Thermal Instability of a Thermonuclear Plasma in a DD Fusion Reactor. IV. Inhomogeneous Plasma

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Abstract

The thermal thermonuclear instability of a deuterium plasma with inhomogeneous temperature and number density in a strong magnetic field is studied theoretically. The distribution of the number density is supposed to be proportional to that of the temperature. By solving the basic equation which governs the reactions, the growth rates in a pure DD fusion reactor and a catalyzed DD fusion reactor are obtained. It is shown that the plasmas with inhomogeneous temperature and number density in the above reactors are unstable to the thermal instability.

Keywords :

DD fusion reactor, thermal thermonuclear instability, deuterium plasma, inhomogeneous plasma, fusion reaction, heat conduction loss, bremsstrahlung loss,

§1. Introduction

The thermal instability of a homogeneous deuterium plasma in a strong magnetic field in a pure DD fusion reactor and a catalyzed DD fusion reactor was studied in Papers I¹⁾ and II²⁾.

On the other hand, the thermal instability in reactors with inhomogeneous temperature was considered in Paper III³⁾.

In this paper, we present an analysis of the instability of a deuterium plasma with inhomogeneous temperature and number density in a strong magnetic field.

The instability of a pure DD fusion reactor is considered in §2.

The instability of a catalyzed DD fusion reactor is discussed in §3.

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§2. Pure DD fusion reactor

First, we discuss the instability in a pure DD fusion reactor.

We assume that (1) the deuterium plasma is isothermal ($T_e = T_i = T$), (2) the energy losses in the plasma are due only to heat conduction across a magnetic field and bremsstrahlung, (3) the Coulomb logarithm λ is constant in space and time, (4) the magnetic field B is uniform in space and constant in time and (5) the ion number density n is proportional to the temperature T : $n \sim T$.

For simplicity, we discuss the plasma of plane layer. In this case, we have the following equation for the energy balance:

$$3n \frac{\partial T}{\partial t} - 2T \frac{\partial n}{\partial t} = \frac{\partial}{\partial x} \left(\frac{A n^2}{T^{1/2}} \frac{\partial T}{\partial x} \right) + C n^2 T^{1.37} - D n^2 T^{1/2}, \quad (1)$$

where

$$A = 4.48 \times 10^{-4} \frac{(\lambda / 10)}{B^2}, \quad (2)$$

$$C = 1.21 \times 10^{-16}, \quad (3)$$

$$D = 3.34 \times 10^{-15}, \quad (4)$$

$$\lambda = 33.48 + \frac{1}{0.8686} \log_{10} \frac{T^3}{n}, \quad (5)$$

$$(T(\text{keV}), \quad n(1/\text{cm}^3), \quad t(\text{sec}), \quad x(\text{cm})).$$

The x-axis is supposed to be perpendicular to B and to the surface of the layer. Substituting the following relation

$$n = n(0) \frac{T}{T(0)}, \quad (6)$$

into eq. (1), we have

$$\left(\frac{T(0)}{n(0)} \right) T \frac{\partial T}{\partial t} = A \frac{\partial}{\partial x} \left(T^{3/2} \frac{\partial T}{\partial x} \right) + P(T), \quad (7)$$

where

$$P(T) = C T^{3.37} - D T^{5/2}; \quad (8)$$

$n(0)$ and $T(0)$ are the ion number density and the temperature at the center $x=0$, respectively.

We assume that the temperature of the 0-th order T_0 depends only on x . Then, we obtain the 0-th order equation from eq. (7):

$$\frac{2A}{5} \frac{d^2}{dx^2} T_0^{5/2}(x) + P(T_0) = 0. \quad (9)$$

Putting

$$T(x, t) = T_0(x) + T_1(x, t); \quad T_0 \gg T_1,$$

and using the 0-th order equation (9), we obtain the first order equation from eq. (7):

$$\left(\frac{T(0)}{n(0)} \right) T_0(x) \frac{\partial T_1(x, t)}{\partial t} = A \frac{\partial^2}{\partial x^2} \left(T_0^{3/2}(x) T_1(x, t) \right) + \frac{dP(T_0)}{dT_0} T_1(x, t). \quad (10)$$

Substituting

$$T_1(x, t) = T_1(x) e^{st}, \quad (11)$$

into eq. (10), we have the following equation:

$$A \frac{d^2}{dx^2} \left(T_0^{3/2}(x) T_1(x) \right) + \frac{dP(T_0)}{dT_0} T_1(x) = s \left(\frac{T(0)}{n(0)} \right) T_0(x) T_1(x). \quad (12)$$

Here, we assume that the perturbation temperature $T_1(x)$ is symmetric with respect to the plane $x=0$:

$$\frac{dT_1(x)}{dx} = 0 \quad \text{at} \quad x=0, \quad (13)$$

and vanishes at the plasma boundaries $x=\pm L$:

$$T_1(x) = 0 \quad \text{at} \quad x = \pm L. \quad (14)$$

When $s=0$, the solutions of the differential equation (12) are given by⁴⁾

$$T_1^{(1)}(x) = \frac{\partial T_0(x)}{\partial T(0)}, \quad (15)$$

and

$$T_1^{(2)}(x) = \frac{dT_0(x)}{dx}. \quad (16)$$

As done in ref. 4), we can express the boundary value problem (12), (13) and (14) in an integral form.

Putting

$$u(x) = \alpha(x) T_1(x), \quad (17)$$

in eq. (12), we have the following equation:

$$\frac{d^2 u(x)}{dx^2} + \frac{\nu(x)}{\alpha(x)} u(x) = \frac{s}{\alpha(x)} \left(\frac{T(0)}{n(0)} T_0(x) \right) u(x), \quad (18)$$

where

$$\alpha(x) = A T_0^{3/2}(x), \quad (19)$$

and

$$\nu(x) = \frac{dP(T_0)}{dT_0}. \quad (20)$$

When $s=0$, the solutions of eq. (18) are given by

$$u_1(x) = \alpha(x) T_1^{(1)}(x), \quad (21)$$

and

$$u_2(x) = \alpha(x) T_1^{(2)}(x). \quad (22)$$

The solution $u(x)$ of the differential equation (18) satisfies the following integral equation:

$$u(x) = c_1 u_1(x) + s \frac{T(0)}{n(0)} \int_0^L dx \xi G(x, \xi) \frac{T_0(\xi)}{\alpha(\xi)} u(\xi), \quad (23)$$

Where c_1 is a constant, and $G(x, \xi)$ is the Green's function given by

$$G(x, \xi) = \begin{cases} a u_1(x) u_2(\xi) & \text{for } 0 \leq x < \xi, \\ a u_1(\xi) u_2(x) & \text{for } \xi < x \leq L. \end{cases} \quad (24)$$

The constant a is given by the following relation:³⁾

$$\frac{1}{a} = \alpha^2(\xi) \left\{ \frac{\partial T_0(\xi)}{\partial T(0)} \frac{d^2 T_0(\xi)}{d\xi^2} - \frac{1}{2} \frac{\partial}{\partial T(0)} \left(\frac{dT_0}{d\xi} \right)^2 \right\}. \quad (25)$$

From the 0-th order equation (9), we obtain

$$\frac{dT_0(x)}{dx} = - \left\{ \frac{2}{A T_0^3(x)} \left(F_3 - \frac{C}{5.87} T_0^{5.87}(x) + \frac{D}{5} T_0^5(x) \right) \right\}^{1/2}, \quad (26)$$

where

$$F_3 = \frac{C}{5.87} T^{5.87}(0) - \frac{D}{5} T^5(0), \quad (27)$$

and, here, the condition

$$\frac{dT_0(x)}{dx} = 0 \quad \text{at} \quad x = 0, \quad (28)$$

is used. Substituting eqs. (9) and (26) into eq. (25), we obtain

$$\frac{1}{a} = -A T^{3/2}(0) P(T(0)), \quad (29)$$

Now, substituting eq. (24) into eq. (23), and using eqs. (17), (21), (22) and (29), we have the following integral equation

$$T_1(x) = a_1 T_1^{(1)}(x) + s \int_0^x d\xi K(x, \xi) T_1(\xi), \quad (30)$$

where a_1 is a constant and

$$K(x, \xi) = \frac{T_0^{5/2}(\xi)}{n(0) T^{1/2}(0) P(T(0))} \left[T_1^{(1)}(x) T_1^{(2)}(\xi) - T_1^{(1)}(\xi) T_1^{(2)}(x) \right]. \quad (31)$$

We can obtain the growth rate s by using the boundary condition $T_1(L)=0$ in eq.(30):

$$s = - \frac{a_1 T_1^{(1)}(L)}{\int_0^L d\xi K(L, \xi) T_1(\xi)}. \quad (32)$$

Putting

$$T_1(x) = a_1 \rho(x), \quad (33)$$

where

$$\rho(0) = 1 \quad \text{and} \quad \rho(L) = 0, \quad (34)$$

we obtain

$$s = - \frac{T_1^{(1)}(L)}{\int_0^L d\xi K(L, \xi) \rho(\xi)} \quad (35)$$

Now, we have the quadrature of the 0-th order equation (26):

$$L - x = \left(\frac{A}{2} \right)^{1/2} \int_0^{T_0(x)} dT_0 \frac{T_0^{3/2}}{\left(F_3 - \frac{C}{5.87} T_0^{5.87} + \frac{D}{5} T_0^5 \right)^{1/2}} \quad (36)$$

where we used the condition

$$T_0(x) = 0 \quad \text{at} \quad x = L. \quad (37)$$

Differentiating both sides of eq. (36) with respect to $T(0)$ for fixed x , we have

$$\begin{aligned} \frac{\partial T_0(x)}{\partial T(0)} = & \left[\frac{2}{A T_0^3(x)} \left(F_3 - \frac{C}{5.87} T_0^{5.87}(x) + \frac{D}{5} T_0^5(x) \right) \right]^{1/2} \\ & \cdot \left\{ \frac{dL}{dT(0)} + \frac{(C T_0^{4.87}(0) - D T_0^4(0)) \left(\frac{A}{2} \right)^{1/2} \int_0^{T_0(x)} dT_0 \frac{T_0^{3/2}}{\left(F_3 - \frac{C}{5.87} T_0^{5.87} + \frac{D}{5} T_0^5 \right)^{3/2}}} \right\}. \end{aligned} \quad (38)$$

From the above equation (38), we have

$$T_1^{(1)}(L) = \left(\frac{\partial T_0(x)}{\partial T(0)} \right)_{x=L} = \left(\frac{2 F_3}{A T_0^3(L)} \right)^{1/2} \frac{dL}{dT(0)} \quad (39)$$

Using eqs. (35), (31), (15), (16), (26), (38) and (39), we have the growth rate s :

$$\begin{aligned} s = & - \left[\frac{T(0)}{2 n(0)} \int_0^L d\xi \rho(\xi) T_0(\xi) \left(F_3 - \frac{C}{5.87} T_0^{5.87}(\xi) + \frac{D}{5} T_0^5(\xi) \right)^{1/2} \right. \\ & \cdot \left. \int_0^{T_0(\xi)} dT_0 \frac{T_0^{3/2}}{\left(F_3 - \frac{C}{5.87} T_0^{5.87} + \frac{D}{5} T_0^5 \right)^{3/2}} \right]^{-1} \cdot \frac{dL}{dT(0)}. \end{aligned} \quad (40)$$

The sign of s is determined by that of $dL/dT(0)$, since we may consider only the case $\rho(x) > 0$.

If we assume that the growth rate s is small, we have an estimate of s as done in ref. 3):

$$s \approx - \frac{2 n(0) P(T(0))}{T^2(0) L} \cdot \frac{dL}{dT(0)} \quad (41)$$

Now, substituting eq. (27) into eq. (36), we have a half thickness L :

$$L = \left(\frac{A}{2}\right)^{1/2} T^{5/2}(0) \int_0^1 d\xi \frac{\xi^{3/2}}{\left[\frac{CT^{5.87}(0)}{5.87} (1 - \xi^{5.87}) - \frac{DT^{5/2}(0)}{5} (1 - \xi^5)\right]^{1/2}}. \quad (42)$$

An approximate formula of L is obtained by dividing the integral of ξ in eq. (42) into two parts as follows:⁵⁾

$$\int_0^1 d\xi = \int_{3/4}^1 d\xi + \int_0^{3/4} d\xi.$$

Then, we obtain the approximate formula of L :

$$L = 1.50 \times 10^{-2} \frac{(\lambda/10)^{1/2}}{B} \left\{ \frac{1}{(CT^{0.87}(0) - D)^{1/2}} + \frac{9\sqrt{3}}{80} \frac{1}{(CT^{0.87}(0)/5.87 - D/5)^{1/2}} \right\}. \quad (43)$$

Since

$$\frac{dL}{dT(0)} < 0, \quad (44)$$

the sign of the growth rate s is positive:

$$s \sim -\frac{dL}{dT(0)} > 0. \quad (45)$$

Thus, we conclude that the plasma is unstable to the thermal instability.

The $T(0)$ -dependence of L for $B=10^5$ G and $\lambda \approx 22$ is given by

$$L = \frac{20.2}{(T^{0.87}(0) - 27.6)^{1/2}} + \frac{9.53}{(T^{0.87}(0) - 32.4)^{1/2}}, \quad (46)$$

and shown in Fig. 1. In this case, for $T(0)=100$ keV, we have from Fig. 1

$$\frac{dL}{dT(0)} \approx -5.5 \times 10^{-2} \text{ cm/keV},$$

$$L \approx 5.87 \text{ cm}.$$

Using eq. (41), we have the order of the growth rate

$$s \approx 6.2 \times 10^{-16} n(0) \quad 1/\text{sec}.$$

When $n \approx 10^{16} \text{ 1/cm}^3$, we have

$$s \approx 6.2 \quad 1/\text{sec}.$$

Similarly, for $T(0)=150$ keV, $B=10^5$ G and $\lambda \approx 22$, we have

$$\frac{dL}{dT(0)} \approx -2.0 \times 10^{-2} \text{ cm/keV},$$

$$L \approx 4.25 \text{ cm},$$

$$s \approx 7.1 \times 10^{-16} n(0) \text{ 1/sec.}$$

§3. Catalyzed DD fusion reactor

Secondly, we discuss the instability in a catalyzed DD fusion reactor. The growth rate s in a catalyzed DD fusion reactor is obtained similarly by the method used in §2.

In this section, we will use

$$C_1 = 6.68 \times 10^{-16}, \quad (47)$$

in place of C in §2.

The $T(0)$ -dependence of a half thickness L of the plane plasma layer for $B=10^5$ G and

$\lambda \approx 22$ is given by

$$L = \frac{8.59}{(T^{0.87}(0) - 5.00)^{1/2}} + \frac{4.06}{(T^{0.87}(0) - 5.87)^{1/2}}, \quad (48)$$

and shown in Fig. 2. In this case, for $T(0)=50$ keV, we have from Fig. 2

$$\frac{dL}{dT(0)} \approx -2.7 \times 10^{-2} \text{ cm/keV},$$

$$L \approx 2.54 \text{ cm}.$$

Then, using eqs. (41), (8) and (47), we have

$$s \approx 2.5 \times 10^{-15} n(0) \text{ 1/sec.}$$

Similarly, for $T(0)=100$ keV, $B=10^5$ G

and $\lambda \approx 22$, we have

$$\frac{dL}{dT(0)} \approx -8.6 \times 10^{-3} \text{ cm/keV},$$

Fig. 1. A half plasma thickness L as a function of temperature $T(0)$ at the center for $B=10^5$ G and $\lambda=22$ in a pure DD fusion reactor.

Fig. 2. A half plasma thickness L as a function of temperature $T(0)$ at the center for $B=10^5$ G and $\lambda=22$ in a catalyzed DD fusion reactor.

$$L \approx 1.79 \text{ cm,}$$

$$s \approx 3.2 \times 10^{-15} n(0) \text{ 1 / sec.}$$

§4. Conclusions

We have obtained the growth rates for the thermal thermonuclear instability of a deuterium plasma with an inhomogeneous temperature and number density in a strong magnetic field.

The growth rate s in a pure DD fusion reactor is given by

$$s = -\frac{2n(0)}{L} (1.21 \times 10^{-16} T^{1.37}(0) - 3.34 \times 10^{-15} T^{1/2}(0)) \cdot \frac{dL}{dT(0)} \text{ 1 / sec,} \quad (49)$$

where $n(0)$ ($1/\text{cm}^3$) is the ion number density at the center, $L(\text{cm})$ a half thickness of the plane plasma layer and $T(0)$ (keV) the temperature at the center. In a catalyzed DD fusion reactor, the growth rate s is given by

$$s = -\frac{2n(0)}{L} (6.68 \times 10^{-16} T^{1.37}(0) - 3.34 \times 10^{-15} T^{1/2}(0)) \cdot \frac{dL}{dT(0)} \text{ 1 / sec.} \quad (50)$$

The growth rate in the plasma with an inhomogeneous temperature and number density is of the same order of magnitude with that in the plasma with an inhomogeneous temperature and a homogeneous number density.³⁾

The $T(0)$ -dependence of L for the strength of the magnetic field $B=10^5$ G and the Coulomb logarithm $\lambda \approx 22$ is shown in Fig. 1 (pure DD fusion reactor) or Fig. 2 (catalyzed DD fusion reactor). Since the sign of s is positive ($s > 0$) because of $dL/dT(0) < 0$ in the temperature region considered in this paper, the plasmas with an inhomogeneous temperature and number density in the pure and catalyzed DD fusion reactors are unstable to the thermal instability.

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