Viscoelastic Slow Flow through a Tapered Channel into a Slit

by Kiyoji Nakamura, Takuji Higashi and Akira Horikawa, Members, TMSJ

Department of Mechanical Engineering, Osaka University, Suita, Osaka

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Abstract

The effect of inlet angles of a tapered channel on viscoelastic flows in a Maxwell model was numerically investigated by the perturbation method using oblique coordinates. These results have shown that the characteristic flow pattern of viscoelastic liquids such as a shape of a wine glass is produced by the sudden contraction into the slit rather than by turning round the corner, and, in addition, that the inlet angle of a tapered channel has little effect on the shape of the main flow.

1. Introduction

In the previous papers^[1, 2], we described flow phenomena of highly viscous polymer solutions or polymer melts from a large reservoir into a slit; a large circulating flow was observed near the slit inlet. This flow is located near the corner when the deformation velocity is small, that is, Weissenberg number is small. However, as Weissenberg number becomes large, this circulating flow grows and moves towards the slit inlet. This shows that the circulating flow may be caused by both the effect of turning round the corner and that of sudden flowing into the slit.

In the present paper, we investigate the influence of slit inlet angles on the main and the circulating flows, and examine two effects mentioned above. The numerical method is the same as in the previous papers^[2, 3]; the perturbation method based on a Maxwell model. Although the shape of the main flow into the slit was observed to be independent of the inlet angle, this was reconfirmed by this numerical method.

2. Fundamental Equations

Motions of incompressible continua obey the following equations:

$$\text{Equation of continuity; } v^i_{,i} = 0 \quad \dots\dots\dots(1)$$

The Cauchy's equation of motion;

$$\rho v^k v^i_{,k} = -g^{ik} p_{,k} + T^{ki}_{,k} \quad \dots\dots\dots(2)$$

where g_{ik} denotes a metric tensor, and $T^{ij}_{,k}$ the covariant derivative such as

$$T^{ij}_{,k} = \frac{\partial T^{ij}}{\partial x^k} + \left\{ \begin{matrix} i \\ km \end{matrix} \right\} T^{mj} + \left\{ \begin{matrix} j \\ km \end{matrix} \right\} T^{im}$$

$\left\{ \begin{matrix} i \\ km \end{matrix} \right\}$ is the Christoffel's 3-index symbol denoted by

$$\left\{ \begin{matrix} i \\ km \end{matrix} \right\} = \frac{1}{2} g^{il} \left(\frac{\partial g_{kl}}{\partial x^m} + \frac{\partial g_{ml}}{\partial x^k} - \frac{\partial g_{km}}{\partial x^l} \right)$$

The viscoelastic constitutive equation of the Maxwell type is

$$T^{ij} + \lambda \frac{D_j T^{ij}}{Dt} = 2\eta D^{ij} \quad \dots\dots\dots(3)$$

where λ denotes the relaxation time, η the coefficient of viscosity and D_j/Dt the Jaumann derivative defined by

$$\frac{D_j T^{ij}}{Dt} = \frac{\partial T^{ij}}{\partial t} + v^k T^{ij}_{,k} - W^i_k T^{kj} + T^{ik} W_k^j$$

$$D_{km} = \frac{1}{2} (v_{k,m} + v_{m,k})$$

$$W_{km} = \frac{1}{2} (v_{k,m} - v_{m,k})$$

Consider a two-dimensional steady flow shown in Fig. 1. The following relations hold between oblique rectilinear coordinates (x^1, x^2) and the Descartes's coordinates ($\bar{x}_1, \bar{x}_2$);

$$\bar{x}_1 = x^1 - x^2 \cos \theta$$

$$\bar{x}_2 = x^2 \sin \theta$$

where θ denotes an angle between x^1 and x^2 axes. The metric tensor g_{ij} in oblique rectilinear coordinates is

$$g_{ij} = \frac{\partial \bar{x}^k}{\partial x^i} \frac{\partial \bar{x}^k}{\partial x^j}$$

that is,

$$g_{11} = g_{22} = 1$$

$$g_{12} = g_{21} = -\cos \theta$$

All the Christoffel's 3-index symbols $\left\{ \begin{matrix} i \\ km \end{matrix} \right\}$ are zeroes, because metric tensors are independent of the coordinate.

3. Physical Component of Fundamental Equations

The physical component of a contravariant vector v^i is given by

$$v_{(i)} = \sqrt{g_{ii}} v^i \quad (\text{not summed for } i) \quad \dots\dots\dots(4)$$

Fig. 1 Schematic diagram of a flow channel

Hence, for a covariant vector v_i ,

$$v_{(i)} = \sqrt{g_{ii}} g^{ik} v_k \quad (\text{not summed for } i)$$

Next, consider the physical component of a tensor, and restrict the tensor within the stress tensor;

$$f^i = T^i_{\cdot j} n^j$$

or

$$f^i = n^j T_j^{\cdot i} \quad \dots\dots\dots(5)$$

where n^j denotes a unit normal vector in an arbitrary plane and f^i a force per unit area of this plane. From this relation and eq. (4), the physical component of the stress tensor is obtained;

$$T_{(ij)} = \sqrt{\frac{g_{ii}}{g_{jj}}} T^i_j \quad (\text{not summed for } i \text{ and } j)$$

or

$${}_{(ji)}T = \sqrt{\frac{g_{ii}}{g_{jj}}} T_j^{\cdot i} \quad (\text{not summed for } i \text{ and } j) \quad \dots\dots\dots(6)$$

$T_{(ij)}$ and ${}_{(ji)}T$ are a right and a left physical component^[4], respectively. In the present paper, let us define the stress tensor by eq. (5), and use ${}_{(ji)}T$ of eq. (6) as the physical component of the stress tensor. We can prove ${}_{(ij)}T = T_{(ij)}$ for orthogonal coordinates, but ${}_{(ij)}T \neq T_{(ij)}$ for oblique coordinates. For a symmetric tensor, we obtain ${}_{(ij)}T = T_{(ji)}$, but $T_{(ij)} \neq T_{(ji)}$ and ${}_{(ij)}T \neq T_{(ji)}$.

Using physical components of eq. (4) and eq. (6), we rewrite fundamental eqs. (1), (2) and (3). Further, in order to obtain a non-dimensional form, we introduce the following relations:

$$\begin{aligned} x_1^* &= \frac{x_1}{H}, & x_2^* &= \frac{x_2}{H}, \\ v_1^* &= \frac{v_{(1)}}{V}, & v_2^* &= \frac{v_{(2)}}{V}, \end{aligned}$$

$$\begin{aligned} p^* &= \frac{H}{\eta V} p, & T_{11}^* &= \frac{H}{\eta V} {}_{(11)}T, & T_{22}^* &= \frac{H}{\eta V} {}_{(22)}T, \\ T_{12}^* &= \frac{H}{\eta V} {}_{(12)}T, & T_{21}^* &= \frac{H}{\eta V} {}_{(21)}T \quad \dots\dots\dots(7) \end{aligned}$$

where V denotes a mean velocity, H is $\bar{H}/\sin \theta$ and $2\bar{H}$ a channel width.

Therefore, eqs. (1), (2) and (3) are

$$\frac{\partial v_1^*}{\partial x_1^*} + \frac{\partial v_2^*}{\partial x_2^*} = 0 \quad \dots\dots\dots(8)$$

$$\begin{aligned} R_e \left\{ v_1^* \frac{\partial v_1^*}{\partial x_1^*} + v_2^* \frac{\partial v_2^*}{\partial x_2^*} \right\} &= \frac{1}{\sin^2 \theta} \left\{ -\frac{\partial p^*}{\partial x_1^*} \right. \\ &+ \frac{\partial T_{11}^*}{\partial x_1^*} + \frac{\partial T_{12}^*}{\partial x_2^*} + \cos \theta \left(-\frac{\partial p^*}{\partial x_2^*} + \frac{\partial T_{21}^*}{\partial x_1^*} \right. \\ &\left. \left. + \frac{\partial T_{22}^*}{\partial x_2^*} \right) \right\} \quad \dots\dots\dots(9a) \end{aligned}$$

$$\begin{aligned} R_e \left\{ v_1^* \frac{\partial v_2^*}{\partial x_1^*} + v_2^* \frac{\partial v_1^*}{\partial x_2^*} \right\} &= \frac{1}{\sin^2 \theta} \left\{ -\frac{\partial p^*}{\partial x_2^*} \right. \\ &+ \frac{\partial T_{21}^*}{\partial x_1^*} + \frac{\partial T_{22}^*}{\partial x_2^*} + \cos \theta \left(-\frac{\partial p^*}{\partial x_1^*} + \frac{\partial T_{11}^*}{\partial x_1^*} \right. \\ &\left. \left. + \frac{\partial T_{12}^*}{\partial x_2^*} \right) \right\} \quad \dots\dots\dots(9b) \end{aligned}$$

$$\begin{aligned} T_{11}^* + W_e \left\{ v_1^* \frac{\partial T_{11}^*}{\partial x_1^*} + v_2^* \frac{\partial T_{11}^*}{\partial x_2^*} - \frac{1}{2 \sin^2 \theta} \left(\frac{\partial v_1^*}{\partial x_2^*} \right. \right. \\ \left. \left. + 2 \cos \theta \frac{\partial v_1^*}{\partial x_1^*} - \frac{\partial v_2^*}{\partial x_1^*} \right) (T_{12}^* + T_{21}^*) \right\} \\ = \frac{2}{\sin^2 \theta} \left\{ \frac{\partial v_1^*}{\partial x_1^*} + \frac{1}{2} \cos \theta \left(\frac{\partial v_1^*}{\partial x_2^*} - \frac{\partial v_2^*}{\partial x_1^*} \right) \right\} \quad \dots\dots\dots(10a) \end{aligned}$$

$$\begin{aligned} T_{22}^* + W_e \left\{ v_1^* \frac{\partial T_{22}^*}{\partial x_1^*} + v_2^* \frac{\partial T_{22}^*}{\partial x_2^*} + \frac{1}{2 \sin^2 \theta} \left(\frac{\partial v_1^*}{\partial x_2^*} \right. \right. \\ \left. \left. + 2 \cos \theta \frac{\partial v_1^*}{\partial x_1^*} - \frac{\partial v_2^*}{\partial x_1^*} \right) (T_{12}^* + T_{21}^*) \right\} \\ = \frac{2}{\sin^2 \theta} \left\{ \frac{\partial v_2^*}{\partial x_1^*} + \frac{1}{2} \cos \theta \left(\frac{\partial v_2^*}{\partial x_1^*} - \frac{\partial v_1^*}{\partial x_2^*} \right) \right\} \quad \dots\dots\dots(10b) \end{aligned}$$

$$\begin{aligned} T_{12}^* + W_e \left\{ v_1^* \frac{\partial T_{12}^*}{\partial x_1^*} + v_2^* \frac{\partial T_{12}^*}{\partial x_2^*} + \frac{1}{2 \sin^2 \theta} \left(\frac{\partial v_1^*}{\partial x_2^*} \right. \right. \\ \left. \left. + 2 \cos \theta \frac{\partial v_1^*}{\partial x_1^*} - \frac{\partial v_2^*}{\partial x_1^*} \right) (T_{11}^* - 2 \cos \theta T_{12}^* - T_{22}^*) \right\} \\ = \frac{1}{\sin^2 \theta} \left(\frac{\partial v_1^*}{\partial x_2^*} + \frac{\partial v_2^*}{\partial x_1^*} - 2 \cos \theta \frac{\partial v_2^*}{\partial x_2^*} \right. \\ \left. - 2 \cos^2 \theta \frac{\partial v_2^*}{\partial x_1^*} \right) \quad \dots\dots\dots(10c) \end{aligned}$$

$$\begin{aligned} T_{21}^* + W_e \left\{ v_1^* \frac{\partial T_{21}^*}{\partial x_1^*} + v_2^* \frac{\partial T_{21}^*}{\partial x_2^*} + \frac{1}{2 \sin^2 \theta} \left(\frac{\partial v_1^*}{\partial x_2^*} \right. \right. \\ \left. \left. + 2 \cos \theta \frac{\partial v_1^*}{\partial x_1^*} - \frac{\partial v_2^*}{\partial x_1^*} \right) (T_{11}^* + 2 \cos \theta T_{21}^* - T_{22}^*) \right\} \\ = \frac{1}{\sin^2 \theta} \left(\frac{\partial v_1^*}{\partial x_2^*} + \frac{\partial v_2^*}{\partial x_1^*} - 2 \cos \theta \frac{\partial v_1^*}{\partial x_1^*} \right. \\ \left. - 2 \cos^2 \theta \frac{\partial v_1^*}{\partial x_2^*} \right) \quad \dots\dots\dots(10d) \end{aligned}$$

where $R_e = \rho V H / \eta$ and $W_e = \lambda V / H$, which are called Reynolds number and Weissenberg number, respectively. Now, restricting the flow to such one as having only very

small inertia force, the left term of eq. (9) is zero. In order to satisfy the equation of continuity (8), we introduce a stream function;

$$v_1^* = \frac{\partial \psi}{\partial x_2^*}, \quad v_2^* = -\frac{\partial \psi}{\partial x_1^*} \quad \dots\dots\dots(11)$$

As for the boundary condition, the velocity is zero on a solid wall and is fully developed upstream far from a slit:

$$v_1^* = \frac{3}{2}(1-x_2^{*2}), \quad v_2^* = 0 \quad \text{at} \quad x_1^* = -\infty \quad \dots\dots\dots(12a)$$

$$v_1^* = v_2^* = 0 \quad \text{at} \quad x_1^* = 0, \quad \text{and} \quad 0 \leq x_2^* \leq h^* \quad \dots\dots\dots(12b)$$

$$v_1^* = v_2^* = 0 \quad \text{at} \quad x_1^* = 0, \quad \text{and} \quad h^* \leq x_2^* \leq 1 \quad \dots\dots\dots(12c)$$

$$\frac{\partial v_1^*}{\partial x_2^*} = v_2^* = 0 \quad \text{at} \quad x_2^* = 0 \quad \dots\dots\dots(12d)$$

$$v_1^* = v_2^* = 0 \quad \text{at} \quad x_2^* = 1 \quad \dots\dots\dots(12e)$$

where h^* denotes a dimensionless quantity h/H or $\bar{h}/\bar{H}$, $2\bar{h}$ a slit width and $2h$ is $2\bar{h}/\sin \theta$.

4. Development of Fundamental Equations

In order to obtain the flow field, we must solve eqs. (8), (9) and (10) simultaneously under the boundary conditions (12). As the constitutive equation of Maxwell type expressed by eq. (10) is not explicit on stress, it is difficult to solve these systems. The purpose of this paper, however, is to analyze the flow when the viscoelastic effect begins to appear. So, such an assumption can be introduced as Weissenberg number W_e is smaller than 1. Then, velocity, stress and stream function can be expanded in power series of Weissenberg numbers as follows;

$$\begin{aligned} v_i^* &= v_i^{(0)} + W_e v_i^{(1)} + W_e^2 v_i^{(2)} + \dots\dots \\ p^* &= p^{(0)} + W_e p^{(1)} + W_e^2 p^{(2)} + \dots\dots \\ T_{ij}^* &= T_{ij}^{(0)} + W_e T_{ij}^{(1)} + W_e^2 T_{ij}^{(2)} + \dots\dots \\ \psi &= \psi^{(0)} + W_e \psi^{(1)} + W_e^2 \psi^{(2)} + \dots\dots \end{aligned} \quad \dots\dots\dots(13)$$

Substituting eq. (13) into eqs. (8), (9), (10), (11) and (12), we rearrange them on Weissenberg number.

4.1 Terms having no Weissenberg number

$$\frac{\partial v_1^{(0)}}{\partial x_1^*} + \frac{\partial v_2^{(0)}}{\partial x_2^*} = 0 \quad \dots\dots\dots(14a)$$

$$\begin{aligned} &\frac{\partial T_{11}^{(0)}}{\partial x_1^*} + \frac{\partial T_{12}^{(0)}}{\partial x_2^*} + \cos \theta \left(\frac{\partial T_{21}^{(0)}}{\partial x_1^*} + \frac{\partial T_{22}^{(0)}}{\partial x_2^*} \right) \\ &= \frac{\partial p^{(0)}}{\partial x_1^*} + \cos \theta \frac{\partial p^{(0)}}{\partial x_2^*} \quad \dots\dots\dots(14b) \end{aligned}$$

$$\begin{aligned} &\frac{\partial T_{21}^{(0)}}{\partial x_1^*} + \frac{\partial T_{22}^{(0)}}{\partial x_2^*} + \cos \theta \left(\frac{\partial T_{11}^{(0)}}{\partial x_1^*} + \frac{\partial T_{12}^{(0)}}{\partial x_2^*} \right) \\ &= \frac{\partial p^{(0)}}{\partial x_2^*} + \cos \theta \frac{\partial p^{(0)}}{\partial x_1^*} \quad \dots\dots\dots(14c) \end{aligned}$$

$$T_{11}^{(0)} = \frac{1}{\sin^2 \theta} \left\{ 2 \frac{\partial v_1^{(0)}}{\partial x_1^*} + \cos \theta \left(\frac{\partial v_1^{(0)}}{\partial x_2^*} - \frac{\partial v_2^{(0)}}{\partial x_1^*} \right) \right\} \quad \dots\dots\dots(14d)$$

$$T_{22}^{(0)} = \frac{1}{\sin^2 \theta} \left\{ 2 \frac{\partial v_2^{(0)}}{\partial x_2^*} + \cos \theta \left(\frac{\partial v_2^{(0)}}{\partial x_1^*} - \frac{\partial v_1^{(0)}}{\partial x_2^*} \right) \right\} \quad \dots\dots\dots(14e)$$

$$\begin{aligned} T_{12}^{(0)} &= \frac{1}{\sin^2 \theta} \left\{ \frac{\partial v_1^{(0)}}{\partial x_2^*} + \frac{\partial v_2^{(0)}}{\partial x_1^*} - 2 \cos \theta \frac{\partial v_2^{(0)}}{\partial x_2^*} \right. \\ &\quad \left. - 2 \cos^2 \theta \frac{\partial v_2^{(0)}}{\partial x_1^*} \right\} \quad \dots\dots\dots(14f) \end{aligned}$$

$$\begin{aligned} T_{21}^{(0)} &= \frac{1}{\sin^2 \theta} \left\{ \frac{\partial v_1^{(0)}}{\partial x_2^*} + \frac{\partial v_2^{(0)}}{\partial x_1^*} - 2 \cos \theta \frac{\partial v_1^{(0)}}{\partial x_1^*} \right. \\ &\quad \left. - 2 \cos^2 \theta \frac{\partial v_1^{(0)}}{\partial x_2^*} \right\} \quad \dots\dots\dots(14g) \end{aligned}$$

$$v_1^{(0)} = \frac{\partial \psi^{(0)}}{\partial x_2^*}, \quad v_2^{(0)} = -\frac{\partial \psi^{(0)}}{\partial x_1^*} \quad \dots\dots\dots(14h)$$

Their boundary conditions are

$$v_1^{(0)} = \frac{3}{2}(1-x_2^{*2}), \quad v_2^{(0)} = 0 \quad \text{at} \quad x_1^* = -\infty$$

$$v_1^{(0)} = v_2^{(0)} = 0 \quad \text{at} \quad x_1^* = 0, \quad 0 \leq x_2^* \leq h^*$$

$$v_1^{(0)} = v_2^{(0)} = 0 \quad \text{at} \quad x_1^* = 0, \quad h^* \leq x_2^* \leq 1$$

$$\frac{\partial v_1^{(0)}}{\partial x_2^*} = v_2^{(0)} = 0 \quad \text{at} \quad x_2^* = 0$$

$$v_1^{(0)} = v_2^{(0)} = 0 \quad \text{at} \quad x_2^* = 1 \quad \dots\dots\dots(15)$$

Upon eliminating $\partial p^{(0)}/\partial x_1^*$ and $\partial p^{(0)}/\partial x_2^*$, eq. (14) can be written by $\psi^{(0)}$ and its derivatives as follows;

$$\left[\frac{\partial^2}{\partial x_1^{*2}} + \frac{\partial^2}{\partial x_2^{*2}} + 2 \cos \theta \frac{\partial^2}{\partial x_1^* \partial x_2^*} \right] \psi^{(0)} = 0 \quad \dots\dots\dots(16)$$

The solution $\psi^{(0)} = \psi^{(0)}(x_1^*, x_2^*)$ of eq. (16) under the boundary condition (15) shows a Newtonian flow, and leads to the zero order approximate term of the viscoelastic flow.

4.2 First order terms of Weissenberg number

Upon substituting eq. (13) into eq. (10), and equating the first order terms of Weissenberg number, then following equations are obtained;

$$\begin{aligned} T_{11}^{(1)} &= \frac{1}{\sin^2 \theta} \left\{ 2 \frac{\partial v_1^{(1)}}{\partial x_1^*} + \cos \theta \left(\frac{\partial v_1^{(1)}}{\partial x_2^*} - \frac{\partial v_2^{(1)}}{\partial x_1^*} \right) \right. \\ &\quad \left. + \frac{1}{2} \left(\frac{\partial v_1^{(0)}}{\partial x_2^*} - \frac{\partial v_2^{(0)}}{\partial x_1^*} + 2 \cos \theta \frac{\partial v_1^{(0)}}{\partial x_1^*} \right) (T_{12}^{(0)} \right. \\ &\quad \left. + T_{21}^{(0)}) \right\} - v_1^{(0)} \frac{\partial T_{11}^{(0)}}{\partial x_1^*} - v_2^{(0)} \frac{\partial T_{11}^{(0)}}{\partial x_2^*} \quad \dots\dots\dots(17a) \end{aligned}$$

$$\begin{aligned} T_{22}^{(1)} &= \frac{1}{\sin^2 \theta} \left\{ 2 \frac{\partial v_2^{(1)}}{\partial x_2^*} + \cos \theta \left(\frac{\partial v_2^{(1)}}{\partial x_1^*} - \frac{\partial v_1^{(1)}}{\partial x_2^*} \right) \right. \\ &\quad \left. - \frac{1}{2} \left(\frac{\partial v_1^{(0)}}{\partial x_2^*} - \frac{\partial v_2^{(0)}}{\partial x_1^*} + 2 \cos \theta \frac{\partial v_1^{(0)}}{\partial x_1^*} \right) (T_{12}^{(0)} \right. \\ &\quad \left. + T_{21}^{(0)}) \right\} - v_1^{(0)} \frac{\partial T_{22}^{(0)}}{\partial x_1^*} - v_2^{(0)} \frac{\partial T_{22}^{(0)}}{\partial x_2^*} \quad \dots\dots\dots(17b) \end{aligned}$$

$$\begin{aligned} T_{12}^{(1)} &= \frac{1}{\sin^2 \theta} \left\{ \frac{\partial v_1^{(1)}}{\partial x_2^*} + \frac{\partial v_2^{(1)}}{\partial x_1^*} - 2 \cos \theta \frac{\partial v_2^{(0)}}{\partial x_2^*} \right. \\ &\quad \left. - 2 \cos^2 \theta \frac{\partial v_2^{(1)}}{\partial x_1^*} - \frac{1}{2} \left(\frac{\partial v_1^{(0)}}{\partial x_2^*} - \frac{\partial v_2^{(0)}}{\partial x_1^*} \right) \right. \\ &\quad \left. + 2 \cos \theta \frac{\partial v_1^{(0)}}{\partial x_1^*} \right\} (T_{11}^{(0)} - 2 \cos \theta T_{12}^{(0)} - T_{22}^{(0)}) \\ &\quad - v_1^{(0)} \frac{\partial T_{12}^{(0)}}{\partial x_1^*} - v_2^{(0)} \frac{\partial T_{12}^{(0)}}{\partial x_2^*} \quad \dots\dots\dots(17c) \end{aligned}$$

$$T_{21}^{(1)} = \frac{1}{\sin^2 \theta} \left\{ \frac{\partial v_1^{(1)}}{\partial x_2^*} + \frac{\partial v_2^{(1)}}{\partial x_1^*} - 2 \cos \theta \frac{\partial v_1^{(1)}}{\partial x_1^*} \right. \\ \left. - 2 \cos^2 \theta \frac{\partial v_1^{(1)}}{\partial x_2^*} - \frac{1}{2} \left(\frac{\partial v_1^{(0)}}{\partial x_2^*} - \frac{\partial v_2^{(0)}}{\partial x_1^*} \right) \right. \\ \left. + 2 \cos \theta \frac{\partial v_1^{(0)}}{\partial x_1^*} \right\} (T_{11}^{(0)} + 2 \cos \theta T_{21}^{(0)} - T_{22}^{(0)}) \\ - v_1^{(0)} \frac{\partial T_{21}^{(0)}}{\partial x_1^*} - v_2^{(0)} \frac{\partial T_{21}^{(0)}}{\partial x_2^*} \dots (17d)$$

Those $v_i^{(0)}$ and $T_{ij}^{(0)}$ are components of velocity and stress in the zero order of Weissenberg number, and are known.

Equations of continuity, motion and boundary conditions are obtained by the same way;

$$\frac{\partial v_1^{(1)}}{\partial x_1^*} + \frac{\partial v_2^{(1)}}{\partial x_2^*} = 0 \dots (18a)$$

$$\frac{\partial T_{11}^{(1)}}{\partial x_1^*} + \frac{\partial T_{12}^{(1)}}{\partial x_2^*} + \cos \theta \left(\frac{\partial T_{21}^{(1)}}{\partial x_1^*} + \frac{\partial T_{22}^{(1)}}{\partial x_2^*} \right) \\ = \frac{\partial p^{(1)}}{\partial x_1^*} + \cos \theta \frac{\partial p^{(1)}}{\partial x_2^*} \dots (18b)$$

$$\frac{\partial T_{21}^{(1)}}{\partial x_1^*} + \frac{\partial T_{22}^{(1)}}{\partial x_2^*} + \cos \theta \left(\frac{\partial T_{11}^{(1)}}{\partial x_1^*} + \frac{\partial T_{12}^{(1)}}{\partial x_2^*} \right) \\ = \frac{\partial p^{(1)}}{\partial x_2^*} + \cos \theta \frac{\partial p^{(1)}}{\partial x_1^*} \dots (18c)$$

Their boundary conditions are

$$v_1^{(1)} = v_2^{(1)} = 0 \quad \text{at} \quad x_1^* = -\infty \\ v_1^{(1)} = v_2^{(1)} = 0 \quad \text{at} \quad x_1^* = 0 \\ v_1^{(1)} = v_2^{(1)} = 0 \quad \text{at} \quad x_2^* = 1 \\ \frac{\partial v_1^{(1)}}{\partial x_2^*} = v_2^{(1)} = 0 \quad \text{at} \quad x_2^* = 0 \dots (19)$$

Stream function $\psi^{(1)}$ is

$$v_1^{(1)} = \frac{\partial \psi^{(1)}}{\partial x_2^*}, \quad v_2^{(1)} = -\frac{\partial \psi^{(1)}}{\partial x_1^*} \dots (20)$$

Eliminate $p^{(1)}$ from eq. (18) using eq. (20), then

$$\left[\frac{\partial^2}{\partial x_1^{*2}} + \frac{\partial^2}{\partial x_2^{*2}} + 2 \cos \theta \frac{\partial^2}{\partial x_1^* \partial x_2^*} \right]^2 \psi^{(1)} = 0 \dots (21)$$

A solution of eq. (21) under the boundary condition (19) is $\psi^{(1)} = 0$; the first order term of Weissenberg number does not influence on the velocity profile. However, as seen from eq. (17), the elastic effect influences already on the first order term of Weissenberg number for stresses $T_{11}^{(1)}$, $T_{22}^{(1)}$ and $T_{12}^{(1)}$.

4.3 Second order terms of Weissenberg number

Substituting eq. (13) into eq. (10) and considering $v_1^{(1)} = v_2^{(1)} = 0$, we rewrite the second order terms of Weissenberg number;

$$T_{11}^{(2)} = \frac{1}{\sin^2 \theta} \left\{ 2 \frac{\partial v_1^{(2)}}{\partial x_1^*} + \cos \theta \left(\frac{\partial v_1^{(2)}}{\partial x_2^*} - \frac{\partial v_2^{(2)}}{\partial x_1^*} \right) \right. \\ \left. + \frac{1}{2} \left(\frac{\partial v_1^{(0)}}{\partial x_2^*} - \frac{\partial v_2^{(0)}}{\partial x_1^*} + 2 \cos \theta \frac{\partial v_1^{(0)}}{\partial x_1^*} \right) (T_{12}^{(1)} \right. \\ \left. + T_{21}^{(1)}) \right\} - v_1^{(0)} \frac{\partial T_{11}^{(1)}}{\partial x_1^*} - v_2^{(0)} \frac{\partial T_{11}^{(1)}}{\partial x_2^*} \dots (22a)$$

$$T_{22}^{(2)} = \frac{1}{\sin^2 \theta} \left\{ 2 \frac{\partial v_2^{(2)}}{\partial x_2^*} + \cos \theta \left(\frac{\partial v_2^{(2)}}{\partial x_1^*} - \frac{\partial v_1^{(2)}}{\partial x_2^*} \right) \right. \\ \left. - \frac{1}{2} \left(\frac{\partial v_1^{(0)}}{\partial x_2^*} - \frac{\partial v_2^{(0)}}{\partial x_1^*} + 2 \cos \theta \frac{\partial v_1^{(0)}}{\partial x_1^*} \right) (T_{12}^{(1)} \right. \\ \left. + T_{21}^{(1)}) \right\} - v_1^{(0)} \frac{\partial T_{22}^{(1)}}{\partial x_1^*} - v_2^{(0)} \frac{\partial T_{22}^{(1)}}{\partial x_2^*} \dots (22b)$$

$$T_{12}^{(2)} = \frac{1}{\sin^2 \theta} \left\{ \frac{\partial v_1^{(2)}}{\partial x_2^*} + \frac{\partial v_2^{(2)}}{\partial x_1^*} - 2 \cos \theta \frac{\partial v_2^{(2)}}{\partial x_2^*} \right. \\ \left. - 2 \cos^2 \theta \frac{\partial v_2^{(2)}}{\partial x_1^*} - \frac{1}{2} \left(\frac{\partial v_1^{(0)}}{\partial x_2^*} - \frac{\partial v_2^{(0)}}{\partial x_1^*} \right) \right. \\ \left. + 2 \cos \theta \frac{\partial v_1^{(0)}}{\partial x_1^*} \right\} (T_{11}^{(1)} - 2 \cos \theta T_{12}^{(1)} - T_{22}^{(1)}) \\ - v_1^{(0)} \frac{\partial T_{12}^{(1)}}{\partial x_1^*} - v_2^{(0)} \frac{\partial T_{12}^{(1)}}{\partial x_2^*} \dots (22c)$$

$$T_{21}^{(2)} = \frac{1}{\sin^2 \theta} \left\{ \frac{\partial v_1^{(2)}}{\partial x_2^*} + \frac{\partial v_2^{(2)}}{\partial x_1^*} - 2 \cos \theta \frac{\partial v_1^{(2)}}{\partial x_1^*} \right. \\ \left. - 2 \cos^2 \theta \frac{\partial v_1^{(2)}}{\partial x_2^*} - \frac{1}{2} \left(\frac{\partial v_1^{(0)}}{\partial x_2^*} - \frac{\partial v_2^{(0)}}{\partial x_1^*} \right) \right. \\ \left. + 2 \cos \theta \frac{\partial v_1^{(0)}}{\partial x_1^*} \right\} (T_{11}^{(1)} + 2 \cos \theta T_{21}^{(1)} - T_{22}^{(1)}) \\ - v_1^{(0)} \frac{\partial T_{21}^{(1)}}{\partial x_1^*} - v_2^{(0)} \frac{\partial T_{21}^{(1)}}{\partial x_2^*} \dots (22d)$$

Here $v_i^{(0)}$ is the velocity of the zero order of Weissenberg number, and $T_{ij}^{(1)}$ the stress of its first order, and they are already known.

Equations of continuity, motion, boundary conditions and stream function can be obtained by changing the suffix of eqs. (18), (19) and (20) from (1) to (2). Then, eliminating $p^{(2)}$, we get

$$\left[\frac{\partial^2}{\partial x_1^{*2}} + \frac{\partial^2}{\partial x_2^{*2}} + 2 \cos \theta \frac{\partial^2}{\partial x_1^* \partial x_2^*} \right]^2 \psi^{(2)} \\ = \frac{\partial^2 v^{(0)}}{\partial x_1^* \partial x_2^*} \left(\frac{\partial A}{\partial x_1^*} + \frac{\partial A}{\partial x_2^*} \right) + \frac{\partial v_1^{(0)}}{\partial x_2^*} \left(\frac{\partial^2 A}{\partial x_1^{*2}} \right. \\ \left. + 2 \frac{\partial^2 T_{12}^{(1)}}{\partial x_1^* \partial x_2^*} \right) + \frac{\partial v_2^{(0)}}{\partial x_1^*} \left(\frac{\partial^2 A}{\partial x_2^{*2}} - 2 \frac{\partial^2 T_{21}^{(1)}}{\partial x_1^* \partial x_2^*} \right) \\ - \frac{C}{\sin^2 \theta} \frac{\partial^2 B}{\partial x_1^* \partial x_2^*} - \frac{B}{2 \sin^2 \theta} \left(2 \frac{\partial^2 C}{\partial x_1^* \partial x_2^*} \right. \\ \left. + \frac{\partial^2 E}{\partial x_1^{*2}} - \frac{\partial^2 D}{\partial x_2^{*2}} \right) - \frac{1}{\sin^2 \theta} \frac{\partial B}{\partial x_1^*} \left(\frac{\partial C}{\partial x_2^*} + \frac{\partial E}{\partial x_1^*} \right) \\ + \frac{1}{\sin^2 \theta} \frac{\partial B}{\partial x_2^*} \left(\frac{\partial D}{\partial x_2^*} - \frac{\partial C}{\partial x_1^*} \right) + \frac{\partial^2 v_1^{(0)}}{\partial x_2^{*2}} \frac{\partial T_{12}^{(1)}}{\partial x_1^*} \\ + \frac{\partial^2 v_2^{(0)}}{\partial x_2^{*2}} \frac{\partial T_{12}^{(1)}}{\partial x_2^*} + 2 \frac{\partial v_2^{(0)}}{\partial x_2^{*2}} \left(\frac{\partial^2 T_{12}^{(1)}}{\partial x_2^{*2}} + \frac{\partial^2 T_{21}^{(1)}}{\partial x_1^{*2}} \right) \\ + \frac{D}{2 \sin^2 \theta} \frac{\partial^2 B}{\partial x_2^{*2}} - \frac{\partial^2 v_1^{(0)}}{\partial x_1^{*2}} \frac{\partial T_{21}^{(1)}}{\partial x_1^*} - \frac{\partial^2 v_2^{(0)}}{\partial x_1^{*2}} \frac{\partial T_{21}^{(1)}}{\partial x_2^*} \\ - \frac{E}{2 \sin^2 \theta} \frac{\partial^2 B}{\partial x_1^{*2}} \dots (23)$$

where

$$A = T_{11}^{(1)} - T_{22}^{(1)} \\ B = \frac{\partial v_1^{(0)}}{\partial x_2^*} - \frac{\partial v_2^{(0)}}{\partial x_1^*} + 2 \cos \theta \frac{\partial v_1^{(0)}}{\partial x_1^*} \\ C = T_{12}^{(1)} + T_{21}^{(1)} \\ D = T_{11}^{(1)} - T_{22}^{(1)} - 2 \cos \theta T_{12}^{(1)} \\ E = T_{11}^{(1)} - T_{22}^{(1)} + 2 \cos \theta T_{21}^{(1)}$$

As the right term of eq. (23) is known, we can obtain the second order velocity profile of Weissenberg number.

5. Velocity Profile

In Fig. 2 are shown stream functions when the choke ratio of slit width to channel width is 1/10, Weissenberg number is 0.06 and inlet angles into the slit are 90° and 70° . Fig. 3 shows the flows when the choke ratio is 1/10, Weissenberg number is 0.04 and inlet angles into the slit are 90° , 80° , 70° and 65° . From these figures we deduce that, although the circulating flow decreases with the inlet angle, the shape of main flow does not change. These results agree with the authors' experiment^[1].

Fig. 2 Stream function (Weissenberg number 0.06, Choke ratio 1/10, Inlet angle into the slit (a) 90° , (b) 70°)

When the inlet angle into the slit is 90° , Fig. 3 shows that the circulating flow occupies almost a triangular territory from near the slit to the right corner of the flow channel. When the inlet angle decreases, the circulating flow moves towards the slit. In the case of 65° the circulating flow does not exist at the right corner, but only near the slit. The main flow looking like a wine-glass, however, changes little. These results suggest that the circulating flow and the main flow looking like a wine-glass may be caused rather by the flow effect of turning round the corner than by the effect of sudden flow into the slit.

In Fig. 4 are shown stream functions when the choke ratio is 1/5 and Weissenberg number is 0.06. As the choke ratio is twice of that of Fig. 2, the effect of flow into the slit is small. Therefore, the circulating flow is near the right corner and not near the slit.

In Fig. 5 are shown flow fields using the velocity vector under the same condition as in Fig. 2; the choke ratio is 1/10 and Weissenberg number 0.06. The starting point of an

Fig. 3 Stream function (Weissenberg number 0.04, Choke ratio 1/10, Inlet angle (a) 90° , (b) 80° , (c) 70° , (d) 65°)

arrow shows its position. These flows have circulating flows centered near the inlet and whirl strongly. On the contrary, as shown in Fig. 6, when the choke ratio is 1/5 and Weissenberg number is 0.1 or 0.15, the circulating velocity is much smaller than that of the main flow. This is caused by the difference of the flow into the slit, so the appearance of the circulating flow may be caused by the sudden contraction into the slit.

6. Stress Distribution

In Figs. 7 and 8 are shown numerical results of dimensionless shear stress and dimensionless normal stress difference, respectively, when the choke ratio is 1/10, Weissenberg number 0.06 and inlet angles 90° and 70° . The chain-dotted line shows the boundary of the main and the circulating flow.

Fig. 4 Stream function (Weissenberg number 0.06, Choke ratio 1/5, Inlet angle (a) 90°, (b) 80°, (c) 70°)

Fig. 5 Velocity profile (Weissenberg number 0.06, Choke ratio 1/10, Inlet angle (a) 90°, (b) 70°)

Fig. 6 Velocity profile (Weissenberg number (a) 0.15, (b) 0.1, Choke ratio 1/5, Inlet angle 90°)

Fig. 7 Stress distribution (Weissenberg number 0.06, Choke ratio 1/10, Inlet angle 90°, (a) Dimensionless shear stress, (b) Dimensionless normal stress difference)

These figures are shown using Descarte's coordinates $\bar{T}_{12}^*$ and $\bar{T}_{11}^* - \bar{T}_{22}^*$. These figures show that, as the inlet angle becomes small, the stress distribution has a tendency to be pushed up a little, but that its shape is much the same.

7. Conclusions

The flow field and the stress distribution were calculated for a viscoelastic fluid using the perturbation method on oblique coordinates. It was found that the wine-glass shape in

Fig. 8 Stress distribution (Weissenberg number 0.06, Choke ratio 1/10, Inlet angle 70° , (a) Dimensionless shear stress, (b) Dimensionless normal stress difference)

the viscoelastic flow was caused by the sudden contraction into a slit. Moreover, the change of the inlet angle into the slit influences little on the shape of the main flow. This result agrees well with the experimental result by the flow visualization^[1].

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