Analysis on the Anisotropic Tensile Properties of Plain Weave Fabrics

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Abstract

The anisotropic tensile property of plain weave fabrics is theoretically analyzed as a development of the shear deformation theory and the biaxial deformation theory developed by the authors.

By this theory, the anisotropic properties of plain weave fabrics in the biaxial tensile field, including uniaxial tensile state, can be calculated. The conditions for this calculation are three mechanical properties of yarns, i.e., tensile, lateral compressive and frictional properties, and fabric structure. For uniaxial anisotropic property, some approximated methods are used in the theory.

When the principal tensile axis is rotated by θ from the structure principal axis on the fabric plane, the tensile forces along the biased principal tensile axis are given by the following formulas for given λ and λ_L , where λ and λ_L are stretch ratios along two principal tensile axes, respectively

$$F_{\theta} = F_1 \cos \psi_1 + F_2 \cos \psi_2 + F_{tr1} \sin \psi_1 + F_{tr2} \sin \psi_2$$

$$F_{\theta L} = F_1 \sin \psi_1 + F_2 \sin \psi_2 - F_{tr1} \cos \psi_1 - F_{tr2} \cos \psi_2$$

where

$$\psi_1 = \tan^{-1} \frac{\tan \theta \lambda_L}{\lambda}$$

$$\psi_2 = \tan^{-1} \frac{\lambda_L}{\lambda \tan \theta}$$

F_1 and F_2 are introduced as a function of λ and λ_L by the biaxial deformation theory. F_{tr1} and F_{tr2} are introduced as a function of λ and λ_L by the biaxial deformation theory and by yarn frictional property described in the shear deformation theory.

For uniaxial:

$$F_{\theta} = \frac{F_{tr1}}{\sin \psi_1} + \frac{F_{tr2}}{\sin \psi_2}$$

and

$$\lambda = \frac{y_{01} \cos \psi_1 + y_{02} \cos \psi_2}{y_{01} \cos \theta + y_{02} \sin \theta}$$

where

$$\psi_2 = \tan^{-1} \left[\frac{y_{01}}{y_{02}} \frac{k_2}{k_1} \tan \psi_1 \right]$$

$$\phi = \frac{\pi}{2} - (\psi_1 + \psi_2)$$

y_{01} and y_{02} are yarn spacing for warp and weft respectively. k_1 and k_2 are constants and introduced from the biaxial deformation theory.

To calculate the relation between F_{θ} and λ , take ψ_1 as a parameter. By varying the value of ψ_1 , the corresponding values of F_{θ} and λ can be calculated. This means that the value of F_{θ} is a value corresponding to λ . Thus, the relation between F_{θ} and λ can be obtained.

Experimental researches are also carried out for confirming theoretical calculation and good agreement between experimental and theoretical results is obtained.

KEY WORDS: FABRICS, PLAIN WEAVES, ANISOTROPY (STRESS STRAIN), STRESS, TENSILE PROPERTIES, UNIAXIAL STRESS, BIAxIAL STRESS, SHEAR (MODE OF DEFORMATION), MECHANICAL PROPERTIES, THEORIES.

1. Introduction

Fabrics have unique mechanical properties, in particular the tensile properties, unlike ordinary continuous materials.

The tensile properties of the fabrics when uniaxially stretched in either direction of warp or weft or biaxially in both directions at a time with a certain stretch ratio could be calculated with high accuracy.^[1-4]

Fig. 1 The principal axes of the structure of woven fabric are shown by X_1 and X_2 , where X_1 axis is along warp and X_2 axis along weft. Axes X_θ and $X_{\theta L}$ are taken along the biaxial tensile direction and given by the rotation of the coordinate system of X_i by θ

The present paper, as illustrated in Fig.1, analyzes the bi- and uniaxial anisotropic tensile properties of plain weave fabrics along the newly drawn axis X_θ rotated by θ from the structural principal axis X_1 .

The anisotropy of the uni-axial tensile properties of plain weave fabric is measured at intervals of 15° from the warp axis X_1 , as shown in Fig.2, and the tensile force markedly varies according to direction. The tensile force required for the extension along the direction rotated by about 45° from X_1 axis is approximately 10^{-1} to 10^{-2} times that of the structural-principal axis in same stretch ratio.

Such an extreme anisotropy is presumably because of the low-resistance of the woven fabrics against shear force.

In 1949, K.Weisse nberg et al^[5,6] published a paper on uni-axial tensile anisotropic properties of fabric. In their study, applying a method of simulating the fabric to a lattice model, they analyzed the fabric as a continuous material. Then, W.F.Kilby^[7] analyzed the tensile anisotropy of the fabric with the aid of this simple lattice model by regarding the fabric as a thin layer which has an anisotropy together with Poissons effect. Fukatani^[8] also used the same method in his study.

Fig. 2 Anisotropy of the uniaxial tensile property, measured sample is No. 3 fabric woven from polyester spun yarn shown in table 1

In their research, the tensile anisotropy was represented by the properties in warp, weft and 45° directions. In other words, it is to treat the fabric as an orthotropic elastic material. However, the fabric is not a continuous material but discontinuous one.

It is doubted that such a method can be used in discontinuous materials.

In this paper, the tensile anisotropic properties are analyzed by expanding the shear deformation theory of plain woven fabric developed by authors and are derived theoretically from the fabric structure and the mechanical properties of yarn used for the fabric.

On the other hand, the anisotropic properties in one or two dimensional-stress field are measured by the biaxial tensile tester^[11] developed by the authors and the theoretical values are compared with the measured one.

2. Theory

2-1. Basic Structure Model

Let us consider a unit structure of plain woven fabric as enclosed by dotted line in Fig. 3. The unit structure includes a crossing point.

Fig.4 shows coordinate systems before and after deformation.

X_1, X_2 axis are taken along structural-central lines of the fabric before deformation which are orthogonal to each other, and x_1, x_2 axes are along them after stretching along X_θ and $X_{\theta L}$. x_1 and x_2 axes are not orthogonal in general.

Fig. 3 An unit structure of plain woven fabric is shown by a dashed line

(a) undeformed state

(b) deformed state

Fig. 4 Coordinate systems

(a) undeformed state

(b) deformed state

Fig. 5 Model of unit structure before and after deformation

As for the assumption and structural model, the biaxial tensile- and shear-deformation theories of the plain woven fabric already developed by authors can be applied.

The assumptions are:

(1) When the biaxial or uniaxial tensile deformation is given along the new X_θ and $X_{\theta L}$ axes, the structural central lines of the fabric do not bend and keep straight lines.

(2) The weave angles of the weft and warp yarns do not be varied by change of the crossing angle. And the yarn tensions are dependent only on the stretch ratios along x_1 and x_2 axes and are not affected by the change of crossing angle.

Three dimensional representations of the unit model are shown in Fig.5 where P , Q , R and S are correspondent to them in Fig.3.

Putting the warp density be n_1 , weft density n_2 , woven shrinkage ratio of warp and weft S_1 and S_2 respectively, then the yarn spacings y_{01} and y_{02} and weave angles θ_{01} and θ_{02} in undeformed state are obtained by

$$y_{01} = \frac{1}{n_2}, \quad y_{02} = \frac{1}{n_1} \quad (1)$$

$$\theta_{01} = \sin^{-1} \left(\frac{1}{S_1 + 1} \right), \quad \theta_{02} = \sin^{-1} \left(\frac{1}{S_2 + 1} \right) \quad (2)$$

Then, the distances between the break points of yarn and the structure central line, h_{mi} ($i=1, 2$) can be obtained by

$$h_{m1} = \frac{y_{01}}{2 \tan \theta_{01}}, \quad h_{m2} = \frac{y_{02}}{2 \tan \theta_{02}} \quad (3)$$

2-2. Anisotropy of the Biaxial Tensile Properties

Let us consider the general deformation state shown in Fig.4(b) or Fig.5(b) where the biaxial tensile force is applied on the unit.

The projections of the unit model on X_1 - X_2 plain or x_1 - x_2 plain are shown in Fig.6, where the dotted lines and the solid lines present the undeformed and deformed state respectively. Now consider the general tensile deformation where the stretch ratio λ is given along the axis X_θ and at the same time λ_L along the axis $X_{\theta L}$, vertical to the axis X_θ . Here let the tensile forces in each direction be F_θ and $F_{\theta L}$.

As seen in Fig.6, the points P , Q , R and S before the deformation move to the points p , q , r , and s after the deformation. The external force F_θ can be divided into F_a and F_b and the forces act at p and r respectively. And in same manner, $F_{\theta L}$ is also divided into F_{aL} and F_{bL} . These forces are in equilibrium with the yarn tension F_i ($i=1, 2$) and forces F_{tri} which arise from the torque required for the rotation of the yarn around a crossing point.

Now let the angles between x_1 axis and the tensile direction be ψ_1 as shown in Fig.6, and define ϕ such that the crossing angle of warp and weft is varied by ϕ from $\pi/2$ due to the tensile deformation of the fabric. Then the relation among these angles, as is clear from Fig.6, is

Fig. 6 Equilibrium of forces acting on a unit structure under biaxially stretched state

given as

$$\psi_1 + \psi_2 = \frac{\pi}{2} - \phi \quad (4)$$

$$\phi = \phi_1 + \phi_2, \quad \phi_1, \phi_2 > 0 \quad (5)$$

The forces F_{tr1} and F_{tr2} are functions of ϕ , as is clear from eqs. (16), (17) and (18).

From the equilibrium of the force, the following formules are obtained:

$$F_{\theta} = F_a + F_b \quad (6)$$

$$F_{\theta L} = F_{aL} + F_{bL} \quad (7)$$

$$F_a = F_1 \cos \psi_1 + F_{tr1} \sin \psi_1 \quad (8)$$

$$F_b = F_2 \cos \psi_2 + F_{tr2} \sin \psi_2 \quad (9)$$

$$F_{aL} = F_1 \sin \psi_1 - F_{tr1} \cos \psi_1 \quad (10)$$

$$F_{bL} = F_2 \sin \psi_2 - F_{tr2} \cos \psi_2 \quad (11)$$

$$\frac{y_1}{2} F_{tr1} = -\frac{y_2}{2} F_{tr2} \quad (12)$$

From these equilibrium equations, F_{θ} and $F_{\theta L}$ under the biaxial stretching at given stretch ratios λ and λ_L can be calculated.

Substituting eqs.(8) and (9) into eq.(6),

$$F_{\theta} = F_1 \cos \psi_1 + F_2 \cos \psi_2 + F_{tr1} \sin \psi_1 + F_{tr2} \sin \psi_2 \quad (13)$$

Also, substitute eqs.(10) and (11) into eq.(7), and

$$F_{\theta L} = F_1 \sin \psi_1 + F_2 \sin \psi_2 - F_{tr1} \cos \psi_1 - F_{tr2} \cos \psi_2 \quad (14)$$

In the case of $\lambda = \lambda_L$, $\theta = 0$, or $\theta = \pi/2$, we have

$$F_{tr1} = F_{tr2} = 0 \quad (15)$$

Otherwise

$$F_{tr1} = \frac{T_r}{y_1} \quad (16)$$

$$F_{tr2} = \frac{T_r}{y_2} \quad (17)$$

Here, T_r , when the crossing angle of the yarn is changing, will take the value approximated by eq.(18). Otherwise, i.e., when θ is 0 or $\pi/2$ and is stretched biaxially with equal stretch ratios, F_{tr1} and F_{tr2} are zero.

How the torque is affected by the degree of variation of crossing angle at the crossing point of warp and weft was already clarified in the shear-deformation theory.^[9,10,12] Namely, considering T_r as a function of F_c and ϕ where F_c is the compressible force between warp and weft at their crossing point and ϕ is the rotated angle from the original rectangle at the point.

$$T_r = \pm T_{r0} \pm (C_1 + C_2 \phi) F_c + (C_3 + C_4 F_c) \phi \quad (18)$$

where T_{r0} , C_1 , C_2 , C_3 and C_4 are all constants and positive signs in the 1st and 2nd terms of the right of eq.(18) is used for the increasing process of ϕ and negative ones for the decreasing process of ϕ . The 1st and 2nd terms represent the frictional component and the 3rd terms elastic component.

Fig. 7 Principle of the measurement of torque caused by rotation of crossing yarn at a crossing point and torque constants are derived from this measurement

The constants included in eq.(18), say torque constants, can be directly obtained by measuring the torque as shown in Fig.7.^[12] Those constants can be also derived from the shear-deformation properties of the fabric.^[13] Here the torque constants are treated as a characteristic value obtained by measurement of the yarn and the theoretical derivation of them is put out of our consideration.

On the other hand, F_1 , F_2 and F_c are given as a function

of λ and λ_L respectively for the calculation of F_θ and $F_{\theta L}$ by eqs.(13) and (14). In the first place the yarn spacings y_{01} and y_{02} become y_1 and y_2 respectively by biaxial stretching of stretch ratios λ and λ_L .

$$y_1 = \sqrt{(y_{01} \sin \theta \lambda_L)^2 + (y_{01} \cos \theta \lambda)^2} \dots\dots\dots(19)$$

$$y_2 = \sqrt{(y_{02} \cos \theta \lambda_L)^2 + (y_{02} \sin \theta \lambda)^2} \dots\dots\dots(20)$$

Also, λ_1 and λ_2 which are the stretch ratios along structural-principal axes can be calculated by

$$\lambda_1 = \frac{y_1}{y_{01}} = \frac{\sqrt{(y_{01} \sin \theta \lambda_L)^2 + (y_{01} \cos \theta \lambda)^2}}{y_{01}} \dots\dots\dots(21)$$

$$\lambda_2 = \frac{y_2}{y_{02}} = \frac{\sqrt{(y_{02} \cos \theta \lambda_L)^2 + (y_{02} \sin \theta \lambda)^2}}{y_{02}} \dots\dots\dots(22)$$

From the assumptions 1 and 2, the tensile stretch forces F_1 and F_2 and the compressive force F_c at the yarn-crossing point can be calculated as functions of λ_1 and λ_2 , by applying the biaxial tensile deformation theory, by the following equations,^[2-4]

$$F_c = \phi(\lambda_1, \lambda_2) \dots\dots\dots(23)$$

$$F_1 = g_1(\lambda_1, \lambda_2) = \frac{F_c \lambda_1 y_{01}}{4(h_{m1} - h_1)} \dots\dots\dots(24)$$

$$F_2 = g_2(\lambda_1, \lambda_2) = \frac{F_c \lambda_2 y_{02}}{4(h_{m2} + h_2)} \dots\dots\dots(25)$$

where h is a function of λ_1 and λ_2 and defined by the following equation.

$$h_i = K_i(\lambda_1, \lambda_2) \quad (i=1, 2) \dots\dots\dots(26)$$

The values of h_{m1} and h_{m2} are calculable from eq.(3), and h_i is the displacement of point o_1 or o_2 in Fig.5 along X_3 axis, and can be calculated from tensile and compressive properties of the yarn and from fabric structure^[3-4] under any biaxial deformation.

ψ_1 and ψ_2 are calculated as following

$$\tan \psi_1 = \frac{\tan \theta \lambda_L}{\lambda} \dots\dots\dots(27)$$

$$\tan \psi_2 = \frac{\lambda_L}{\lambda \tan \theta} \dots\dots\dots(28)$$

Accordingly, the angle ϕ can be given by substituting eqs.(27) and (28) into eq.(4).

$$\phi = \frac{\pi}{2} - \left\{ \tan^{-1} \left(\frac{\tan \theta \lambda_L}{\lambda} \right) + \tan^{-1} \left(\frac{\lambda_L}{\lambda \tan \theta} \right) \right\} \dots\dots\dots(29)$$

After calculating F_c by eq.(23) and ϕ by eq.(29), then F_{tr1} and F_{tr2} are obtainable as following. Substituting eqs.(8) and (19) into eq.(16) to obtain,

$$F_{tr1} = \frac{\pm T_{r0} \pm C_1 F_c + (C_3 + C_4 F_c \pm C_2 F_c) \phi}{\sqrt{(y_{01} \sin \theta \lambda_L)^2 + (y_{01} \cos \theta \lambda)^2}} \dots\dots\dots(30)$$

And, substituting eqs.(18) and (20) into eq.(17),

$$F_{tr2} = \frac{\pm T_{r0} \pm C_1 F_c + (C_3 + C_4 F_c \pm C_2 F_c) \phi}{\sqrt{(y_{02} \cos \theta \lambda_L)^2 + (y_{02} \sin \theta \lambda)^2}} \dots\dots\dots(31)$$

Finally F_θ and $F_{\theta L}$ are derived by eqs.(13) and (14), where F_c , F_1 and F_2 can be calculated by the biaxial tensile

deformation theory^[3] of the fabric. Here, note that F_θ and $F_{\theta L}$ are tensile loads per unit structure and the load per unit length, f_θ and $f_{\theta L}$ correspondingly, are obtained as follows:

$$f_\theta = n F_\theta \dots\dots\dots(32)$$

$$f_{\theta L} = n_L F_{\theta L} \dots\dots\dots(33)$$

where,

$$n = \frac{1}{y_{01} \sin \theta + y_{02} \cos \theta} \dots\dots\dots(34)$$

$$n_L = \frac{1}{y_{01} \cos \theta + y_{02} \sin \theta} \dots\dots\dots(35)$$

2-3. Anisotropy in Uniaxial Tensile Properties

Uniaxial tensile deformation is a special case of the biaxial deformation where the force $F_{\theta L}$ is kept zero during the deformation.

This situation makes the analysis very difficult, so that some approximations are required. The equilibrium state of forces in unit structure is as shown in Fig.8. The external force F_θ consists of forces F_a and F_b acting on points p and q respectively. F_{aL} and F_{bL} are both disappeared because of uniaxial extension.

$$F_\theta = F_a + F_b \dots\dots\dots(6)$$

$$F_{\theta L} = F_{aL} + F_{bL} = 0 \dots\dots\dots(36)$$

Fig. 8 Forces in equilibrium state of an unit structure under the uniaxially stretched state

The equilibrium equations (10) and (11) can be rewritten by using the relation of eq.(36), that is,

$$F_1 \sin \psi_1 - F_{tr1} \cos \psi_1 = 0 \dots\dots\dots(37)$$

$$F_2 \sin \psi_2 - F_{tr2} \cos \psi_2 = 0 \dots\dots\dots(38)$$

Accordingly, in the case of uniaxial deformation, the equilibrium equations presented by formule (6), (36), (8), (9), (37), (38) and (12) can be established.

$$F_{\theta} = F_1 \cos \psi_1 + F_2 \cos \psi_2 + F_{tr1} \sin \psi_1 + F_{tr2} \sin \psi_2 \dots (13)$$

From eq.(37),

$$F_1 = \frac{F_{tr1} \cos \psi_1}{\sin \psi_1} \dots (39)$$

and from eq.(38),

$$F_2 = \frac{F_{tr2} \cos \psi_2}{\sin \psi_2} \dots (40)$$

It is noted that the fabric is deformed biaxially even though the bulk stretching is uniaxial.

By substituting eqs.(39) and (40) into eq.(13), F_{θ} can be derived as:

$$F_{\theta} = \frac{F_{tr1}}{\sin \psi_1} + \frac{F_{tr2}}{\sin \psi_2} \dots (41)$$

where F_{tr1} and F_{tr2} can be presented by eqs.(16) and (17), and T_r can be approximated by eq.(18).

In general, woven fabrics is very extensible by small extension force toward the biased direction from warp or weft direction in the uniaxial extension. (Fig.2)

Accordingly, F_1 and F_2 are small and, thereby, the tension on the yarn is extremely small.

This fact implies that the biaxial deformations along warp and weft directions of the fabric caused by this uniaxial extension should be very small. In these small deformation region it has become clear that the relation between F_c and F_1 or between F_c and F_2 can be approximated by a linear relation^[2-4] such that,

$$F_c \approx k_1 F_1 \quad F_c \approx k_2 F_2 \dots (42)$$

where k_1 and k_2 are both constants calculated by the biaxial tensile deformation theory^[2-4]. But the values of k_1 and k_2 are depend on the deformation mode of the biaxial stretching. Thus it is assumed that the deformation mode is very close to the homogeneous biaxial deformation, that is, the case of $\lambda_1 = \lambda_2$.

Substituting eq.(42) into eq.(18),

$$T_r = \pm T_{r0} \pm (C_1 + C_2 \phi \pm C_4 \phi) k_1 F_1 + C_3 \phi \\ = \pm T_{r0} \pm (C_1 + C_2 \phi \pm C_4 \phi) k_2 F_2 + C_3 \phi \dots (43)$$

And substituting eq.(43) into eq.(16),

$$F_1 = \frac{F_{tr1} y_1 \pm T_{r0} - C_3 \phi}{\pm (C_1 + C_2 \phi \pm C_4 \phi) k_1} \dots (44)$$

Also, substituting eq.(43) into eq.(17),

$$F_2 = \frac{F_{tr2} y_2 \pm T_{r0} - C_3 \phi}{\pm (C_1 + C_2 \phi \pm C_4 \phi) k_2} \dots (45)$$

From eqs.(44) and (39),

$$F_{tr1} = \frac{\pm T_{r0} + C_3 \phi}{y_1 - \frac{\pm (C_1 + C_2 \phi \pm C_4 \phi) k_1}{\tan \psi_1}} \dots (46)$$

From eqs.(45) and (40),

$$F_{tr2} = \frac{\pm T_{r0} + C_3 \phi}{y_2 - \frac{\pm (C_1 + C_2 \phi \pm C_4 \phi) k_2}{\tan \psi_2}} \dots (47)$$

By substituting eqs.(46) and (47) into the equilibrium eq.(12) and by rearranging the terms,

$$y_2 - \frac{\pm (C_1 + C_2 \phi \pm C_4 \phi) k_2}{\tan \psi_2} \\ = \frac{y_2}{y_1} \left\{ y_1 - \frac{\pm (C_1 + C_2 \phi \pm C_4 \phi) k_1}{\tan \psi_1} \right\} \dots (48)$$

From eq.(48), the following is obtained:

$$\tan \psi_2 = \frac{y_1}{y_2} \frac{k_2}{k_1} \tan \psi_1 \\ \psi_2 = \tan^{-1} \left(\frac{y_1}{y_2} \frac{k_2}{k_1} \tan \psi_1 \right) \dots (49)$$

In the light of eq.(49), the relation between ψ_1 and ψ_2 during the uniaxial tensile deformation can be obtained. It is noticed that this relation between ψ_1 and ψ_2 is independent of θ . This implies that if the specimen is stretched along X_{θ} axis uniaxially there will take place the bending of the specimen in the plain so as to keep the relation of eq.(49) (refer to Fig.19).

Substituting eqs.(46) and (47) into eq.(41), and eqs.(48) and (49) into eq.(41), F_{θ} is approximately as given

$$F_{\theta} = \frac{\pm T_{r0} + C_3 \phi}{y_1 - \frac{\pm (C_1 + C_2 \phi \pm C_4 \phi) k_1}{\tan \psi_1}} \\ \left\{ \frac{1}{\sin \psi_1} + \frac{y_1}{y_2} \frac{1}{\sin \left\{ \tan^{-1} \left(\frac{y_1}{y_2} \frac{k_2}{k_1} \tan \psi_1 \right) \right\}} \right\} \dots (50)$$

The relation between F_{θ} and ψ_1 can be obtained from eq.(50), and the relation between ψ_1 and ψ_2 is given by eq.(49), where

$$y_1 = y_{01} \lambda_1, \quad y_2 = y_{02} \lambda_2 \dots (51)$$

As shown previously, the fabric can be stretched easily in biased direction from its structural-principal axes by small amount of force, so that the stretching of the yarn should be negligible during the deformation. So that, assume

$$\lambda_1 = \lambda_2 \approx 1 \dots (52)$$

then eq.(51) will be approximated as

$$y_1 = y_{01}, \quad y_2 = y_{02} \dots (53)$$

By using the relations of eq.(53), eq.(50) becomes very simple as

$$F_{\theta} = \frac{\pm T_{r0} + C_3 \phi}{y_{01} - \frac{\pm (C_1 + C_2 \phi \pm C_4 \phi) k_1}{\tan \psi_1}} \\ \left\{ \frac{1}{\sin \psi_1} + \frac{y_{01}}{y_{02}} \frac{1}{\sin \left\{ \tan^{-1} \left(\frac{y_{01}}{y_{02}} \frac{k_2}{k_1} \tan \psi_1 \right) \right\}} \right\} \dots (54)$$

The load per unit specimen width, f_{θ} , can be calculated by the following equation where n is the number of unit

Fig. 10 Strip-biaxial extension along X_θ axis

The constants k_1 and k_2 are calculated by the biaxial deformation theory and also shown in Table 2.

3-1. Anisotropy of Biaxial Tensile Properties

Calculation of the tensile load F_θ and constraint load $F_{\theta L}$ is done at first under the given condition that the fabric is stretched at tensile ratio λ toward the axis X_θ kept $\lambda_L = 1$ as shown in Fig.10.

From the given condition,

$$\lambda_L = 1 \quad (57)$$

Substituting eqs.(59) into (21) and (22), obtained

$$\lambda_1 = \frac{y_1}{y_{01}} = \frac{\sqrt{(y_{01} \sin \theta)^2 + (y_{01} \cos \theta \lambda)^2}}{y_{01}} \quad (58)$$

$$\lambda_2 = \frac{y_2}{y_{02}} = \frac{\sqrt{(y_{02} \cos \theta)^2 + (y_{02} \sin \theta \lambda)^2}}{y_{02}} \quad (59)$$

The tensile ratios λ_1 and λ_2 of the fabric along the axes x_1 and x_2 at tensile ratio λ along X_θ axes are calculable from the yarn spacings y_{01} , y_{02} and biased angle θ by eqs.(58) and (59) respectively. As a particular case, if $\theta = 45^\circ$

$$\lambda_1 = \frac{\sqrt{\frac{1}{2} y_{01}^2 (1 + \lambda^2)}}{y_{01}} \quad (60)$$

$$\lambda_2 = \frac{\sqrt{\frac{1}{2} y_{02}^2 (1 + \lambda^2)}}{y_{02}} \quad (61)$$

F_c , F_1 and F_2 can be calculated by the biaxial tensile deformation theory,^[1-4] using the structural constants of the fabric shown in Table 1 and the mechanical properties of the yarns shown in Fig.9. (For the details of the calculating method, refer to^[4]).

Then, substituting eq.(57) into eqs.(27) and (28),

$$\tan \psi_1 = \frac{\tan \theta}{\lambda} \quad (62)$$

$$\tan \psi_2 = \frac{1}{\lambda \tan \theta} \quad (63)$$

In particular, when $\theta = 45^\circ$,

$$\tan \psi_1 = \frac{1}{\lambda}, \quad \tan \psi_2 = \frac{1}{\lambda} \quad (64)$$

i.e.,

$$\psi_1 = \psi_2 \quad (65)$$

Angle ϕ is obtained by substituting eqs.(62) and (63) into eq.(4)

$$\phi = \frac{\pi}{2} - \left\{ \tan^{-1} \left(\frac{\tan \theta}{\lambda} \right) + \tan^{-1} \left(\frac{1}{\lambda \tan \theta} \right) \right\} \quad (66)$$

Fig. 11 Anisotropy in the tensile properties of No. 1 cotton fabric under strip-biaxial extension

Fig. 12 Anisotropy in the tensile properties of No.2 wool fabric in strip-biaxial extension

Fig. 13 Anisotropy in the tensile properties of No.3 fabric woven from polyester spun yarn in strip-biaxial extension

and when $\Theta=45^\circ$, substituting eq.(64) into eq.(4),

$$\phi = \frac{\pi}{2} - 2\left(\tan^{-1}\frac{1}{\lambda}\right) \dots\dots\dots(67)$$

Thus, F_c and ϕ can be calculated. Using the torque constants T_{r0} , C_1 , C_3 and C_4 (in this case, $C_2=0$), F_{tr1} and F_{tr2} are calculable from eqs.(30) and (31) by incorporating eqs.(58) and (59), that is,

$$F_{tr1} = \frac{\pm T_{r0} \pm C_1 F_c + (C_3 + C_4 F_c) \phi}{\lambda_1 y_{01}} \dots\dots\dots(68)$$

$$F_{tr2} = \frac{\pm T_{r0} \pm C_1 F_c + (C_3 + C_4 F_c) \phi}{\lambda_2 y_{02}} \dots\dots\dots(69)$$

Substituting F_{tr1} and F_{tr2} calculated above into eqs. (13) and (14), and putting $\lambda_L=1$, F_θ and $F_{\theta L}$ are calculable.

If n and n_L are calculated by eqs.(34) and (35), then the load f_θ per unit width and constraint load $f_{\theta L}$ can be obtained. The results of this calculations for $\Theta=22.5^\circ$, 45° and 67.5° are shown in Figs. 11 to 13 by dotted lines. On the other hand, the measurement of these properties have been carried out using the biaxial tensile tester^[11] where the tensile velocity is selected as 50%/min. And the results are shown in same figures by solid lines for comparison. In these figures the tensile properties in the structural

Fig. 14 Deformation modes

principal axes X_1 and X_2 , i.e., in the case of $\theta=0^\circ, 90^\circ$ are shown in the same figure.

As shown in Fig.14, warp direction of the specimens is taken as $\theta=0^\circ$, arrow mark indicates the tensile direction.

Fig.15 is the photographic view. A good agreement between theoretical and experimental values is obtained all

over the full region from small stretching to large stretching.

Samples shown in Figs. 12 and 13 have yielding property and yet well-agreement between theoretical and measured values is observed after yielding. In Fig.16, these results are shown on polar coordinate as contour line of λ .

3-2. Anisotropy of Uniaxial Tensile Properties

Taking the approximation of $\lambda_1=\lambda_2 \approx 1$, i.e.,

$$y_{01} \approx y_1, \quad y_{02} \approx y_2 \dots \dots \dots (53)$$

The anisotropy of the uniaxial stretching property is calculated by eqs.(54) and (56).

Giving ψ_1 as a parameter and substituting the approximated relation of eq.(53) into eq.(49), ψ_2 is calculated by

$$\psi_2 \approx \tan^{-1} \left(\frac{y_{01} k_2}{y_{02} k_1} \tan \psi_1 \right) \dots \dots \dots (70)$$

and ϕ is calculated by

$$\phi = \frac{\pi}{2} - (\psi_1 + \psi_2) \dots \dots \dots (4)$$

The relationship between ψ_1 , ψ_2 and ϕ can be decided regardless of θ , and the upper and the lower limits of ψ_1 and ψ_2 can be decided by constants k_1 and k_2 obtainable from fabric structures y_{01} and y_{02} and homogeneous

(a) $\theta=22.5^\circ$ (b) $\theta=45^\circ$ (c) $\theta=67.5^\circ$
Fig. 15 The appearance of the samples under sing of the strip biaxial extension ($\lambda_L=1$, $\lambda=1.2$)

Fig. 16 Anisotropy in the tensile properties of No. 1, No. 2 and No. 3 samples in strip-biaxial extension

biaxial tensile properties.

F_θ can be calculated substituting the torque constants and the value of k_1 , which are shown in Table 2, into eq.(71) for any given parameter ψ_1 .

$$F_\theta = \frac{\pm Tr_0 + C_3\phi}{y_{01} - \frac{\pm (C_1 \pm C_4\phi)k_1}{\tan \psi_1}} \left[\frac{1}{\sin \psi_1} + \frac{y_{01}}{y_{02}} \frac{1}{\sin \psi_2} \right] \dots (71)$$

λ can be also calculated by eq.(72) for any given parameter ψ_1 .

$$\lambda = \frac{y_{01} \cos \psi_1 + y_{02} \cos \psi_2}{y_{01} \cos \theta + y_{02} \sin \theta} \dots (72)$$

In this way many sets of λ and F_θ corresponding to many values of ψ_1 are obtained. From this, the relation between λ and F_θ is decided. Besides, the load f_θ per unit width can be calculated by nF_θ .

However, in this case, n is

$$n = \frac{1}{y_{01} \sin \theta + y_{02} \cos \theta} \dots (34)$$

Accordingly, f_θ can be decided according to the angle θ .

Using constants of the samples (Tables 1 and 2), the calculation curve when the fabric stretched in the direction is $\theta=45^\circ$ is shown with dotted lines in Fig.17.

Experimental conditions used here was the following: specimen size is 2×20 cm cut at intervals of 15° or 22.5° , taking the warp direction as 0° ; width to length ratio of sample, 1:10, and tensile speed is 50%/min. The solid lines of Fig.17 show the experimental results. In this case, there is disagreement between experimental and theoretical values in relatively large-stretched region in which the $f_\theta - \lambda$ curve takes a steep curve. The reason is presumably that the yarn stretching can not be neglected. Accordingly, as far as the approximation of eq.(53) is concerned, i.e., in the small deformation region, the both curves are well agreed.

There is also a stepwise rise-up in the beginning of the deformation as seen is the shear deformation properties. This is a usual phenomenon seen in the uniaxial tensile properties in biased direction from warp or weft. This can be also theoretically calculated. The difference in the initial tensile properties between the specimens of Fig.17, comes from the difference of the torque constants of those specimen.

In the similar comparison, Fig.18 shows the case of 45° . In this case, in particular, hysteresis calculation was also carried out. The material is the polyester spun yarn fabric.

As previously mentioned, for the recovery process of theoretical calculation is used the negative symbol of $-$ in eq.(71).

As is clear from Fig.19, when the fabric is stretched toward the direction of 45° , both ψ_1 and ψ_2 take almost same values. The fabric is stretched evenly toward either direction of warp and weft. Recall the approximation that k_1 and k_2 are used for the value in the case of homogeneous

Fig. 17 Uniaxial tensile properties along X_θ axis. ($\theta=45^\circ$)

Fig. 18 Hysteresis in the uniaxial tensile property of fabric (polyester No.3 spun) along X_θ ($\theta=45^\circ$) axis

biaxial stretching. Relatively highly accurate approximation is estimated around 45° direction. In Fig.19, when $\theta=45^\circ$, the fabric is stretched straight. But when θ takes the other angle, there occurs some bending of specimen as shown in the figure which was expected by formula (70).

The pure uniaxial experiment is expected to be difficult in practice except in the case of 45° .

4. Discussion

4-1. Accuracy of the Approximated Solution for Anisotropic Property of the Uniaxial Tensile Deformation

Values of k_1 and k_2 in eq.(49) are measured to be equal to those obtained from the homogeneous-biaxial extension of the fabric toward the axes x_1 and x_2 .

However, as is clear from the photographic observation, this assumption is reasonable for the tensile defor-

Fig. 19 The appearance of the samples under testing of uniaxial extension (left $\theta = 22.5^\circ$, center $\theta = 45^\circ$, right $\theta = 67.5^\circ$ in each figure)

Fig. 20 Anisotropy in the tensile properties of No. 3 sample in uniaxial extension ($\theta = 15^\circ, 75^\circ$ are corrected using the corrected theory)

mation in the vicinity of $\theta = 45^\circ$. But as it moves from $\theta = 45^\circ$ to 0° , the tensile ratio along the axis x_1 is greater than that of x_2 axis, and to 90° , x_2 is greater than x_1 .

Accordingly, when θ is 0° or 90° , it seems to be a better approximation to use k_1 and k_2 obtained by strip-biaxial

Fig. 21 Deformation of the fabric is approximated being equal to homogeneous-biaxial extension for $\theta = 45^\circ \pm 22.5^\circ$, strip-biaxial deformation along warp for $0^\circ < \theta < 22.5^\circ$ and strip-biaxial deformation along weft for $67.5^\circ < \theta < 90^\circ$

Fig. 22 Effective length of the specimen

extension rather than homogeneous one. A rule regarding to the selection of the values of k_1 and k_2 is shown in Fig.21. And this set of k values can be obtained by biaxial deformation theory^[8]. Substituting these values into eq.(49), ψ_1 and ψ_2 are obtained, and f_θ is calculated as function of λ . An example of the calculated results is shown with experimental data in Fig.20. When θ is less than 15° or larger than 75° , the agreement is much better than the previous method.

From this facts, for the calculation of the uniaxial tensile property, it seems to be necessary to select the use of the

(a) uniaxial extension

(b) strip-biaxial extension

Fig. 23 Anisotropy in the tensile properties of No. 3 fabric woven from polyester spun yarn

values of k_1 and k_2 in accordance with θ .

From the view point of experiment, the uniaxial testing is very difficult. One of the reason is the bending of specimen shown in Fig.19 and the problem to decide the effective length. Fig.22 shows the effective length taken in this experiment.

4-2. Comparison of Anisotropic Properties of Biaxial and Uniaxial Extension.

In general, extreme anisotropy is observed in the property of uniaxial extension but the anisotropy becomes very small in biaxial extension as shown in Fig.23.

4-3. Relation between the Anisotropy in Tensile Properties and the Shear Deformation Properties

S.M. Spivak^[14] et al. tried to predict shear properties of the fabric in simple shear^[15] by the measurement of the tensile property along $\theta=45^\circ$, and concluded being impossible to do it.

From this theory, this result is easily predicted because the uniaxial tensile properties is very hard to solve without some approximation.

However, the theory presented here predicts that the torque constants which characterize the shear property can be calculated from the anisotropic properties of biaxial extension, that is, F_θ and $F_{\theta L}$ measured by the biaxial experiment of any θ and F_1 and F_2 measured by the experiment of $\theta=0^\circ$ are substituted into eqs.(13) and (14) to obtain F_{tr1} and F_{tr2} . The value of F_c is also calculated by the biaxial deformation theory^[3].

From eqs.(16), (17) and (18) torque constants, T_{r0} , C_1 , C_2 , C_3 and C_4 can be determined.

These constants characterize the shear property of plain woven fabrics.^[3,9,10]

5. Conclusion

The anisotropy of the biaxial tensile property including the uniaxial property has been solved theoretically in this

paper by applying the shear deformation theory of plain woven fabrics already developed.

In this theory, the fabric structure and the following properties of yarns such that tensile property, compressible property against side pressure, and torque property at crossing point of warp and weft, are given.

On the other hand, the measurement of the anisotropy has been carried out and the results have been compared with theoretically calculated results and experimental ones.

This theory is constructed by very small approximations except the case of uniaxial deformation.

The agreement between theory and experiment is very good all over the full region of the stretching for any value of θ . But uniaxial tensile deformation is a special case of the biaxial tensile deformation, and some approximation is required in this case. And from this, some disagreement between them is observed in the stretching region of relatively high stretch ratio.

As is clear from the analysis presented here, the infinitesimal theory of anisotropic-continuous materials can not be applied to analyze the stress-strain properties of fabrics.

But this theory presented here can be used for investigations and design of fabrics instead of those conventional analyses.

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