A DESIGN METHOD OF L/D CONTROL PROGRAM
FOR ATMOSPHERIC RE-ENTRY
Yoshio YAMADA
Kakuda Propulsion Center
National Space Development Agency of Japan
1 kokuzo, Jinjiro, Kakuda-shi, Miyagi 981-15, Japan

Abstract

It is required for a vehicle to control lift to drag ratio (L/D) during atmospheric re-entry to keep aerodynamic heat-rate or axial load factor at constant level.
The L/D programs under such constraints are analytically solved directly from the equations of motion.
A transfer flight mode to connect the constant heat-rate mode and constant axial load mode is introduced and also solved.

Nomenclature

V: relative velocity to atmosphere
t: time
g: acceleration of gravity=go( re/r )²
go: 9.8m/s²
ρ: density of atmosphere
ρ₀: density of atmosphere on surface of the earth
r: distance from the center of the earth to vehicle
re: radius of the earth
H: altitude, r-re
B: ballistic coefficient
θ: flight path angle
L: lift
D: drag
G: axial load factor
Q: aerodynamic heat-rate
GL,QL: design limit of G,Q

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RN: nose radius

K: coefficient

Tw: wall temperature at equilibrium condition of aerodynamic heating and surface radiation

e: coefficient of radiation

Ksb: Stephen-Boltzmann constant

1. Introduction

A ballistic re-entry vehicle will encounter severe aerodynamic heating and axial load during atmospheric deceleration from orbital velocity.

Affording a lift by means of winged body or lifting body, the heat-rate and axial load could be controlled within the design limitations.

The environments of atmospheric re-entry could be relaxed by giving a constant L/D to the vehicle, however this method often gives phugoid motion to the vehicle.

Eliminating phugoid motion, the continuous control of L/D which satisfy the limitations is indispensable.

The numerical methods such as "Closed-Form Guidance Law" (Ref.1) are used in many cases for solving L/D program to satisfy the constraints.

In this paper, an analytical solution of L/D program is directly derived from the equation of motions under the constraints regarding heat-rate and axial load.

2. Equations of Motion and Constraints

The equations of a re-entry vehicle in great circle plane are described as follows;

\[
\frac{dV}{dt} = g \sin \theta - \frac{\rho V^2}{2B} \tag{1}
\]

\[
V \frac{d\theta}{dt} = (g - \frac{V^2}{r}) \cos \theta - \frac{\rho V^2}{2B} \left( \frac{L}{D} \right) \tag{2}
\]

\[
\frac{dr}{dt} = -V \sin \theta \tag{3}
\]

The coordinate system used here is depicted in Fig.1.

The velocity with respect to inertial coordinate system differs from the relative velocity to the atmosphere of the rotating earth, however both velocities are treated as same variable in the equations above because the difference of them is small enough for the region under consideration.
The limitation of heat-rate to surface of the vehicle could be expressed in terms of velocity and air density as shown in equation (5) (ref 2).

The axial load (G's) by aerodynamic drag is shown in the equation (4).

\[
\begin{align*}
GL & \geq G = \frac{1}{g} \cdot \frac{\rho V^2}{2B} \\
\dot{Q} & \geq \dot{Q} = K \sqrt{\rho / \delta} \quad V^2 \approx \epsilon K s b T w^4
\end{align*}
\]

3. Solution of Lift/Drag Control Program

Since the constraints are described in velocity and air density which is only a function of altitude, it is convenient to describe the equations of motion in terms of velocity and radius vector by eliminating the time.

The equation (1) and (2) can be rewritten in the form of the equation (6) and (7) by using the equation (3).

\[
\begin{align*}
V \frac{dV}{dr} & = -g + \frac{1}{\sin \theta} \cdot \frac{\rho V^2}{2B} \\
V^2 \frac{d\theta}{dr} & = \left( g - \frac{V^2}{r} \right) \cot \theta + \frac{1}{\sin \theta} \cdot \frac{\rho V^2 (L)}{2B} (D)
\end{align*}
\]

Solutions for both constraints are derived in same expression, the left side term \( V \frac{dV}{dr} \) of the equation (6) is expressed as the equation (8) by differenciating the equation (4) and (5) when the vehicle flies along the limitations in the equation (4) and (5).

\[
V \frac{dV}{dr} = -\frac{V^2}{n \rho} \cdot \frac{d\rho}{dr}
\]

where \( n = 2 \) for \( G = GL \)

\( n = 6 \) for \( Q = \dot{Q}L \)

Substituting the equation (8) into (6), the flight path angle \( \theta \) becomes as follows;

\[
\sin \theta = \frac{\rho V^2}{2B} \left( g - \frac{V^2}{n \rho} \cdot \frac{d\rho}{dr} \right)
\]
The equation (9) can be expanded to any case if constraints are expressed in the form of \( \rho V^a = \text{constant} \). \( L/D \) can be computed for arbitrary altitude by the equation (10) derived from the equation (7).

\[
\frac{L}{D} = (V^2 \frac{d\theta}{dr} + (g - \frac{V^2}{r}) \cot \theta) \sin \theta / (G g_0)
\] (10)

The velocity \( V \) is given by the equation (4) or (5), and \( d \theta/dr \) is derived by differentiating the equation (9) as follows;

\[
\rho \cdot V^2 \frac{d\theta}{dr} = 2B \tan \theta \sin \theta \left[ \frac{n-2}{n} \cdot \frac{g}{\rho} \frac{d\rho}{dr} - \frac{V^2}{n \rho} \frac{d^2\rho}{dr^2} + \frac{1}{\rho} \frac{d^2\rho}{dr^2} \right]
\] (11)

4. Transfer Flight Mode

High altitude and high velocity region in early phase of re-entry might be constrained by the limitation of the heat-rate and later phase might be constrained by the limitation of axial load.

As shown in the equation (8), it is obvious that the flight path angle is discontinuous with respect to the altitude where the governing law shown in the equation (4) is changed to the equation (5) instantaneously. Hence, to keep the continuity of the flight path angle, the transfer flight mode must be newly introduced between the constant heat-rate mode and the constant axial load mode.

The flight path must satisfy following boundary conditions in the transfer flight mode.

1. Velocity and flight path angle are continuous at both points of flight mode change.
2. Heat-rate and axial load are below design limitations throughout the transfer flight.

The continuity of flight path angle leads that \( dV/dr \) is also continuous at the altitude of changing flight mode. Any constitutive law could be applicable as long as above requirements are satisfied. A polynomial which includes four coefficients is selected as the simplest example.

\[
V = aH^3 + bH^2 +cH+d
\] (12)

The coefficients \( a, b, c, d \) are determined by the boundary conditions at the connecting points.

\[
V_1 = aH_1^3 + bH_1^2 + cH_1 + d
\]
\[
V_2 = aH_2^3 + bH_2^2 + cH_1 + d
\]
\[
V_1 \cdot (3aH_1^2 + 2bH_1 + c) = g - \frac{1}{\sin \theta_1} \cdot \frac{\rho_1 \cdot V_1^2}{2B} = B'
\] (13)
\[
V_2 \cdot (3aH_2^2 + 2bH_2 + c) = g - \frac{1}{\sin \theta_2} \cdot \frac{\rho_2 \cdot V_2^2}{2B} = C'
\]
Subscripts 1 and 2 mean the points of initiation and termination of the transfer flight mode respectively.
Assuming \( H_1 \) and \( H_2 \) are given, one can easily solve the equations above.

\[
a = \frac{2A - B' - C'}{(H_1 - H_2)^2}
\]

\[
b = \frac{1}{2} \left\{ \frac{B' - C'}{H_1 - H_2} - 3a \frac{H_1 + H_2}{(H_1 + H_2)} \right\}
\]

\[c = B' - (3aH_1 + 2b)H_1\]

\[d = V_1 - (aH_1 + b)H_1 + c \frac{H_1}{H_1 - H_2}\]

Where \( A = (V_1 - V_2) / (H_1 - H_2) \)

Once these coefficients are determined, flight path angle \( \sin \theta \) and \( L/D \) can be derived immediately in the same procedure of deriving the equation (8), (9).

\[
\sin \theta = G_0 / (g + V \frac{dV}{dr})
\]

Where \( dV/dr = 3aH^2 + 2bH + c \)

\[
L = \left( V^2 \frac{d\theta}{dr} + (g - V^2) \cot \theta \right) \frac{\sin \theta}{G_0}
\]

Where \( \frac{d\theta}{dr} = [- \{ (\frac{dV}{dr})^2 + V \frac{d^2V}{dr^2} \} / (g + V \frac{dV}{dr})]
\]

\[\frac{d^2V}{dr^2} = 6aH + 2b \]

5. Mathematical Model of Air Density

Mathematical model of air density must be continuous and differentiable at arbitrary altitude for calculating flight path angle and \( L/D \) in the method stated above.

An exponential model such as \( \rho = \rho_0 \exp(-\beta H) \) seems to be suitable in this regard, however this model has significant error in the region where most part of atmospheric deceleration occur.

A polynomial approximation shown in the equation (17) (ref.3) is used for calculation of air density and its derivatives.

\[
\rho / \rho_0 = S_0^{-4}
\]

Derivatives of air density are as follows;

\[
d\rho / dr = -4\rho S_1 / S_0
\]

\[
d^2\rho / dr^2 = 4\rho \left\{ 5 \left( S_1 / S_0 \right)^2 - S_2 / S_0 \right\}
\]

Where \( S_0 = \sum_{i=0}^{11} A_i H^i \)

\[
S_1 = \sum_{i=0}^{11} A_i \cdot i \cdot H^{i-1}
\]

(18)
\[ S_2 = \sum_{i=1}^{11} A_i \cdot i \cdot (i-1) \cdot H^{i-2} \]

Coefficients \( A_i \) are shown in Table 1.

**Table 1. Coefficients of air Density Model**

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<th>( A_i )</th>
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</table>

6. An Example of Calculation

An example of calculation is depicted in Fig. 2 and Fig. 3. The re-entry flight of a vehicle with controlled L/D is shown in solid lines and that of a ballistic vehicle without lift is shown in dotted lines for comparison.

The vehicle initiates atmospheric re-entry at the altitude of 90km with the velocity of 7.8km/s.

Aerodynamic heating reaches the limitation of heat-rate at approximately 80km, then L/D program is applied until 50km altitude under the flight mode of constant heat-rate. Required L/D for this mode reaches the maximum value of 1.2.

The altitude range where the transfer mode take place is selected between 50km and 40km. Required L/D for this mode varies from 1.1 to 0.4.

Then flight mode changes to the constant axial load mode and it terminates at the altitude of 30km where required L/D reaches zero.

The ballistic flight is applied to altitude below 30km for convenience of computation.

Fig. 3 shows that heat-rate and axial load of the flight under controlled L/D are kept at low level constantly compared with that of the ballistic flight.

Velocity and flight path angle of ballistic flight are calculated from the equation (6) and (7) by Runge -Kutta-Gill method.
7. Conclusion

Analytical solutions of L/D program to satisfy constant heat-rate, constant axial load and transfer mode in atmospheric re-entry are derived and an example of calculation for a hypothetical flight is presented.

The method shown here needs no iterative process nor feedback type computation by flight simulation.

Reference