \frac{\partial T_1(x,t)}{\partial t} = An^2 \frac{\partial^2}{\partial x^2} \left( \frac{T_1(x,t)}{T_{1/2}^2(x)} \right) + \frac{dP(T_0)}{dT_0} T_1(x,t).
\]

If we put

\( T_1(x,t) = T_1(x)e^{ut} \),

we obtain the following equation from eq. (8):

\[
An^2 \frac{d^2}{dx^2} \left( \frac{T_1(x,t)}{T_{1/2}^2(x)} \right) + \frac{dP(T_0)}{dT_0} T_1(x,t) = 3ns T_1(x).
\]
We assume that the perturbation $T_1(x)$ of the temperature is symmetric with respect to $x = 0$:

$$\frac{dT_1(x)}{dx} = 0 \quad \text{at} \quad x = 0 ,$$

(11)

and vanishes at the plasma boundaries $x = \pm L$:

$$T_1(x) = 0 \quad \text{at} \quad x = \pm L .$$

(12)

When $s = 0$, the differential equation (10) was solved by Kolesnichenko, Reznik and Yavorskij,\textsuperscript{3}) who obtained the following solutions:

$$T_1^{(1)}(x) = \frac{\partial T_0(x)}{\partial T(0)} ,$$

(13)

$$T_1^{(2)}(x) = \frac{dT_0(x)}{dx} ,$$

(14)

where $T(0)$ is the temperature at the center $x = 0$. As easily seen, the first solution $T_1^{(1)}(x)$ satisfies the boundary condition (11), but the second solution $T_1^{(2)}(x)$ does not. The function $T_0(x)$ is the solution of the 0-th order equation (7) under the following boundary conditions: $dT_0(x)/dx = 0$ at $x = 0$ and $T_0(x) = 0$ at $x = \pm L$. Here, the value of $L$ depends on the temperature at the center $T(0) : L = L(T(0))$. As done in ref. 3), we can express the boundary value problem (10), (11) and (12) in an integral form. If we put in eq. (10)

$$u(x) = \alpha(x) T_1(x) ,$$

(15)

where

$$\alpha(x) = \frac{An}{3 T_0^{1/2}} ,$$

(16)

we have the following equation:

$$\frac{d^2 u(x)}{dx^2} + \frac{\nu(x)}{\alpha(x)} u(x) = \frac{s}{\alpha(x)} u(x) ,$$

(17)

where

$$\nu(x) = \frac{1}{3n} \frac{dP(T_0)}{dT_0} .$$

(18)

The solutions of eq. (17) with $s = 0$ are given by

$$u_1(x) = \alpha(x) T_1^{(1)}(x) ,$$

(19)

$$u_2(x) = \alpha(x) T_1^{(2)}(x) .$$

(20)

We assume that a Green's function $G(x, \xi)$ satisfies the following equation:

$$\frac{d^2}{dx^2} G(x, \xi) + \frac{\nu(x)}{\alpha(x)} G(x, \xi) = \delta(x-\xi) .$$

(21)
Then, the solution \( u(x) \) of eq. (17) satisfies the following integral equation:

\[
    u(x) = c_1 u_1(x) + s \int_0^L d \xi \ G(x, \xi) \ \frac{u(\xi)}{a(\xi)},
\]

where \( c_1 \) is a constant. Putting

\[
    G(x, \xi) = \begin{cases} 
    a \ u_1(x) \ u_2(\xi), & \text{for } 0 \leq x < \xi, \\
    a \ u_1(\xi) \ u_2(x), & \text{for } \xi < x \leq L,
    \end{cases}
\]

and using the condition that is derived from the integration of both sides of eq. (21) with respect to \( x \) for the small interval \( \xi - \varepsilon \leq x \leq \xi + \varepsilon \):

\[
    \lim_{\varepsilon \to 0} \left\{ \frac{dG}{dx} \bigg|_{x=\xi+\varepsilon} - \frac{dG}{dx} \bigg|_{x=\xi-\varepsilon} \right\} = 1,
\]

we have

\[
    \frac{1}{a} = u_1(\xi) \ u'_2(\xi) - u'_1(\xi) \ u_2(\xi).
\]

Substituting eqs. (19) and (20) into the above equation (25), we obtain

\[
    \frac{1}{a} = a^2(\xi) \left\{ \frac{\partial T_0(\xi)}{\partial T(0)} \ \frac{d^2 T_0(\xi)}{d\xi^2} - \frac{1}{2} \ \frac{\partial}{\partial T(0)} \ \left( \frac{d T_0(\xi)}{d\xi} \right)^2 \right\}.
\]

Now, from the equation (7) of the 0-th order, we have \(^5\)

\[
    \frac{d^2 T_0(x)}{dx^2} = T_0^{1/2}(x) \left\{ \frac{1}{2T_0^{3/2}(x)} \left( \frac{d T_0(x)}{dx} \right)^2 - \frac{P(T_0)}{A n^2} \right\},
\]

and

\[
    \frac{d T_0(x)}{dx} = -\left\{ \frac{2}{A} \ T_0(x) \ \left( F - \frac{C}{1.87} T_0^{1.87}(x) + D T_0(x) \right) \right\}^{1/2},
\]

where

\[
    F = \frac{C}{1.87} \ T_0^{1.87}(0) - D T(0),
\]

and we used the condition

\[
    \frac{d T_0(x)}{dx} = 0 \quad \text{at} \quad x = 0.
\]

Substituting eqs. (27) and (28) into eq. (26), we obtain

\[
    \frac{1}{a} = -\frac{A}{9 T_0^{1/2}(0)} \ P(T_0)
\]

Now, substituting eq. (23) into eq. (22), we have
Using eqs. (15), (19) and (20), we can write eq. (32) as follows:

\[ T_I(x) = c_I T_I^{(1)}(x) + sa \left[ \int_0^x d\xi \alpha(\xi) T_I^{(1)}(\xi) T_I^{(2)}(x) T_I(\xi) \right. \]

\[ \left. + \int_x^L d\xi \alpha(\xi) T_I^{(1)}(\xi) T_I^{(2)}(\xi) T_I(\xi) \right] . \]  

(33)

Transforming the integral of the second term in the square brackets in eq. (33) as follows:

\[ \int_x^L d\xi = \int_0^L d\xi - \int_0^x d\xi , \]  

(34)

we obtain

\[ T_I(x) = a_I T_I^{(1)}(x) + sa \int_0^x d\xi \alpha(\xi) \left( T_I^{(1)}(\xi) T_I^{(2)}(x) - T_I^{(1)}(\xi) T_I^{(2)}(\xi) \right) T_I(\xi) , \]  

(35)

where \( a_I \) is a constant. Substituting eqs. (16) and (31) into eq. (35), we have the following integral equation:

\[ T_I(x) = a_I T_I^{(1)}(x) + s \int_0^x d\xi K(x, \xi) T_I(\xi) , \]  

(36)

where

\[ K(x, \xi) = \frac{3n}{P(T(0))} \frac{T_I^{1/2}(0)}{T_I^{1/2}(\xi)} \left[ T_I^{(1)}(x) T_I^{(2)}(\xi) - T_I^{(1)}(\xi) T_I^{(2)}(x) \right] . \]  

(37)

The growth rate \( s \) is obtained from the boundary condition (12) at \( x = L \):

\[ T_I(L) = a_I T_I^{(1)}(L) + s \int_0^L d\xi K(L, \xi) T_I(\xi) = 0 . \]  

(38)

Then, we have

\[ s = -\frac{a_I T_I^{(1)}(L)}{\int_0^L d\xi K(L, \xi) T_I(\xi)} . \]  

(39)

If we put

\[ T_I(x) = a_I \rho(x) , \]  

(40)

where

\[ \rho(0) = 1 \quad \text{and} \quad \rho(L) = 0 , \]  

(41)
we obtain
\[ s = - \frac{T_1^{(1)}(L)}{\int_0^L d\xi \ K(L, \xi) \ p(\xi)}. \] (42)

Now, we have the quadrature of the equation of the 0-th order (7):
\[ L - x = \left( \frac{A}{2} \right)^{1/2} \int_0^x dT_0 \ \frac{1}{T_0^{1/2} (F - \frac{C}{1.87} T_0^{1.87})^{1/2} + D T_0^{1/2}}. \] (43)

where we used the condition
\[ T_0(x) = 0 \quad \text{at} \quad x = L. \] (44)

Differentiating both sides of the above equation (43) with respect to \( T(0) \) for fixed \( x \), we have
\[ \frac{\partial T_0(x)}{\partial T(0)} = \left[ \frac{2}{A} T_0(x) \ (F - \frac{C}{1.87} T_0^{1.87} + D T_0(x)) \right]^{1/2} \left\{ \frac{dL}{dT(0)} \right\}
+ \left( \frac{C T_0^{0.87} (0) - D}{2} \right) \left( \frac{A}{2} \right)^{1/2} \int_0^x dT_0 \ \frac{1}{T_0^{1/2} (F - \frac{C T_0^{1.87}}{1.87}) + D T_0^{3/2}}. \] (45)

From the above equation (45), we obtain
\[ T_1^{(1)}(L) = \left( \frac{\partial T_0(x)}{\partial T(0)} \right)_{x = L} = \left( \frac{2}{A} T_0(L) F \right)^{1/2} \frac{dL}{dT(0)}. \] (46)

The kernel \( K(L, \xi) \) is given by
\[ K(L, \xi) = \frac{3n T_0^{1/2}(0)}{P(T(0)) T_0^{1/2}(\xi)} \left[ \left( \frac{\partial T_0(x)}{\partial T(0)} \right)_{x = L} \frac{dT_0(\xi)}{dT(0)} \right] - \left( \frac{dT_0(x)}{dx} \right)_{x = L}. \] (47)

Substituting eqs. (46), (28) and (45) into the above equation (47), we obtain
\[ K(L, \xi) = \frac{3}{2n} \left( \frac{2}{A} T_0(L) F \right)^{1/2} \left( F - \frac{C T_0^{1.87}(\xi)}{1.87} \right)
+ D T_0(\xi)) \right)^{1/2} \int_0^{T_0(\xi)} dT_0 \ \frac{1}{T_0^{1/2} (F - \frac{C T_0^{1.87}}{1.87} + D T_0)^{3/2}}. \] (48)
Substituting eqs. (46) and (48) into eq. (42), we obtain the growth rate

\[ s = - \left[ \frac{3}{2n} \int_0^L d\xi \rho(\xi) \left( F - \frac{CT_0^{1.87}(\xi)}{1.87} + DT_0(\xi) \right) \right]^{1/2} \cdot \int_0^L dT_0 \frac{dL}{dT} \frac{I}{T_0^{1/2} \left( F - \frac{CT_0^{1.87}}{1.87} + DT_0 \right)^{3/2}} \] \quad (49)

The sign of \( s \) is determined by that of \( dL/dT(0) \) since we may consider only the case \( \rho(x) > 0 \).

If we assume that the growth rate \( s \) is small, we have an approximate solution of the integral equation (36) as follows:

\[ T_1(x) \approx a_1 T_1^{(1)}(x) \quad (50) \]

Substituting

\[ \rho(x) = T_1^{(1)}(x) = \frac{\partial T_0(x)}{\partial T(0)} \quad (51) \]

into eq. (49) and using the approximation

\[ \frac{\partial T_0(x)}{\partial T(0)} \approx \frac{T_0(x)}{T(0)} \quad (52) \]

we have an estimate of \( s \):

\[ s = - \frac{2P(T(0))}{3nL} \cdot \frac{dL}{dT(0)} \quad (53) \]

In order to see the \( T(0) \)-dependence of \( L \) analytically, we use the following approximation

\[ P_c = C' n^2 T^{3/2} \quad (54) \]

in place of the first term in the r.h.s. of eq. (2). Determining the constant \( C' \) from the following condition

\[ C T^{1.37} = C' T^{3/2} \quad \text{at} \quad T = 100 \text{ keV} \quad (55) \]

we have

\[ C' = 6.65 \times 10^{-17} \quad (56) \]

When we use the approximation (54), we have the equation of the 0-th order in place of eq. (7):

\[ \frac{d}{dx} \left( \frac{An^2}{T_0^{1/2}(x)} \frac{dT_0(x)}{dx} \right) + C' n^2 T_0^{3/2}(x) - Dn^2 T_0^{1/2}(x) = 0 \quad (57) \]
Then, we have the quadrature of the above equation (57):

\[
x = \left( \frac{A}{2} \right)^{1/2} \int_{0}^{T(0)} T_{0}(\xi) \frac{1}{T_{0}^{1/2} \left( F' - \frac{C'}{2} T_{0}^{2} + DT_{0} \right)^{1/2}} dT_{0}, \tag{58}
\]

where

\[
F' = \frac{C'}{2} T^{2}(0) - DT(0). \tag{59}
\]

Using the boundary condition (44) in eq. (58), we have

\[
L = \left( \frac{A}{2} \right)^{1/2} \int_{0}^{T(0)} dT_{0} \frac{1}{T_{0}^{1/2} \left( F' - \frac{C'}{2} T_{0}^{2} + DT_{0} \right)^{1/2}}. \tag{60}
\]

If we put \( \xi = T_{0}/T(0) \) in eq. (60), we have

\[
L = \left( \frac{A}{C'} T(0) \right)^{1/2} \int_{0}^{1} d\xi \frac{1}{\left[ \xi (1-\xi) (\xi + 1 - b) \right]^{1/2}}, \tag{61}
\]

where

\[
b = \frac{2D}{C'} \frac{T(0)}{T(0)} = \frac{100}{T(0)}. \tag{62}
\]

Putting \( \xi = 1 - u^2 \) in eq. (61), we obtain

\[
L = 2 \left( \frac{A}{C'} T(0) \right)^{1/2} K(k), \tag{63}
\]

where \( K(k) \) is the complete elliptic integral of the first kind

\[
K(k) = \int_{0}^{1} du \frac{1}{\left[ (1-u^2) (1-k^2 u^2) \right]^{1/2}}, \tag{64}
\]

\[
k = \frac{1}{(2-b)^{1/2}} = \frac{1}{\left( 2 - \frac{100}{T(0)} \right)^{1/2}}. \tag{65}
\]

Substituting eqs. (3), (56) and (65) into eq. (63), we obtain

\[
L = \frac{3.67 \times 10^6 (\lambda / 10)^{1/2}}{B (T(0) - 50)^{1/2}} K(k). \tag{66}
\]

Differentiating both sides of the above equation (66) with respect to \( T(0) \), we have

\[
\frac{dL}{dT(0)} = -\frac{3.67 \times 10^6 (\lambda / 10)^{1/2}}{2B (T(0) - 50)^{3/2}} \left\{ K(k) + \frac{50}{T(0) - 50} \cdot \frac{dK(k)}{d(k^2)} \right\}. \tag{67}
\]
Since
\[ K(k) > 0 \quad \text{and} \quad \frac{dK}{d(k^2)} > 0 \quad , \] (68)
we obtain
\[ \frac{dL}{dT(0)} < 0 \quad . \] (69)
Therefore, the sign of the growth rate \( s \) is positive :
\[ s \sim - \frac{dL}{dT(0)} > 0 \quad . \] (70)
We conclude that the plasma is unstable for the thermal instability.

The \( T(0) \)-dependence of \( L \) for \( B = 10^5 G \) and \( \lambda \approx 22 \) is given in Fig. 1. In this case, for \( T(0) = 150 \text{ keV} \), we have from Fig. 1
\[ \frac{dL}{dT(0)} \approx - 8.5 \times 10^{-2} \text{ cm/keV} \quad , \]
\[ L \approx 11.7 \text{ cm} \quad . \]
Using eq. (53), we obtain the order of the growth rate \( s \) for \( T(0)=150 \text{ keV} \), \( B = 10^5 G \) and \( \lambda \approx 22 \) :
\[ s \approx 3.6 \times 10^{-16} n \quad \text{1/sec} \quad . \]
When \( n \approx 10^{16} \quad 1/\text{cm}^3 \), we have
\[ s = 3.6 \quad \text{1/sec} \quad . \]
Similarly, for \( T(0)=200 \text{ keV} \), \( B = 10^5 G \), and \( \lambda \approx 22 \), we have
\[ \frac{dL}{dT(0)} \approx = - 3.5 \times 10^{-2} \text{ cm/keV} \quad , \]
\[ L \approx 9.01 \text{ cm} \quad , \]
\[ s \approx 4.9 \times 10^{-16} n \quad \text{1/sec} \quad . \]

§ 3. Catalyzed DD fusion reactor

Secondly, we discuss the instability in a catalyzed DD fusion reactor. The growth rate \( s \) in a catalyzed DD fusion reactor is obtained similarly by the method done in § 2. Using eqs. (I.2.13) and (II.6), we have
\[ P_{fc} = C_I n^2 T^{1.37}, \quad \text{(71)} \]

where
\[ C_I = 6.68 \times 10^{-16}. \quad \text{(72)} \]

Using \( C_I \) in place of \( C \) in § 2, we obtain the growth rate
\[ s = - \left[ \frac{3}{2n} \int_0^L d\xi \rho(\xi) \left( F_I - \frac{C_I T_0^{1.87}(\xi)}{1.87} + DT_0(\xi) \right) \right]^{1/2} \cdot \int_0^{T_0(\xi)} \frac{1}{T_0^{1/2} \left( F_I - \frac{C_I T_0^{1.87}}{1.87} + DT_0 \right)^{3/2}} \right]^{-1} \frac{dL}{dT(0)}, \quad \text{(73)} \]

where
\[ F_I = \frac{C_I T_0^{1.87}(0)}{1.87} - DT(0), \quad \text{(74)} \]
\[ L = \left( \frac{A}{2} \right)^{1/2} \int_0^{T(0)} \frac{1}{dT} \left( F_I - \frac{C_I T_0^{1.87} + DT_0}{1.87} \right)^{1/2}. \quad \text{(75)} \]

Now, we have an estimate of
\[ s \approx - \frac{2P(T(0))}{3nL} \cdot \frac{dL}{dT(0)}, \quad \text{(76)} \]

where
\[ P(T(0)) = C_I n^2 T^{1.37}(0) - Dn^2 T^{1/2}(0). \quad \text{(77)} \]

When we use the approximation
\[ P_{fc} = C_I' n^2 T^{3/2}, \quad \text{(78)} \]
we have
\[ C_I' = 3.67 \times 10^{-16}, \quad \text{(79)} \]
in place of \( C' \) in § 2. In place of \( b \) in § 2, we have
\[ b_I = \frac{2D}{C_I T(0)} = \frac{18.2}{T(0)} \quad \text{(80)} \]

Then, we have
\[ L = \frac{1.56 \times 10^6 (\lambda/10)^{1/2}}{B(T(0) - 9.1)^{1/2}} K(k_I), \quad \text{(81)} \]

where
\[ k_I = \frac{1}{\left( 2 - \frac{18.2}{T(0)} \right)^{1/2}} \quad \text{(82)} \]
We conclude that the plasma in the catalyzed DD fusion reactor is also unstable for the thermal instability.

The $T(0)$-dependence of $L$ for $B = 10^5 G$ and $\lambda \approx 22$ is given in Fig. 2. In this case, for $T(0) = 50$ keV, we have from Fig. 2

$$\frac{dL}{dT(0)} \approx -1.0 \times 10^{-1} \text{ cm/keV} ,$$
$$L \approx 7.12 \text{ cm} ,$$
$$s \approx 1.1 \times 10^{-15} n \text{ 1/sec} .$$

When $n \approx 10^{15} 1/\text{cm}^3$, we have
$$s \approx 1.1 \text{ 1/sec} .$$

Similarly, for $T(0) = 100$ keV, $B = 10^5 G$ and $\lambda \approx 22$, we have

$$\frac{dL}{dT(0)} \approx -2.8 \times 10^{-2} \text{ cm/keV} ,$$
$$L \approx 4.62 \text{ cm} ,$$
$$s \approx 1.3 \times 10^{-15} n \text{ 1/sec} .$$

§ 4. Conclusions

We have obtained the growth rates for the thermal thermonuclear instability of the deuterium plasma with an inhomogeneous temperature in a strong magnetic field. The growth rate $s$ in a DD fusion reactor is given by

$$s = - \frac{2n}{3L} \left\{ 1.21 \times 10^{-16} (T(0))^{1.37} - 3.34 \times 10^{-15} (T(0))^{1/2} \right\} \cdot \frac{dL}{dT(0)} \text{ 1/sec} ,$$

where $n(\text{cm}^{-3})$ is the ion number density, $L(\text{cm})$ a half thickness of the plane plasma layer and $T(0)$(keV) the temperature at the center. In a catalyzed DD fusion reactor, the growth rate $s$ is given by

$$s = - \frac{2n}{3L} \left\{ 6.68 \times 10^{-16} (T(0))^{1.37} - 3.34 \times 10^{-15} (T(0))^{1/2} \right\} \cdot \frac{dL}{dT(0)} \text{ 1/sec} .$$

The $T(0)$-dependence of $L$ for the strength of the magnetic field $B = 10^5 G$ and the Coulomb logarithm $\lambda \approx 22$ is given in Fig. 1 (DD fusion reactor) and Fig. 2 (catalyzed DD fusion reactor).
Since \( s > 0 \) because of \( dL/dT(0) < 0 \) in the temperature region considered in this paper, the plasmas with an inhomogeneous temperature in a DD fusion reactor and in a catalyzed DD fusion reactor are unstable for the thermal instability.

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References