Fluttering of Flexible Bodies

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Abstract

In order to clarify the fluttering behavior of flexible bodies in a stream, the wave form was assumed as \( y = ax \sin(pt+qx) \), and from the energy conservation principle, \( p \) and \( q \) were determined as

\[
p = -0.415V \sqrt{(p/\rho_s)(b/l)}(C_x+2k_0)
\]

\[
q = 1.6911/\lambda
\]

On the other hand, fabric or film strips were observed in a vertical wind tunnel, and it was shown that the wave form assumed agreed with the observed form within a limited range, and that the frequency calculated from the above-mentioned \( p \) could give the rough estimate of the real flutter frequency.

However, the wave form observed was very complicated, and especially so at the tail part of the strip.


1. Introduction

Many theoretical and experimental studies have been carried out on the vibration of rigid bodies, because the latter will suffer critical damage if they vibrate in resonance with the outer stimulus. But very few have been studied, excepting the vibration of chord fixed on both ends, on such flexible bodies as threads or clothes, because they do not induce any danger even if they flutter in a stream. However, it is usually observed that banners or flags flutter in a wind, that a cloth-carp, a Japanese festival symbol in May, swims in the sky.

Similar phenomena are also found in the textile industry when the yarn is sucked by air, or when the cloth is washed in water-stream. They are similar in such character as the vibration of bodies having little flexural rigidity. In this paper will be studied the fluttering of a body, fixed at one end and flexible as mentioned above.

2. Theory of Fluttering

2.1 Symbols

Symbols used are summarized as follows:

- \( a \): Amplitude coefficient
- \( b \): Width specimen
- \( C_N \): Drag coefficient normal to the specimen
- \( C_x \): Drag coefficient along \( x \) axis
- \( C_y \): Drag coefficient along \( y \) axis
- \( D_x \): Drag along \( x \) axis
- \( D_y \): Drag along \( y \) axis
- \( f \): Vibration frequency of the specimen per min.
- \( h \): Specimen thickness
- \( K \): Total kinetic energy of the specimen
- \( k \): Coefficient of the circular frequency of the specimen
- \( k_0 \): Proportional coefficient of \( C_y \)
- \( l \): Specimen length
- \( p \): Circular frequency of the specimen
- \( q \): Wave length coefficient
- \( \theta \): Inclination of the specimen-segment to \( x \) axis
- \( \nu \): Dynamic viscosity of the fluid
- \( \rho \): Density of the fluid per unit volume
- \( \rho_1 \): Density of the specimen per unit volume
- \( \rho_0 \): Density of the specimen per unit area
- \( \rho_1 \): Linear density of the specimen
- \( S \): Specimen area on which drag acts
- \( t \): Time
- \( V \): Fluid velocity along \( x \) axis
- \( W \): Total potential energy of the specimen
- \( x \): Coordinate of the specimen-segment when the specimen does not vibrate
- \( y \): Displacement of the specimen-segment normal to \( x \) axis

2.2 Energy Equation

A small body \( dS \) put in a stream with an attack angle \( \theta \)
has, as shown in Fig. 1, drag \( dD_x \) along the stream and drag \( dD_y \) vertical to the stream, the velocity of which is uniform and \( V \). In this case, they are generally defined as

\[
\begin{align*}
    dD_x &= C_x \frac{1}{2} \rho V^2 dS \\
    dD_y &= C_y \frac{1}{2} \rho V^2 dS
\end{align*}
\]

Now, the flutter wave form of a flexible body, fixed its one end at 0, is assumed as shown by a broken line in Fig. 1 and is put

\[
y = a_v \cdot \sin (pt + q_l) \quad \text{ .................................. (2)}
\]

under the assumption that the vibration is only in two dimensions, and that the amplitude of the travelling wave increases linearly when one end of the specimen is fixed and the other is free.

If the body is perfectly flexible, and the effect due to both the internal friction and the gravity could be neglected, there holds the energy conservation principle to the whole vibrating system as

\[
W = \sum (\text{Total potential energy}) + \sum (\text{Total kinetic energy}) = \text{constant} \quad \text{ ............. (3)}
\]

This \( W \) comprises of both the potential energy \( W_x \) due to the access of any specimen-segment to origin 0 from the place when the specimen does not vibrate, and the potential energy \( W_y \) due to its displacement along \( y \) axis. If the amplitude is small, the small access \( dx \) to origin 0 can be put as

\[
J_x = \frac{1}{2} \int_0^z \left( \frac{dy}{dx} \right)^2 dx \quad \text{ .................................. (4)}
\]

So the work done by \( dD_x \) is

\[
dW_x = dD_x \cdot dy = \frac{b}{4} C_x \rho V^2 dx \cdot dy \quad \text{ .................................. (5)}
\]

On the other, the work done by \( dD_y \) is

\[
dW_y = dD_y \cdot dy = \frac{b}{2} C_y \rho V^2 dx \cdot dy \quad \text{ .................................. (6)}
\]

In such a special case as when we could put \( C_x = \text{const.} \), and \( C_y = k_0 \theta = k_0 \frac{dy}{dx} \) and \( k_0 = \text{const.} \), independent of \( x \), these will be discussed later, the total energy along the whole length \( l \) is

\[
W = \int dW_x + \int dW_y
\]

where

\[
\begin{align*}
    W_x &= \frac{b}{4} C_x \rho V^2 \int_0^z \int_0^z \left( \frac{dy}{dx} \right)^2 dx \cdot dy \\
    W_y &= \frac{b}{2} C_y \rho V^2 \int_0^z \int_0^z \left( \frac{dy}{dx} \right)^2 dx \cdot dy
\end{align*}
\]

Substituting (2), it goes

\[
W = \frac{b}{4} \rho V^2 \left( C_x + 2 k_0 \right) \int_0^z \int_0^z \left( \frac{dy}{dx} \right)^2 dx \cdot dy
\]

The kinetic energy is obtained by neglecting the small velocity component along \( x \) axis, and putting

\[
y = a_v x \cos (pt + q_l) \quad \text{ .................................. (7)}
\]

So we have

\[
K = \frac{1}{2} \rho \int_0^x a_v x \rho \cos (pt + q_l) \cdot \frac{a_v^2}{2V^2} \left( \frac{q_l^4}{4} - \frac{q_l^2}{4} + \frac{1}{8} \right) \sin 2 (pt + q_l) + \frac{q_l^2}{8} \cos 2 (pt + q_l) + \sin \frac{2pt}{8} \quad \text{ .................................. (8)}
\]

Substituting eqs. (7) and (8) into eq. (3),

\[
\frac{\rho s^2}{k} A + B + C \sin 2pt + \frac{\rho s^2}{k} B + D \cos 2 pt + E = \text{constant} \quad \text{........................................} \quad \text{ (10)}
\]

where

\[
\begin{align*}
    A &= \frac{1}{q_l} \left( \left( \frac{q_l^2}{4} - \frac{1}{8} \right) \cos 2 q_l \cdot \frac{q^2}{4} - \frac{7}{8} \sin 2 q_l + \frac{1}{8} \right) \\
    B &= \frac{1}{q_l} \left( \left( \frac{q_l^2}{4} - \frac{1}{8} \right) \sin 2 q_l + \frac{q^2}{4} \cos 2 q_l + \frac{7}{8} \right) \\
    C &= q_l \frac{1}{2} \left( -1 - q_l^2 \right) \sin 2 q_l \\
    D &= - \frac{1}{16} \left( 1 - 2 q_l^2 \right) \cos 2 q_l \\
    E &= \frac{q^2}{24} + \frac{q^4}{4} + \frac{q^6}{8} \frac{q_l^2}{6} \\
    k &= \frac{1}{4} \left( \frac{q^2}{k} \right) \left( C_x + 2 k_0 \right) V^2
\end{align*}
\]
In order that eq. (10) is always correct irrespective of time, the coefficients preceding \( \sin 2pt \) and \( \cos 2pt \) must be zero. Hence,
\[
\frac{p^2}{k} A + C = 0
\]
\[
\frac{p^2}{k} B + D = 0
\]
So, we have
\[
A \cdot D - B \cdot C = 0
\]

From eq. (13), we have \( q_1 = 0 \) or 1.6911. The former corresponds to such a vibration as the body is straight and swings like a door. This does not mean a flexible body, and is discarded. So, \( q_1 \) should be 1.6911. Thus, from eq. (12),
\[
p^2 = 0.68718k
\]
This gives plus and minus value to \( p \). But, plus \( p \) means the wave travelling backwards against the stream, and does not fit to the problem now being considered. Therefore,
\[
p = -0.415 \sqrt{\left( \frac{\rho}{\rho_s} \right) \left( \frac{b}{l} \right) (C_x + 2k_0)}
\]
and the wave form is
\[
y = ax \sin \left( -0.415 \sqrt{\left( \frac{\rho}{\rho_s} \right) \left( \frac{b}{l} \right) (C_x + 2k_0)} + 1.6911 x/l \right)
\]
This indicates that \( f \) is independent of the specimen width \( b \).

3. Flutter Experiment

3.1 Specimens and Measuring Method
To affirm the wave form assumed by eq. (2), the flutter experiment was carried out with strips shown in Table 1. There, \( A \) and \( B \) are very flexible and thought liable to twist in the stream, giving somewhat abnormal wave form. So a little bit brittle specimen \( C \) is also tried. Each specimen is vertically suspended on a horizontal piano wire spun in a wind tunnel, the cross-section of which is \( 0.21 \times 0.11 \text{ m}^2 \) and the length of which is 1.4 m. The wave form and the flutter frequency were observed stroboscopically. As there was large fluctuation in the flutter frequency, the latter was observed every 10 sec., and the mean of 60 readings was assumed as the representative value.

<table>
<thead>
<tr>
<th>Table 1 Specimens Used</th>
</tr>
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<tbody>
<tr>
<td>A</td>
</tr>
<tr>
<td>Material</td>
</tr>
<tr>
<td>Width (cm)</td>
</tr>
<tr>
<td>Areal density (mg/cm²)</td>
</tr>
<tr>
<td>Length (cm)</td>
</tr>
<tr>
<td>Color</td>
</tr>
<tr>
<td>Surface</td>
</tr>
</tbody>
</table>

![Fig. 2 Wave form by equ. (14) \((a = 1/2)\)](image)

Fig. 3 Superposed wave form
(i) Specimen A  (ii) Specimen C
3.2 Experimental Results

Fig. 3(i) is the wave form with specimen A, Fig. 3(ii) being that with specimen C, taken stroboscopically by a long exposure. They contain some abnormal forms, but are roughly triangular, showing the soundness of the assumption underlying eq. (2).

Fig. 4 shows the wave pattern with specimen C, in which (i) and (ii) agree well with the pattern shown in Fig. 2. But as the amplitude increases, the specimen tail touches the wall of the wind tunnel as shown in Fig. 4(iii). This causes an abnormal form, and the intermediate form such as shown in Fig. 4(iv), and the two waves such as shown in Fig. 4(v) are originated. In case of a very flexible specimen, together with the flutter, twisting takes part in the vibration such as shown in Fig. 5, in (iii) and (iv) of which the steep swing-up of the tail part is observed. The latter is the cause of the deviation of the wave form from the sectorial region. If the specimen length is short, some whirling motion around the vertical axis is also observed. So the flutter phenomena are very complicated with flexible bodies.

Fig. 6 shows the relation between the wind velocity and the observed flutter frequency. As mentioned above, the frequency is independent of the specimen width in case of
the same material, so are plotted in the figure both observations with $b = 4$ cm and 8 cm. From the figure, it is presumed that the linear relation holds between the wind velocity and the flutter frequency with short specimen such as $l = 10$ cm. However, as the specimen becomes long, the relation tends to curvilinear, and this tendency is conspicuous when the wind velocity is over 3 m/sec. The flutter frequency is also very unstable there, and in some cases the observed values appeared upside down when the specimen length was increased.

The flutter phenomena were rather stable as shown in Fig. 7 with specimen C having a little flexural rigidity. But the lines there do not converge to zero when there is no wind. This may be because of its flexural rigidity which should be zero in the theory. Fig. 8 indicates the variation of flutter frequency observed with specimen C. As is natural, the variation increases as the specimen is lengthened and the standard deviation is about 60/min when $l = 30$ cm. Compared it with the mean flutter frequency there, about 380/min, the coefficient of variation at that time is very large and about 16%.

4. Discussion

4.1 Wave Form and Amplitude

The wave form assumed by eq. (2) agrees well with the experiment when the specimen is short and the wind speed is low. But as the latters increase, the wave is made very complicated by the tail-swing-up, yawing, or twisting.

One of the reasons for the tail-swing-up may be because of the abrupt drop of $C_y$ beyond the critical angle of attack. This is illustrated in Fig. 9, in which $C_x$ and $C_y$ are shown with a circular cylinder\textsuperscript{3}, a square plate\textsuperscript{4}, and a symmetrical aerofoil. In case of a circular cylinder, $C_y$ increases gradually. But with flat bodies, $C_y$ increases up to a certain angle of attack, beyond which it drops abruptly. So, if a specimen-segment exceeds that critical angle of attack, the force pushing that part towards x axis decreases, whereas the segment preceding it and having smaller angle of attack than the critical angle has the pushing force towards x axis, larger than that of the former. So the segment exceeding the critical angle is left behind, and the segment slope becomes steeper, resulting in the tail-swing-up.

If the specimen has a little flexural rigidity, the tail-swing-up is restricted due to this rigidity, forming less turbulent wave than the cloth. The neat wave form shown in Fig. 4 is due to this.

The amplitude coefficient, $a$, could not yet be determined by the energy equation. But the dimensional analysis gives

$$\text{Amplitude} = k_1\left(\frac{\rho}{\rho_0}\right)^a\left(\frac{Vh}{L}\right)^b\left(\frac{h}{b}\right)^c\left(\frac{h}{L}\right)^d (17)$$

Each bracket in the right hand side of eq. (17) stands for density, Reynold’s number, thickness ratio, slenderness
ratio and specimen length, respectively. This formula shows clearly the effect of each factor on the amplitude. But if it is rearranged by 1, it gives $l \frac{r}{r}$ which is still undetermined as $r$ is unknown. The max. amplitude of specimen C was observed as shown in Fig. 10, from which $y$ could be roughly assumed to be 2.

4.2 $C_y$ and Flutter Frequency

Hoerner shows the cylinder drag as follows:

$$C_x = 1.1 \sin^3 \theta + 0.2$$

$$C_y = 1.1 \sin^2 \theta \cos \theta$$

Also, with a plate,

$$C_x = C_N \sin \theta$$

$$C_y = C_N \cos \theta$$

where $C_N$ is the drag coefficient normal to the plate. The drag coefficient with an aerofoil can be determined by the wind tunnel test. These are plotted in Fig. 9, which indicates that $C_y$ is roughly proportional to $\theta$ within a limited range of the attack angle with flat bodies. This proportional constant is put $k_0$ in this paper. But $k_0$ should be different depending on whether the body is perfectly flat or is slightly curvilinear although any small part of the body can be assumed flat. The flutter problem considered here corresponds to the latter case. But there is no available data for $k_0$ in such a case, and $k_0$ is assumed constant over the whole length of the specimen.

As for an example, $k_0$ is assumed equal to 6.20, which is the highest value observable in Fig. 9. On the other, $C_x$ may be put equal to zero compared with $k_0$, although it increases gradually as the shape changes from curvilinear to cylindrical. So, we have

$$C_x + 2k_0 \div 2k_0 = 12.40$$

Thus, from equ. (16),

$$f/V = 13.9 V \div \sqrt{\frac{1 - \frac{\rho}{\rho_s}}{l}}$$

The observed $f/V$ and the calculated $f/V$ by equ. (20) are compared in Table 2. With specimen A, there is good agreement. With specimen C, the observed value is much smaller than the calculated one. This may be because the assumption of perfect flexibility is not satisfied and much energy is lost for internal strain. There could be found no good reason for the discrepancy with specimen B. One of the reasons may be, as B is much heavier than A, due to the inertia force which causes the tail-swing-up as shown in Fig. 5. The swing-up induces large $C_x$ which however is omitted in equ. (20). Fairthorne showed also much larger $C_x$ than the value assumed from the frictional coefficient in case of the flag flutter. Anyway, $C_x$ and $C_y$ exert large influence upon the frequency calculation, and it is a problem left for future consideration how large they are assumed.

5. Conclusion

The flutter frequency is theoretically obtained with a flat flexible body, exposed in a stream and fixed at one end. It is compared with the value obtained by experiments. The main results are as follows:

(i) The wave form assumed as $y = ax \sin(pt + qx)$ agrees well with the observed one within a limited range.

(ii) By the energy conservation principle, $p$ and $q$ above-mentioned are determined as

$$p = -0.415 V \div \sqrt{\frac{1}{l} (C_x + 2k_0)}$$

$$q = 1.6911/l$$

(iii) $p$ above obtained explains well the tendency observed by Fairthorne, as it contains the density ratio $(\rho/\rho_s)$, the slenderness ratio $(b/l)$, the drag coefficient, etc.

(iv) But in reality, the flutter phenomena are very complicated due to abrupt large amplitude, tail-swing-up or three-dimensional vibration. These cause large variation of flutter frequency observed.

(v) However, the flutter frequency theoretically obtained can give a rough estimate of observed values.

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Literature Cited


