Principle of Measuring Fabric Stiffness by Clark Method

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Abstract

This paper outlines the way to measure the bending length of comparatively stiff fabrics by the Clark method, and presents the theoretical basis of this method. We derived the relation theoretically between the bending length and the hanging length of the specimen at the critical angle at which swing back occurs on the assumption that the test specimen is completely elastic. This theory is applicable to measure the bending length by the Clark Stiffness Tester, which correlates well to the 41.5° cantilever method. But the Clark method gives approximately 10% smaller values than the 41.5° cantilever method.

1. Introduction

The flexural rigidity of an elastic beam or a plate is defined as a proportional constant in the Euler-Bernoulli law which describes the proportionality between bending moment and curvature. This relation can be applied to fiber assemblies such as woven fabrics when the deformation is small in spite of their inhomogeneity, discontinuity and anisotropy. Various measuring methods and testers for the flexural rigidity of textile fabrics have been devised according to this principle.

The most reasonable condition for measurement is when the beam deflection is pure bending so that the curvature and the bending moment along the beam may be constant. In the methods devised by Isshi^[1], Popper and Backer^[2], Carson^[3], and Kawabata^[4], measurement is performed in the pure bending state. The Livesy and Owen^[5] method now widely used assumes approximately the pure bending of a very small test piece. But these methods are so complicated and troublesome in operation that more practical methods are used for routine tests. The principle of these routine methods depends upon the non-linear relationship between load and deflection of an elastic beam. This is theoretically analysed as a simple or heavy elastica by F. T. Peirce^[6] and others. The Gurley tester and the loop compression method^[7] (JIS L1079) belong to the simple elastica type with a concentrated load, and the Peirce cantilever or heart loop method belong to the heavy elastica type with distributed load or gravity.

The Clark method, which is our concerns here, is a modified type of the heavy cantilever method. It was devised by Clark^[8] in 1935 to measure paper stiffness. In Japan, this was applied to measure the stiffness of textile fabrics, and standardized as JIS (JIS L1003 for rayon fabrics, L1004 for cotton fabrics, L1005 for viscose rayon fabrics, L1031 for vinylon fabrics, L1032 for nylon filaments fabrics, and L1079 for fabrics from chemical fibers).

The quantities obtained by this method are qualitative and their physical meanings are not necessarily clear. Therefore, we can not use this method to obtain the bending length or flexural rigidity. The purpose of this paper is to discuss the Clark method from the theoretical point of view, and to make it applicable to the calculation of the bending length.

2. Principle of Clark Method

The principle of the Clark method is shown schematically in Fig. 1. The bottom end of a test strip of length l and width b is vertically fixed to a horizontal plate. When it is shorter than the critical buckling length, it can be considered as a long column under gravity. But when it is longer than that, it buckles and takes the form of a vertical heavy cantilever as (1) in Fig. 1. Now, consider a post-buckling form (I) which,

Fig. 1 Principle of the Clark method

for example, deflects to the right side with respect to the vertical line. Rotate a clamping plate by an angle α anticlockwise. When α increases and approaches a critical value α_c , the test strip becomes unstable and turns over to the opposite side as (2) in Fig. 1. When $\alpha < \alpha_c$, there are two stable states for the deflected strip as (1) and (2). But when $\alpha > \alpha_c$, there is only one stable state as (3) in Fig. 1. So the critical angle α_c could be defined as a bifurcation angle from two stable states to one stable state.

The state (II) in Fig. 1 is obtainable by rotating clockwise and is symmetrical to (I) with respect to a vertical line. Since this method is based on the assumptions that the test strip is pure elastic, that the stress-strain relationships in tension and compression are the same, that the neutral plane does not move by bending, and that critical values in states (I) and (II) in Fig. 1 are identical. The angle α_c is a function of the hanging length L of the strip and increases as L . Now, we designate critical values corresponding to states (I) and (II) by α_{cI} and α_{cII} , respectively. In a routine test, we define the stiffness^[9] by the critical length L_0 when $\alpha_{cI} + \alpha_{cII}$ becomes $90^\circ \pm 2^\circ$. So long as the above assumption is correct, two critical values α_{cI} and α_{cII} should be identical. But for woven fabrics or paper, two values are generally different.

In this paper, the strip is assumed to be perfectly elastic and flexible, and $\alpha_{cI} = \alpha_{cII}$.

3. Theoretical Analysis

If a heavy cantilever is fixed at the bottom end perpendicular to a plane inclined by an angle α from the horizon, as already mentioned, it takes two stable deflected forms of equilibrium (I) and (II) in Fig. 1, provided that the inclined angle α is smaller than the critical value α_c .

We take such coordinates the origin of which is at the free end of the cantilever in equilibrium and y coordinate is parallel to and x coordinate perpendicular to the inclined plane respectively as in Fig. 2. Let w_x and w_y are x and y components of a uniformly distributed load or linear density of the cantilever w , respectively, i.e.

$$\begin{aligned} w_x &= w \cos \alpha \\ w_y &= w \sin \alpha \end{aligned} \quad \dots\dots\dots(1)$$

Now, let s be the length from the origin o along the deflected cantilever, θ an angle between the x coordinate and the tangent at s . Then, putting

$$\theta - \alpha = \phi \quad \dots\dots\dots(2)$$

we have the equation of equilibrium of the cantilever^[10],

$$EI \frac{d^2 \theta}{ds^2} = w_y s \cos \theta - w_x s \sin \theta = -ws \sin \phi \quad \dots\dots\dots(3)$$

where EI is the flexural rigidity of the test strip.

This differential equation may be written in a nondimensional form by introducing the bending length $c(\sqrt[3]{EI/w})$, that is,

$$q = s/c, \quad q^3 = r \quad \dots\dots\dots(4)$$

Substituting eq. (4) to eq. (3), we have

$$\frac{d^2 \phi}{dq^2} = -q \sin \phi \quad \dots\dots\dots(5)$$

The boundary conditions are:

$$\left. \begin{aligned} \left(\frac{d\theta}{ds} \right)_{s=0} &= \frac{1}{c} \left(\frac{d\phi}{dq} \right)_{q=0} = 0 \\ (\phi)_{q=0} &= \phi_0 = \theta_0 - \alpha \end{aligned} \right\} \quad \dots\dots\dots(6)$$

Expanding the function $\phi(q)$ into a Maclaurin's series, we have

$$\begin{aligned} \phi(q) &\doteq \phi_0 - \frac{r}{6} \sin \phi_0 + \frac{r^2}{360} \sin 2\phi_0 \\ &\quad + \frac{r^3}{5184} \sin^3 \phi_0 - \frac{r^3}{25920} \cos \phi_0 \sin 2\phi_0 \\ &\quad \dots\dots\dots(7) \end{aligned}$$

Fig. 2 Coordinate systems

In order to calculate much higher terms, we may use Bell's polynomials with little improved accuracy. So, we take the terms only up to $\phi^{(11)}$ and ignore higher terms.

When α increases and approaches the critical angle α_c , the bending moment of the cantilever at the built-in end becomes zero, that is,

$$(d\phi/ds)_{s=L} = 0 \quad \dots\dots\dots(8)$$

Differentiating eq. (7) with respect to q , we have

$$\begin{aligned} \frac{d\phi}{dq} &\doteq -\frac{q^2}{2} \sin \phi_0 + \frac{q^5}{60} \sin 2\phi_0 \\ &\quad + \frac{q^8}{576} \sin^3 \phi_0 - \frac{q^8}{2880} \cos \phi_0 \sin 2\phi_0 \quad \dots\dots\dots(9) \end{aligned}$$

Putting $L/C = q_s$, $q_s^3 = r_s$ and remembering $d\phi/dq = 0$, we have

$$-\frac{r_s}{5} \cos \phi_0 + \frac{r_s^2}{95} \sin^2 \phi_0 - \frac{r_s^2}{240} \cos^2 \phi_0 - 3 = 0 \quad \dots\dots\dots(10)$$

Since $\theta = 0$ at $s = L$, from eqs. (1) and (7) we have

$$\theta_0 = -\frac{r_s}{6} \sin \phi_0 - \frac{r_s^2}{360} \sin 2\phi_0 - \frac{r_s^3}{5184} \sin^3 \phi_0 + \frac{r_s^3}{25920} \cos \phi_0 \sin 2\phi_0 \dots \dots \dots (11)$$

As eq. (10) is a quadratic equation with respect to $\cos \phi_0$,

$$\cos \phi_0 = \frac{48}{7r_s} \left(1 - \sqrt{\frac{7}{460.8} r_s^2 - 3.375} \right) \dots \dots \dots (12)$$

Since ϕ_0 varies in the range $0 < \phi_0 < \pi$, we put a negative sign for the parentheses in eq. (12). Substituting eq. (12) to eq. (11), we obtain θ_0 which leads to easy calculation of α_c for a given q_s by eq. (1). Eq. (12) is an approximate equation, and cannot be applied to both the range $L/c < 2.461$ by the square root restriction and the range of larger L/c by the deformation instability. The relationship between L/c and α_c obtained by this method is shown in Fig. 3, which shows that this test is only applicable to the range $2.46 < L/c < 3.5$. The value of L/c corresponding to $\alpha = 0$ is the critical buckling value for a vertical column with a uniformly distributed load. Theoretical values on the buckling of this type differ each other i.e. $(L/c)^3 = 7.83$ by S. Timoshenko^[11], 7.84 by R. Frisch-Fay^[12] and M. Kuranishi^[13], 7.85 in a handbook^[14]. The dotted line in Fig. 3 is an extrapolation to $\sqrt[3]{7.84}$. Fig. 3 shows that L/c corresponding to $\alpha_c = 45^\circ$ is about 2.48.

Fig. 3 Relationship between α_c and L/c —theoretical curve

4. Experiment and Discussion

Our experiment compatible with JIS used a tester specially prepared, which is similar to commercially available testers in principle, except a clamp with a magnet. Values obtained are largely different, depending on the bending direction, clockwise or counterclockwise, and spread widely. So, we adopt the minimum value because the critical angle is a starting point toward the unstable state. The relationship between L and α_c is shown in Fig. 4.

From Fig. 4, it is found that α_c increases with L and its rate also increases slightly with L for woven fabrics. However, in the case of some kinds of commercially available

Fig. 4 Relationship between α_c and L

- rayon plane fabric
- ▲ worsted plane fabric
- + polyester-cotton fabric, 1/2 twill
- vinyon plane fabric
- △ paper 1
- × paper 2
- cotton fabric, Bedfordcord
- ▼ hair cloth

Fig. 5 Relationship between α_c and L/c —experimental and theoretical

paper, this relationship is almost linear as shown by Onogi and Ando. Dividing L_0 corresponding to $\alpha_c = 45^\circ$ in Fig. 3 by 2.84, we can obtain the bending length c . Fig. 5 shows L/c versus α_c as well as the theoretical curves. Data in the range of $L/c = 2.5$ –2.85 fit to the theoretical curve, especially for paper ($\times$) and hair cloth ($\blacktriangledown$). But some materials are less amenable to theory due to behaviour deviated from perfect elasticity and curling tendency owing to the inhomogeneous structure. These experimental facts show that the method works better for larger values of c and L/c . However, a long specimen tends to twist even if L/c is comparatively small. So, it is desirable to use a specimen shorter than 15 cm.

Relationship between the bending lengths obtained by the Clark method and by the 41.5° cantilever tester shows a close correlation as shown in Fig. 6. However, the Clark method gives about 10% smaller values than the latter, which is considered most popular and versatile test. In order to obtain values more close to the latter, we may use $L/c = 2.56$ in place of 2.84 with no special physical meaning.

Onogi and Ando^[9] defined a measure of stiffness as $L_0^3 w$

Fig. 6 Comparison between the Clark and the 41.5 cantilever method

Fig. 7 Comparison between the Clark and the vibrating reed method

and this is adopted in JIS. As L_0 corresponds to the bending length c , $L_0^3 w$ is proportional to the flexural rigidity. Fig. 7 shows the values by the vibrating reed method versus those obtained by the Clark method^[9]. Although data are not numerous and their plots are widely distributed, the relationship is approximately linear. But, the values by the Clark method are about 25% smaller than those by the vibrating reed method.

5. Conclusion

Theoretical analysis of the relationship between rotating angle α and hanging length was done from the point of elastic stability of a heavy cantilever under the assumption that the specimen is pure elastic. The result gave a new method to calculate the bending length by dividing this length by 2.48. The experiment carried out to verify this theory roughly agreed with the theory. There was a close correlation between the values by the 41.5° cantilever method and those by this one, but the latter were about 10% smaller than the former.

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