Flow of Viscoelastic Fluid in a Curved Pipe

Part 1: Velocity Distribution of a Flow in a Toroidal Pipe of Circular Cross-Section

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Based on the Journal of the Textile Machinery Society of Japan, Transactions, Vol. 34, No. 1, T1-T11 (1981-1)

Abstract

The steady flow of viscoelastic fluid in a toroidal pipe of circular cross-section is analytically solved with the White-Metzner constitutive equation by a perturbation method. The results are as follows:

(1) The flow rate of a secondary flow in the cross-sectional plane increases with decreasing the viscosity index n , and with increasing the elasticity index s , the Reynolds number Re and the Weissenberg number We .

(2) The position of the maximum axial flow velocity shifts on the central plane more outward from the pipe center for smaller n and larger Re in the case of Power law fluid, and for larger s in the case of White-Metzner fluid.

(3) Owing to the centrifugal and elastic forces applied on the fluid, the stream line starting at a pipe center shifts outward on the central plane. The larger the values of s , Re and We are, the larger the amount of the outward shift is.

Hence, the effects of Re and We on the flow are shown to be similar to an available experimental result as well as earlier theoretical results. Also, it is found that the values n and s , which represent the degrees of decrement of the fluid viscosity and the first normal stress coefficient with the shear-rate respectively, have significant effects on the flow at high shear-rate to which the earlier analyses could not be applied.

1. Introduction

The steady flow of viscoelastic fluid such as polymer melts or solutions in a curved pipe is important because it is a fundamental phenomenon in textile, high polymer, mechanical industries and so on. For a Newtonian fluid, Dean^[1-2] firstly analyzed a laminar flow in a toroidal pipe of circular cross-section with small radius-ratio (ratio of radii of pipe curvature to pipe cross-section). The toroidal pipe is one of the most basic configurations among curved pipes. Nowadays the flow in a helically coiled pipe is theoretically solved^[3]. On the other hand, the laminar flow of non-Newtonian fluid in a toroidal pipe of circular cross-section with small radius-ratio was analyzed by Jones^[4] for Reiner-Rivlin-Prager model, by Clegg and Power^[5] for Bingham model, by Thomas and Walters^[6] for Oldroyd-Walters-Fredrickson model, and by Jones^[7] for Coleman-Noll's third order fluid which contains Reiner-Rivlin-Prager fluid and Oldroyd-Walters-Fredrickson model as its special case. In the analyses the perturbation method that a straight pipe flow is a zeroth order approximation was applied. Namely, Jones^[4], Clegg and Power^[5] expanded by applying an exact method^[1] for Newtonian fluid. Thomas and

Walters^[6], Jones^[7] applied an approximate method^[2] for relatively large Reynolds number. They showed that two symmetric spirals with respect to the central plane containing the center axis of the curved pipe appear in the Bingham fluid as well as in Newtonian fluid, but the stream line of secondary flow is significantly different from that of Newtonian fluid because a plug flow is in the vicinity of the center axis. For the viscoelastic fluids it is found that two symmetric spirals appear as well as in the viscous fluid, but that the flow rate of secondary flow is larger and the axial pitch of spirals is shorter than in the viscous fluid^[4,6,7]. These predictions agreed qualitatively with experiments by Barnes and Walters^[7]. However, the distributions of axial flow velocities of Reiner-Rivlin-Prager fluid and Oldroyd-Walters-Fredrickson model in a straight pipe of circular cross-section do not show non-Newtonian viscosity, but show Poiseuille distributions. The Coleman-Noll's third order fluid can not be applied to the industrially useful flow at high shear-rate where the viscosity and the normal stress coefficient decrease with the increment of shear-rate, because its normal stress coefficient is constant independent of the shear-rate in a simple shear flow.

The viscosity of real fluids is finite at zero shear-rate, but de Waele-Ostwald fluid (Power law fluid) has such defects as its viscosity becomes infinity at zero shear-rate for pseudo-plastic fluid. Also, it is difficult in most cases to solve governing equations, because the constitutive equation is nonlinear. However, de Waele-Ostwald fluid shows non-Newtonian viscosity in a flow through a straight pipe of circular cross-section, and from this usefulness it is often applied to this flow. So, it is thought that the White-Metzner model^[8] used by Denn^[9], the viscosity and the first normal stress coefficient of which are obedient to a power law for shear-rate, is useful to analyze a flow of viscoelastic fluid at relatively high shear-rate that is industrially necessary.

Hence, the present paper deals with the steady flow of incompressible viscoelastic fluid in a toroidal pipe of circular cross-section with small radius-ratio, by the use of the White-Metzner constitutive equation^[8].

2. Coordinate System and Fundamental Equations

2.1 Coordinate system

The coordinate system shown in Fig. 1 is used in the analysis. The center axis of a curved pipe of circular cross-section of radius a is on the xy plane of Cartesian coordinate system (x, y, z) , and the point C is the cross-sectional center of the curved pipe cut by a plane that makes an angle x^3 with xz plane. The curvature center of the curved pipe is at the origin O of Cartesian coordinate system and the radius of curvature of the pipe is denoted by R^* . The position of a point P inside the pipe is specified by the orthogonal curvilinear coordinate system (x^1, x^2, x^3) , where x^1 is a radial position from the pipe center, x^2 an angle made with a line parallel to z axis through the point C by the line CP. Then, the orthogonal curvilinear coordinate system (x^1, x^2, x^3) is related to Cartesian coordinate system (x, y, z) as follows:

Fig. 1 Coordinate system

$$\left. \begin{aligned} x &= (R^* + x^1 \sin x^2) \cos x^3 \\ y &= (R^* + x^1 \sin x^2) \sin x^3 \\ z &= x^1 \cos x^2 \end{aligned} \right\} \dots (1)$$

and the metric tensor is

$$\left. \begin{aligned} g_{11} &= 1, \quad g_{22} = (x^1)^2, \quad g_{33} = (R^* + x^1 \sin x^2)^2 \\ g_{12} = g_{21} = g_{13} = g_{31} = g_{23} = g_{32} &= 0 \end{aligned} \right\} \dots (2)$$

2.2 Fundamental equations

Cauchy's equation of motion and the equation of continuity governing the steady flow of incompressible fluid in a curved pipe under a constant pressure gradient in the flow direction are written in tensor forms

$$\rho v^j v^i_{,j} = -p^*_{,i} + p^{ij}_{,j} \dots (3)$$

$$v^j_{,j} = 0 \dots (4)$$

where ρ is the fluid density, v^j the flow velocity vector, p^* the isotropic pressure and p^{ij} the deviatoric stress tensor. The following White-Metzner model^[8] is applied to constitutive equation:

$$p^{ij} + \lambda (4 |IIe^*|)^{\frac{s-n-1}{2}} \frac{\partial p^{ij}}{\partial t} = 2\eta (4 |IIe^*|)^{\frac{n-1}{2}} e^{ij} \dots (5)$$

where λ , η , s and n are material constants and s and n are named as the elasticity index and the viscosity index, respectively. The elasticity index s is between about 1 and 2, the viscosity index n between 0.2 and 1^[9] and ordinarily $s > n$ ^[10]. Eq. (5) represents Maxwell model in case that $s = 2$ and $n = 1$, Power law fluid in case that $\lambda = 0$ and Newtonian fluid in case that $\lambda = 0$ and $n = 1$. The notation e^{ij} is the rate-of-deformation tensor, IIe^* the second invariant of rate-of-deformation tensor and $\partial p^{ij}/\partial t$ the Oldroyd time derivative for deviatoric stress tensor. They are given by

$$e^{ij} = \frac{1}{2} (g^{kj} v^i_{,k} + g^{ik} v^j_{,k}) \dots (6)$$

$$4 |IIe^*| = 2[(e^1_1)^2 + (e^2_2)^2 + (e^3_3)^2] + 4[e^1_2 e^2_1 + e^2_3 e^3_2 + e^3_1 e^1_3] \dots (7)$$

$$\frac{\partial p^{ij}}{\partial t} = v^k p^{ij}_{,k} - v^i_{,k} p^{kj} - v^j_{,k} p^{ik} \dots (8)$$

In eq. (5), $\lambda(4|IIe^*|)^{(s-n-1)/2}$ is the relaxation time. The second term of its left-hand side represents an elastic stress, and $\eta(4|IIe^*|)^{(n-1)/2}$ is the viscosity. The right-hand side represents a viscous stress.

After the above variables are converted to physical components, in order to analyze in non-dimensional variables, the lengths are non-dimensionalized by the pipe radius a , the velocities being by the velocity component W on the pipe center in x^3 direction, and the stresses being by ρW^2 as follows:

$$\left. \begin{aligned} R &= R^*/a \\ x_1 &= x^1/a, \quad x_2 = x^2, \quad x_3 = x^3 \\ v_i &= \sqrt{g_{ii}} v^i/W \\ p &= p^*/(\rho W^2), \quad p_{ij} = \sqrt{g_{ii}} \sqrt{g_{jj}} p^{ij}/(\rho W^2) \\ \Pi_* &= \Pi^*/(W/a)^2 \\ e_{ij} &= \sqrt{g_{ii}} \sqrt{g_{jj}} e^{ij}/(W/a) \\ \frac{\delta p}{\delta t_{ij}} &= \sqrt{g_{ii}} \sqrt{g_{jj}} \frac{\delta p^{ij}}{\delta t}/(\rho W^3/a) \end{aligned} \right\} \dots (9)$$

where Einstein's summation convention is not applied. In eq. (9), g_{ij} represents the covariant metric tensor, but x_i is non-dimensional value of coordinate, and $v_i, p_{ij}, e_{ij}, \delta p/\delta t_{ij}$ are not covariant tensors but non-dimensional physical components. All variables with subscript in the followings represent non-dimensional ones. The velocity W is approximately equal to the velocity at the center axis of the straight pipe as shown in later results.

Cauchy's equation of motion and the equation of continuity are written in non-dimensional forms

$$\left. \begin{aligned} v_1 \frac{\partial v_1}{\partial x_1} + \frac{v_2}{x_1} \frac{\partial v_1}{\partial x_2} - \frac{v_2^2}{x_1} - \frac{\sin x_2}{h} v_3^2 \\ = -\frac{\partial p}{\partial x_1} + \frac{\partial p_{11}}{\partial x_1} + \left(\frac{1}{x_1} + \frac{\sin x_2}{h} \right) p_{11} \\ + \frac{1}{x_1} \frac{\partial p_{12}}{\partial x_2} + \frac{\cos x_2}{h} p_{12} - \frac{p_{22}}{x_1} - \frac{\sin x_2}{h} p_{33} \\ v_1 \frac{\partial v_2}{\partial x_1} + \frac{v_2}{x_1} \frac{\partial v_2}{\partial x_2} + \frac{v_1 v_2}{x_1} - \frac{\cos x_2}{h} v_3^2 \\ = -\frac{1}{x_1} \frac{\partial p}{\partial x_2} + \frac{\partial p_{12}}{\partial x_1} + \left(\frac{2}{x_1} + \frac{\sin x_2}{h} \right) p_{12} \\ + \frac{1}{x_1} \frac{\partial p_{22}}{\partial x_2} + \frac{\cos x_2}{h} p_{22} - \frac{\cos x_2}{h} p_{33} \\ v_1 \frac{\partial v_3}{\partial x_1} + \frac{v_2}{x_1} \frac{\partial v_3}{\partial x_2} + \frac{\sin x_2}{h} v_1 v_3 + \frac{\cos x_2}{h} v_2 v_3 \\ = -\frac{1}{h} \frac{\partial p}{\partial x_3} + \frac{\partial p_{13}}{\partial x_1} + \left(\frac{1}{x_1} + \frac{2 \sin x_2}{h} \right) p_{13} \\ + \frac{1}{x_1} \frac{\partial p_{23}}{\partial x_2} + \frac{2 \cos x_2}{h} p_{23} \end{aligned} \right\} \dots (10)$$

$$\frac{\partial v_1}{\partial x_1} + \frac{v_1}{x_1} + \frac{1}{x_1} \frac{\partial v_2}{\partial x_2} + \frac{\sin x_2}{h} v_1 + \frac{\cos x_2}{h} v_2 = 0 \dots (11)$$

where

$$h = R + x_1 \sin x_2 \dots (12)$$

because the flow is independent of x_3 except for the isotropic pressure p .

The constitutive equation is written as

$$p_{ij} + W_* M^{\frac{s-n-1}{2}} \frac{\delta p}{\delta t_{ij}} = \frac{2}{R_*} M^{\frac{n-1}{2}} e_{ij} \dots (13)$$

where W_* is the Weissenberg number and R_* is the Reynolds number. They are defined as

$$\left. \begin{aligned} W_* &= \lambda(W/a)^{s-n} \\ R_* &= \rho W^{2-n} a^n / \eta \end{aligned} \right\} \dots (14)$$

The variables M, e_{ij} and $\delta p/\delta t_{ij}$ are

$$M \equiv 4|\Pi_*| = 2(e_{11}^2 + e_{22}^2 + e_{33}^2) + 4(e_{12}^2 + e_{23}^2 + e_{31}^2)$$

$$e_{11} = \frac{\partial v_1}{\partial x_1}, \quad e_{22} = \frac{1}{x_1} \frac{\partial v_2}{\partial x_2} + \frac{v_1}{x_1}$$

$$e_{33} = v_1 \frac{\sin x_2}{h} + v_2 \frac{\cos x_2}{h}, \quad e_{12} = \frac{1}{2} \left(\frac{1}{x_1} \frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} - \frac{v_2}{x_1} \right)$$

$$e_{23} = \frac{1}{2} \left(\frac{1}{x_1} \frac{\partial v_3}{\partial x_2} - v_3 \frac{\cos x_2}{h} \right), \quad e_{13} = \frac{1}{2} \left(\frac{\partial v_3}{\partial x_1} - v_3 \frac{\sin x_2}{h} \right)$$

$$\frac{\delta p}{\delta t_{11}} = v_1 \frac{\partial p_{11}}{\partial x_1} + \frac{v_2}{x_1} \frac{\partial p_{11}}{\partial x_2} - 2 \left(p_{11} \frac{\partial v_1}{\partial x_1} + \frac{p_{12}}{x_1} \frac{\partial v_1}{\partial x_2} \right)$$

$$\begin{aligned} \frac{\delta p}{\delta t_{22}} &= v_1 \frac{\partial p_{22}}{\partial x_1} + \frac{v_2}{x_1} \frac{\partial p_{22}}{\partial x_2} \\ &\quad - 2 \left(\frac{v_1}{x_1} p_{22} - \frac{v_2}{x_1} p_{12} + \frac{\partial v_2}{\partial x_1} p_{12} + \frac{p_{22}}{x_1} \frac{\partial v_2}{\partial x_2} \right) \end{aligned}$$

$$\begin{aligned} \frac{\delta p}{\delta t_{33}} &= v_1 \frac{\partial p_{33}}{\partial x_1} + \frac{v_2}{x_1} \frac{\partial p_{33}}{\partial x_2} - 2 \left\{ \frac{\sin x_2}{h} (v_1 p_{33} - v_3 p_{13}) \right. \\ &\quad \left. + \frac{\cos x_2}{h} (v_2 p_{33} - v_3 p_{23}) + p_{13} \frac{\partial v_3}{\partial x_1} + \frac{p_{23}}{x_1} \frac{\partial v_3}{\partial x_2} \right\} \end{aligned}$$

$$\begin{aligned} \frac{\delta p}{\delta t_{12}} &= v_1 \frac{\partial p_{12}}{\partial x_1} + \frac{v_2}{x_1} \frac{\partial p_{12}}{\partial x_2} - \frac{v_1}{x_1} p_{12} \\ &\quad + \frac{v_2}{x_1} p_{11} - p_{12} \frac{\partial v_1}{\partial x_1} - \frac{p_{22}}{x_1} \frac{\partial v_1}{\partial x_2} - p_{11} \frac{\partial v_2}{\partial x_1} - \frac{p_{12}}{x_1} \frac{\partial v_2}{\partial x_2} \end{aligned}$$

$$\begin{aligned} \frac{\delta p}{\delta t_{13}} &= v_1 \frac{\partial p_{13}}{\partial x_1} + \frac{v_2}{x_1} \frac{\partial p_{13}}{\partial x_2} + \frac{\sin x_2}{h} (v_3 p_{11} - v_1 p_{13}) \\ &\quad + \frac{\cos x_2}{h} (v_3 p_{12} - v_2 p_{13}) - p_{13} \frac{\partial v_1}{\partial x_1} - \frac{p_{23}}{x_1} \frac{\partial v_1}{\partial x_2} \end{aligned}$$

$$- p_{11} \frac{\partial v_3}{\partial x_1} - \frac{p_{12}}{x_1} \frac{\partial v_3}{\partial x_2}$$

$$\begin{aligned} \frac{\delta p}{\delta t_{23}} &= v_1 \frac{\partial p_{23}}{\partial x_1} + \frac{v_2}{x_1} \frac{\partial p_{23}}{\partial x_2} - \frac{v_1}{x_1} p_{23} + \frac{v_2}{x_1} p_{13} \\ &\quad - \frac{\sin x_2}{h} (v_1 p_{23} - v_3 p_{12}) - \frac{\cos x_2}{h} (v_2 p_{23} - v_3 p_{22}) \end{aligned}$$

$$- p_{13} \frac{\partial v_2}{\partial x_1} - \frac{p_{23}}{x_1} \frac{\partial v_2}{\partial x_2} - p_{12} \frac{\partial v_3}{\partial x_1} - \frac{p_{23}}{x_1} \frac{\partial v_3}{\partial x_2}$$

.....(15)

3. Equations for Calculation

3.1 Series expansion

To solve eqs. (10)–(15), all variables are expanded into a series by powers of radius-ratio $1/R$ as follows:

$$\left. \begin{aligned} v_i &= v_i^{(0)} + \frac{1}{R} v_i^{(1)} + \frac{1}{R^2} v_i^{(2)} + \dots \\ p &= p^{(0)} + \frac{1}{R} p^{(1)} + \frac{1}{R^2} p^{(2)} + \dots \\ p_{ij} &= p_{ij}^{(0)} + \frac{1}{R} p_{ij}^{(1)} + \frac{1}{R^2} p_{ij}^{(2)} + \dots \\ e_{ij} &= e_{ij}^{(0)} + \frac{1}{R} e_{ij}^{(1)} + \frac{1}{R^2} e_{ij}^{(2)} + \dots \\ \frac{\delta p}{\delta t_{ij}} &= \frac{\delta p^{(0)}}{\delta t_{ij}} + \frac{1}{R} \frac{\delta p^{(1)}}{\delta t_{ij}} + \frac{1}{R^2} \frac{\delta p^{(2)}}{\delta t_{ij}} + \dots \\ M &= M^{(0)} + \frac{1}{R} M^{(1)} + \frac{1}{R^2} M^{(2)} + \dots \\ \frac{1}{h} &= \frac{1}{R} - \frac{1}{R^2} x_1 \sin x_2 + \frac{1}{R^3} x_1^2 \sin^2 x_2 + \dots \end{aligned} \right\} \dots (16)$$

Substitution of eq. (16) into eqs. (10), (11), (13), (15) and comparison of coefficients of the equal power of $1/R$, yield equations for $v_i^{(m)}$, $p^{(m)}$ ($m = 0, 1, 2, \dots$).

3.2 Flow in a straight pipe

The zeroth approximate solution is obtained from zeroth coefficient equations. It represents the flow in a straight pipe ($R \rightarrow \infty$) and is given by

$$\left. \begin{aligned} v_1^{(0)} &= v_2^{(0)} = 0, \quad v_3^{(0)} = 1 - x_1^{\frac{n+1}{n}} \\ p^{(0)} &= -\frac{2}{R_*} \left(\frac{n+1}{n} \right)^n R x_3 \\ e_{13}^{(0)} &= -\frac{1}{2} \frac{n+1}{n} x_1^{\frac{1}{n}}, \quad e_{ij}^{(0)} = 0 \text{ (except for } e_{13}^{(0)}) \\ M^{(0)} &= \left(\frac{n+1}{n} \right)^2 x_1^{\frac{2}{n}} \\ \frac{\partial p^{(0)}}{\partial t_{33}} &= -\frac{2}{R_*} \left(\frac{n+1}{n} \right)^{n+1} x_1^{\frac{n+1}{n}} \\ \frac{\partial p^{(0)}}{\partial t_{ij}} &= 0 \text{ (except for } \frac{\partial p^{(0)}}{\partial t_{33}}) \\ p_{13}^{(0)} &= -\frac{1}{R_*} \left(\frac{n+1}{n} \right)^n x_1, \quad p_{33}^{(0)} = \frac{W_*}{R_*} \left(\frac{n+1}{n} \right)^s x_1^{\frac{s}{n}} \\ p_{ij}^{(0)} &= 0 \text{ (except for } p_{13}^{(0)}, p_{33}^{(0)}) \end{aligned} \right\} \dots\dots\dots (17)$$

where $R x_3$ represents the length of pipe axis, and it is assumed that $p^{(0)} = 0$ at $x_3 = 0$.

3.3 First approximate solution

Cauchy's equation of motion, the equation of continuity and the constitutive equation for the first approximate solution are from eq. (17)

$$\left. \begin{aligned} -v_3^{(0)} \sin x_2 &= -\frac{\partial p^{(1)}}{\partial x_1} + \frac{\partial p_{11}^{(1)}}{\partial x_1} + \frac{p_{11}^{(1)}}{x_1} \\ &+ \frac{1}{x_1} \frac{\partial p_{12}^{(1)}}{\partial x_2} - \frac{p_{22}^{(1)}}{x_1} - p_{33}^{(0)} \sin x_2 \\ -v_3^{(0)} \cos x_2 &= -\frac{1}{x_1} \frac{\partial p^{(1)}}{\partial x_2} + \frac{\partial p_{12}^{(1)}}{\partial x_1} + 2 \frac{p_{12}^{(1)}}{x_1} \\ &+ \frac{1}{x_1} \frac{\partial p_{22}^{(1)}}{\partial x_2} - p_{33}^{(0)} \cos x_2 \\ v_1^{(1)} \frac{\partial v_3^{(0)}}{\partial x_1} &= -\frac{1}{R} \frac{\partial p^{(1)}}{\partial x_3} + \frac{\partial p_{13}^{(1)}}{\partial x_1} + \frac{p_{13}^{(1)}}{x_1} \\ &+ \frac{1}{x_1} \frac{\partial p_{33}^{(1)}}{\partial x_2} + \frac{x_1 \sin x_2}{R} \frac{\partial p^{(0)}}{\partial x_3} + 2 p_{13}^{(0)} \sin x_2 \end{aligned} \right\} \dots\dots\dots (18)$$

$$\frac{\partial v_1^{(1)}}{\partial x_1} + \frac{v_1^{(1)}}{x_1} + \frac{1}{x_1} \frac{\partial v_2^{(1)}}{\partial x_2} = 0 \dots\dots\dots (19)$$

$$\left. \begin{aligned} p_{11}^{(1)} &= \frac{2}{R_*} M^{(0)\frac{n-1}{2}} e_{11}^{(1)}, \quad p_{22}^{(1)} = \frac{2}{R_*} M^{(0)\frac{n-1}{2}} e_{22}^{(1)} \\ p_{12}^{(1)} &= \frac{2}{R_*} M^{(0)\frac{n-1}{2}} e_{12}^{(1)} \\ p_{13}^{(1)} &= \frac{2}{R_*} M^{(0)\frac{n-1}{2}} \left(\frac{n-1}{2} \frac{M^{(1)}}{M^{(0)}} e_{13}^{(0)} + e_{13}^{(1)} \right) \\ &- W_* M^{(0)\frac{s-n-1}{2}} \frac{\partial p^{(1)}}{\partial t_{13}} \end{aligned} \right\}$$

$$\left. \begin{aligned} p_{23}^{(1)} &= \frac{2}{R_*} M^{(0)\frac{n-1}{2}} e_{23}^{(1)} - W_* M^{(0)\frac{s-n-1}{2}} \frac{\partial p^{(1)}}{\partial t_{23}} \\ p_{33}^{(1)} &= -W_* M^{(0)\frac{s-n-1}{2}} \left(\frac{s-n-1}{2} \frac{M^{(1)}}{M^{(0)}} \frac{\partial p^{(0)}}{\partial t_{33}} + \frac{\partial p^{(1)}}{\partial t_{33}} \right) \end{aligned} \right\} \dots\dots\dots (20)$$

where

$$\left. \begin{aligned} e_{11}^{(1)} &= \frac{\partial v_1^{(1)}}{\partial x_1}, \quad e_{22}^{(1)} = \frac{1}{x_1} \frac{\partial v_2^{(1)}}{\partial x_2} + \frac{v_2^{(1)}}{x_1} \\ e_{33}^{(1)} &= 0, \quad e_{12}^{(1)} = \frac{1}{2} \left(\frac{1}{x_1} \frac{\partial v_1^{(1)}}{\partial x_2} + \frac{\partial v_2^{(1)}}{\partial x_1} - \frac{v_2^{(1)}}{x_1} \right) \\ e_{13}^{(1)} &= \frac{1}{2} \left(\frac{\partial v_3^{(1)}}{\partial x_1} - v_3^{(0)} \sin x_2 \right) \\ e_{23}^{(1)} &= \frac{1}{2} \left(\frac{1}{x_1} \frac{\partial v_3^{(1)}}{\partial x_2} - v_3^{(0)} \cos x_2 \right) \\ M^{(1)} &= 8 e_{13}^{(0)} e_{13}^{(1)} \\ \frac{\partial p^{(1)}}{\partial t_{11}} &= \frac{\partial p^{(1)}}{\partial t_{22}} = \frac{\partial p^{(1)}}{\partial t_{12}} = 0 \\ \frac{\partial p^{(1)}}{\partial t_{13}} &= v_{(1)} \frac{\partial p_{13}^{(0)}}{\partial x_1} - p_{13}^{(0)} \frac{\partial v_1^{(1)}}{\partial x_1} - p_{11}^{(1)} \frac{\partial v_3^{(0)}}{\partial x_1} \\ \frac{\partial p^{(1)}}{\partial t_{23}} &= p_{13}^{(0)} \frac{v_2^{(1)}}{x_1} - p_{13}^{(0)} \frac{\partial v_2^{(1)}}{\partial x_1} - p_{12}^{(1)} \frac{\partial v_3^{(0)}}{\partial x_1} \\ \frac{\partial p^{(1)}}{\partial t_{33}} &= v_1^{(1)} \frac{\partial p_{33}^{(0)}}{\partial x_1} \\ &- 2 \left(p_{13}^{(0)} \frac{\partial v_3^{(1)}}{\partial x_1} + p_{13}^{(1)} \frac{\partial v_3^{(0)}}{\partial x_1} - v_3^{(0)} p_{13}^{(0)} \sin x_2 \right) \end{aligned} \right\} \dots\dots\dots (21)$$

The boundary condition is given by

$$\left. \begin{aligned} v_i^{(1)} &= 0 \quad \text{on the pipe wall } x_1 = 1. \\ v_i^{(1)} &= (\text{finite}) \quad \text{inside the pipe } 0 \leq x_1 < 1 \\ p^{(1)} &= 0 \quad \text{at the pipe center } x_1 = 0 \end{aligned} \right\} \dots\dots\dots (22)$$

The solution of eqs. (18)–(22) can be represented by

$$\left. \begin{aligned} v_1^{(1)} &= V_1(x_1) \sin x_2 \\ v_2^{(1)} &= V_2(x_1) \cos x_2 \\ v_3^{(1)} &= V_3(x_1) \sin x_2 \\ p^{(1)} &= P(x_1) \sin x_2 \end{aligned} \right\} \dots\dots (23)$$

Substituting eq. (23) into eqs. (18)–(22), eliminating P from the first and the second equations of eq. (18) and eliminating V_2 by the use of eq. (19) yield the following differential equation for $V_1(x_1)$:

$$\begin{aligned} x_1^4 V_1'''' + \frac{8n-2}{n} x_1^3 V_1''' + \frac{10n^2-8n+1}{n^2} x_1^2 V_1'' \\ + \frac{-4n^2+1}{n^2} x_1 V_1' = 2R_* \left(\frac{n+1}{n} \right)^{2-n} \left(x_1^{\frac{n+1}{n}} - x_1^{\frac{n-1}{n}} \right) \\ + W_* \left(\frac{n+1}{n} \right)^{s-n+1} \frac{s}{n} x_1^{\frac{s+n+1}{n}} \dots\dots\dots (24) \end{aligned}$$

where the prime denotes differentiation with respect to x_1 .
The general solution of eq. (24) in case that $n \neq 1$ is

$$V_1 = c_1 + c_2 x_1^{\frac{1-n+\sqrt{17n^2-2n+1}}{2n}} + c_4 x_1^{\frac{1-n-\sqrt{17n^2-2n+1}}{2n}} + c_5 x_1^{\frac{1-n}{n}} + R_e \left(\frac{n+1}{n} \right)^{1-n} \frac{n^3}{2(3n+1)(n^2+4n+1)} x_1^{\frac{n+1}{n}} - R_e \left(\frac{n+1}{n} \right)^{1-n} \frac{n^3}{6(2n+1)(4n^2+9n+3)} x_1^{\frac{n+1}{n}} + W_e \left(\frac{n+1}{n} \right)^{s-n+1} \times \frac{s n^3}{(s+n+1)(s+2n)(s^2+3sn+s-2n^2+2n)} x_1^{\frac{s+n+1}{n}} \dots (25)$$

where c_1, c_2, c_4 and c_5 are the integration constants. In case that $n = 1$, $c_5 x_1^{(1-n)/n}$ in eq. (25) is replaced by $c_5 \ln x_1$. After V_1 is obtained, V_2 becomes from the equation of continuity (19)

$$V_2 = x_1 V_1' + V_1 \dots (26)$$

The constants c_4 and c_5 are

$$c_4 = c_5 = 0 \dots (27)$$

because the physical values must be finite at the pipe center $x_1 = 0$. The constants c_1 and c_2 can be obtained for the flow velocities to vanish on the pipe wall.

On substituting the above results into the third equation of eq. (18), the equation for $V_3(x_1)$ is given by

$$x_1^2 V_3'' + \frac{2n-1}{n} x_1 V_3' - \frac{1}{n} V_3 = 2 \frac{n-1}{n} x_1 + \frac{-3n^2+5n+4}{n^2} x_1^{\frac{2n+1}{n}} - R_e \left(\frac{n+1}{n} \right)^{2-n} \frac{c_1}{n} x_1^{\frac{n+2}{n}} - R_e \left(\frac{n+1}{n} \right)^{2-n} \frac{c_2}{n} x_1^{\frac{n+5+\sqrt{17n^2-2n+1}}{2n}} - R_e \left(\frac{n+1}{n} \right)^{3-2n} \frac{n^2}{2(3n+1)(n^2+4n+1)} x_1^{\frac{3n+4}{n}} + R_e \left(\frac{n+1}{n} \right)^{3-2n} \frac{n^2}{6(2n+1)(4n^2+9n+3)} x_1^{\frac{4n+5}{n}} - W_e \left(\frac{n+1}{n} \right)^{s-n} \frac{s-n-1}{n^2} c_1 x_1^{\frac{s-n}{n}} + W_e \left(\frac{n+1}{n} \right)^{s-n} \left\{ 12n^3+2n-2-5sn+3s+(3s-2n-2)\sqrt{17n^2-2n+1} \right\} \frac{c_2}{2n^3} x_1^{\frac{2s-3n+1+\sqrt{17n^2-2n+1}}{2n}} + W_e R_e \left(\frac{n+1}{n} \right)^{s-2n+1} \frac{3n^2+n-2+5sn+6s}{2(3n+1)(n^2+4n+1)} x_1^{\frac{s+n+2}{n}} - W_e R_e \left(\frac{n+1}{n} \right)^{s-2n+1} \times \left\{ \frac{s(n+1)^2}{(s+n+1)(s+2n)(s^2+3sn+s-2n^2+2n)} \right\}$$

$$+ \frac{7n^2+7n+8sn+9s}{6(2n+1)(4n^2+9n+3)} \left\{ x_1^{\frac{s+2n+3}{n}} + W_e \left(\frac{n+1}{n} \right)^{2s-2n+1} \times \frac{s(4s^2+3sn+2s+n^2-n-2)}{(s+n+1)(s+2n)(s^2+3sn+s-2n^2+2n)} x_1^{\frac{2s+1}{n}} \right\} \dots (28)$$

The general solution of eq. (28) is

$$V_3 = c_3 x_1^{\frac{1}{n}} + x_1 + \frac{-3n^2+5n+4}{2n(3n+1)} x_1^{\frac{2n+1}{n}} - R_e \left(\frac{n+1}{n} \right)^{-n} \frac{c_1}{2n} x_1^{\frac{n+2}{n}} - R_e \left(\frac{n+1}{n} \right)^{2-n} \frac{nc_2}{5n^2+3n+4+(n+2)\sqrt{17n^2-2n+1}} \times x_1^{\frac{n+5+\sqrt{17n^2-2n+1}}{2n}} - R_e \left(\frac{n+1}{n} \right)^{1-2n} \frac{n^2}{24(3n+1)(n^2+4n+1)} x_1^{\frac{3n+4}{n}} + R_e \left(\frac{n+1}{n} \right)^{1-2n} \frac{n^2}{120(2n+1)(4n^2+9n+3)} x_1^{\frac{4n+5}{n}} - W_e \left(\frac{n+1}{n} \right)^{s-n} \frac{c_1}{s} x_1^{\frac{s-n}{n}} - W_e \left(\frac{n+1}{n} \right)^{s-n} \times \{ -12n^2-2n+2+5sn-3s+(-3s+2n+2) \times \sqrt{17n^2-2n+1} \} / \{ 10n^2-2n+2s^2-4sn+2(s-n) \times \sqrt{17n^2-2n+1} \} \frac{c_2}{n} x_1^{\frac{2s-3n+1+\sqrt{17n^2-2n+1}}{2n}} + W_e R_e \left(\frac{n+1}{n} \right)^{s-2n+1} \frac{n^2(3n^2+n-2+5sn+6s)}{2(3n+1)(s+n+1)} \times \frac{1}{(s+2n+2)(n^2+4n+1)} x_1^{\frac{s+n+2}{n}} - W_e R_e \left(\frac{n+1}{n} \right)^{s-2n+1} \left\{ \frac{s n^2 (n+1)^2}{(s+n+1)(s+2n+2)} \right\} \times \frac{1}{(s+3n+3)(s+2n)(s^2+3sn+s-2n^2+2n)} + \frac{n^2(7n^2+7n+8sn+9s)}{6(2n+1)(4n^2+9n+3)} \times \frac{1}{s^2+5sn+5s+6n^2+12n+6} \left\{ x_1^{\frac{s+2n+3}{n}} + W_e \left(\frac{n+1}{n} \right)^{2s-2n+1} \frac{n^2(4s^2+3sn+2s+n^2-n-2)}{2(s+n+1)(s+2n)(2s+n+1)} \right\} \times \frac{1}{s^2+3sn+s-2n^2+2n} x_1^{\frac{2s+1}{n}} + c_6 x_1^{-1} \dots (29)$$

where c_3 and c_6 are the integration constants. Because v_3 is finite at $x_1 = 0$, c_6 is

$$c_6 = 0 \dots (30)$$

The constant c_3 is determined for V_3 to vanish on the pipe wall.

3.4 Flow velocity, isotropic pressure, deviatric stress and stream line

From the results obtained hereunto, the flow velocity and the isotropic pressure are as follows:

$$v_1 = \frac{1}{R} V_1(x_1) \sin x_2$$

$$= \frac{1}{R} \left[c_1 + c_2 x_1^{\frac{1-n+\sqrt{17n^2-2n+1}}{2n}} + R_* \left(\frac{n+1}{n} \right)^{1-n} \frac{n^3}{2(3n+1)(n^2+4n+1)} x_1^{\frac{n+1}{2n}} - R_* \left(\frac{n+1}{n} \right)^{1-n} \frac{n^3}{6(2n+1)(4n^2+9n+3)} x_1^{\frac{n+1}{3n}} + W_* \left(\frac{n+1}{n} \right)^{s-n+1} \frac{sn^3}{(s+n+1)(s+2n)} \times \frac{1}{s^2+3sn+s-2n^2+2n} x_1^{\frac{s+n+1}{n}} \right] \sin x_2 \quad \dots\dots (31)$$

$$v_2 = \frac{1}{R} V_2(x_1) \cos x_2$$

$$= \frac{1}{R} \left[c_1 + \frac{n+1+\sqrt{17n^2-2n+1}}{2n} c_2 x_1^{\frac{1-n+\sqrt{17n^2-2n+1}}{2n}} + R_* \left(\frac{n+1}{n} \right)^{1-n} \frac{n^3(3n+2)}{2(3n+1)(n^2+4n+1)} x_1^{\frac{n+1}{2n}} - R_* \left(\frac{n+1}{n} \right)^{1-n} \frac{n^2(4n+3)}{6(2n+1)(4n^2+9n+3)} x_1^{\frac{n+1}{3n}} + W_* \left(\frac{n+1}{n} \right)^{s-n+1} \frac{sn^2(s+2n+1)}{(s+n+1)(s+2n)} \times \frac{1}{s^2+3sn+s-2n^2+2n} x_1^{\frac{s+n+1}{n}} \right] \cos x_2 \quad \dots\dots (32)$$

$$v_3 = 1 - x_1^{\frac{n+1}{n}} + \frac{1}{R} V_3(x_1) \sin x_2$$

$$= 1 - x_1^{\frac{n+1}{n}} + \frac{1}{R} \left[c_3 x_1^{\frac{1}{n}} + x_1 + \frac{-3n^2+5n+4}{2n(3n+1)} x_1^{\frac{2n+1}{n}} - R_* \left(\frac{n+1}{n} \right)^{-n} \frac{c_1}{2n} x_1^{\frac{n+2}{n}} - R_* \left(\frac{n+1}{n} \right)^{2-n} \frac{nc_2}{5n^2+3n+4+(n+2)\sqrt{17n^2-2n+1}} \times x_1^{\frac{n+5+\sqrt{17n^2-2n+1}}{2n}} - R_* \left(\frac{n+1}{n} \right)^{1-2n} \frac{n^2}{24(3n+1)(n^2+4n+1)} x_1^{\frac{3n+4}{n}} + R_* \left(\frac{n+1}{n} \right)^{1-2n} \frac{n^2}{120(2n+1)(4n^2+9n+3)} x_1^{\frac{4n+5}{n}} - W_* \left(\frac{n+1}{n} \right)^{s-n} \frac{c_1}{s} x_1^{\frac{s-n}{n}} - W_* \left(\frac{n+1}{n} \right)^{s-n} \times \{-12n^2-2n+2+5sn-3s+(-3s+2n+2) \times \sqrt{17n^2-2n+1}\} / \{10n^2-2n+2s^2-4sn+2(s-n) \times \sqrt{17n^2-2n+1}\} \frac{c_2}{n} x_1^{\frac{2s-3n+1+\sqrt{17n^2-2n+1}}{2n}} + W_* R_* \left(\frac{n+1}{n} \right)^{s-2n+1} \frac{n^2(3n^2+n-2+5sn+6s)}{2(3n+1)(s+n+1)} \right] \sin x_2$$

$$\times \frac{1}{(s+2n+2)(n^2+4n+1)} x_1^{\frac{s+n+2}{n}} - W_* R_* \left(\frac{n+1}{n} \right)^{s-2n+1} \left\{ \frac{sn^2(n+1)^2}{(s+n+1)(s+2n+2)} \times \frac{1}{(s+3n+3)(s+2n)(s^2+3sn+s-2n^2+2n)} + \frac{n^2(7n^2+7n+8sn+9s)}{6(2n+1)(4n^2+9n+3)} \times \frac{1}{s^2+5sn+5s+6n^2+12n+6} \right\} x_1^{\frac{s+2n+3}{n}} + W_* \left(\frac{n+1}{n} \right)^{2s-2n+1} \frac{n^2(4s^2+3sn+2s+n^2-n-2)}{2(s+n+1)(s+2n)(2s+n+1)} \times \frac{1}{s^2+3sn+s-2n^2+2n} x_1^{\frac{2s+1}{n}} \right] \sin x_2 \quad \dots\dots\dots (33)$$

$$p = -\frac{2}{R_*} \left(\frac{n+1}{n} \right)^n R x_3 + \frac{1}{R} P(x_1) \sin x_2$$

$$= -\frac{2}{R_*} \left(\frac{n+1}{n} \right)^n R x_3 + \frac{1}{R} \left[\frac{1}{R_*} \left(\frac{n+1}{n} \right)^{n-1} \frac{7n^2+1+(n+1)\sqrt{17n^2-2n+1}}{2n^2} \times c_2 x_1^{\frac{-n-1+\sqrt{17n^2-2n+1}}{2n}} + x_1 - \frac{2n(5n+1)}{(3n+1)(n^2+4n+1)} \times x_1^{\frac{2n+1}{n}} + \frac{n(3n+1)(n+3)}{2(2n+1)(4n^2+9n+3)} x_1^{\frac{3n+2}{n}} - \frac{W_*}{R_*} \left(\frac{n+1}{n} \right)^s \frac{n(s+4n)(s-n+1)}{(s+2n)(s^2+3sn+s-2n^2+2n)} \times x_1^{\frac{s+n}{n}} \right] \sin x_2 \quad \dots\dots\dots (34)$$

where the constants c_1 , c_2 and c_3 are

$$c_1 = R_* \left(\frac{n+1}{n} \right)^{1-n} n \{ 3(13n+5)(n+1)^3 \sqrt{17n^2-2n+1} - 129n^5 - 331n^4 - 274n^3 - 142n^2 - 69n - 15 \} / \{ 48(2n+1)(3n+1)(n^2+4n+1)(4n^2+9n+3) \} + W_* \left(\frac{n+1}{n} \right)^{s-n+1} sn \{ (s+n+1) \sqrt{17n^2-2n+1} + sn - s - 7n^2 - 1 \} / \{ 8(s+n+1)(s+2n)(s^2+3sn + s - 2n^2 + 2n) \}$$

$$c_2 = (1 - n - \sqrt{17n^2-2n+1}) \left[R_* \left(\frac{n+1}{n} \right)^{4-n} \times \frac{n^4(13n+5)}{16(2n+1)(3n+1)(n^2+4n+1)(4n^2+9n+3)} + W_* \left(\frac{n+1}{n} \right)^{s-n+1} \times \frac{sn}{8(s+2n)(s^2+3sn+s-2n^2+n)} \right]$$

$$c_3 = -\frac{3n^2+7n+4}{2n(3n+1)} + R_* \left(\frac{n+1}{n} \right)^{-n} \frac{1}{n} \left\{ \frac{c_1}{2} + \frac{(n+1)^2 c_2}{5n^2+3n+4+(n+2)\sqrt{17n^2-2n+1}} \right\} + R_* \left(\frac{n+1}{n} \right)^{1-2n} \left\{ \frac{n^2}{24(3n+1)(n^2+4n+1)} \right\}$$

$$\begin{aligned}
& -\frac{n^2}{120(2n+1)(4n^2+9n+3)} \Big\} \\
& + W_e \left(\frac{n+1}{n} \right)^{s-n} \left[\frac{c_1}{s} + \{ -12n^2 - 2n + 2 + 5sn - 3s \} \right. \\
& + \{ (-3s + 2n + 2) \sqrt{17n^2 - 2n + 1} \} / \{ 10n^2 - 2n + 2s^2 \\
& - 4sn + 2(s-n) \sqrt{17n^2 - 2n + 1} \} \frac{c_2}{n} \Big] \\
& - W_e R_e \left(\frac{n+1}{n} \right)^{s-2n+1} \\
& \times \left\{ \frac{n^2(3n^2+n-2+5sn+6s)}{2(3n+1)(s+n+1)(s+2n+2)(n^2+4n+1)} \right. \\
& - \frac{sn^2(n+1)^2}{(s+n+1)(s+2n+2)(s+3n+3)(s+2n)} \\
& \times \frac{1}{s^2+3sn+s-2n^2+2n} - \frac{n^2}{6(2n+1)} \\
& \times \frac{7n^2+7n+8sn+9s}{(4n^2+9n+3)(s^2+5sn+5s+6n^2+12n+6)} \Big\} \\
& - W_e \left(\frac{n+1}{n} \right)^{2s-2n-1} \frac{n^2(4s^2+3sn+2s+n^2-n-2)}{2(s+n+1)(s+2n)} \\
& \times \frac{1}{(2s+n+1)(s^2+3sn+s-2n^2+2n)} \\
& \dots\dots\dots(35)
\end{aligned}$$

The deviatric stress is as follows:

$$\begin{aligned}
p_{11} &= \frac{1}{R} \frac{2}{R_e} \left(\frac{n+1}{n} \right)^{n-1} x_1^{\frac{n-1}{n}} V_1' \sin x_2 \\
p_{22} &= -\frac{1}{R} \frac{2}{R_e} \left(\frac{n+1}{n} \right)^{n-1} x_1^{\frac{n-1}{n}} V_1' \sin x_2 \\
p_{12} &= \frac{1}{R} \frac{1}{R_e} \left(\frac{n+1}{n} \right)^{n-1} x_1^{\frac{n-1}{n}} (V_2' - V_1') \cos x_2 \\
p_{13} &= -\frac{1}{R_e} \left(\frac{n+1}{n} \right)^n x_1 \\
& + \frac{1}{R} \left[\frac{n}{R_e} \left(\frac{n+1}{n} \right)^{n-1} x_1^{\frac{n-1}{n}} \left(V_3' - 1 + x_1^{\frac{n+1}{n}} \right) \right. \\
& + \left. \frac{W_e}{R_e} \left(\frac{n+1}{n} \right)^{s-1} x_1^{\frac{s-n-1}{n}} (V_1 - 3x_1 V_1') \right] \sin x_2 \\
p_{23} &= \frac{1}{R} \left[\frac{1}{R_e} \left(\frac{n+1}{n} \right)^{n-1} x_1^{\frac{n-1}{n}} \left(\frac{V_3}{x_1} - 1 + x_1^{\frac{n+1}{n}} \right) \right. \\
& + \left. \frac{W_e}{R_e} \left(\frac{n+1}{n} \right)^{s-1} x_1^{\frac{s-1}{n}} \left(-\frac{V_1}{x_1} + 2\frac{V_2}{x_1} - 2V_2' \right) \right] \cos x_2 \\
p_{33} &= \frac{W_e}{R_e} \left(\frac{n+1}{n} \right)^s x_1^{\frac{s}{n}} \\
& + \frac{1}{R} \left[-2\frac{W_e}{R_e} \left(\frac{n+1}{n} \right)^{s-1} s x_1^{\frac{s-1}{n}} \left(V_3' - 1 + x_1^{\frac{n+1}{n}} \right) \right. \\
& - \frac{W_e^2}{R_e} \left(\frac{n+1}{n} \right)^{2s-n-1} x_1^{\frac{2s-2n-1}{n}} \left(\frac{s+2n}{n} V_1 \right. \\
& \left. \left. - 6x_1 V_1' \right) \right] \sin x_2 \\
& \dots\dots\dots(36)
\end{aligned}$$

To determine the stream line of the secondary flow on the cross-sectional plane $x_3 = (\text{const.})$ of the pipe, the stream function ϕ is defined as

$$\left. \begin{aligned} v_1 &= -\frac{1}{x_1} \frac{\partial \phi}{\partial x_2}, & v_2 &= \frac{\partial \phi}{\partial x_1} \\ \phi &= 0 & \text{at } x_1 &= 1 \end{aligned} \right\} \dots(37)$$

Then, the stream function is given by

$$\phi = \frac{1}{R} x_1 V_1(x_1) \cos x_2 \dots\dots\dots(38)$$

The stream line on the central plane $x_2 = \pm\pi/2$ is obtained by numerically integrating

$$x_2 = \frac{1}{\sin x_2} \int_0^{x_1} \frac{1-x_1^{\frac{n+1}{n}}}{V_1} dx_1 \dots\dots\dots(39)$$

because the equation for the stream line on $x_2 = \pm\pi/2$ is

$$\frac{dx_1}{v_1} = \frac{h dx_2}{v_2} \dots\dots\dots(40)$$

4. Calculated Examples of Stream Lines and Axial Flow Velocity

Eq. (38) for the stream function ϕ showing the stream line of the secondary flow on the cross-sectional plane $x_3 = (\text{const.})$ of the pipe can be rewritten as

$$\phi = \frac{1}{R} \left[\phi_a(n, x_1) R_e + \phi_b(n, s, x_1) W_e \right] \cos x_2 \dots(41)$$

The distributions of the coefficients ϕ_a and ϕ_b in eq. (41) for some viscosity index n and elasticity index s are shown in Fig. 2. Values of the stream function ϕ are illustrated in Fig. 3. Points P_i and P_o are intersections of line OC and the pipe wall, and points P_u and P_l are intersections of a line parallel to z axis through the central point C and the pipe wall as shown Fig. 1. The symbol $\times$ represents a central point of the spiral and the arrow shows a flow direction. The stream lines are drawn in quarter parts of a circle, because they are symmetric with respect to the lines $P_i P_o$ and $P_u P_l$ in case that the stream function is given by eq. (41) represented in the series to the first term. In the case of Power law fluid the maximum value of ϕ_a and the flow rate of the secondary flow are larger for smaller n as shown in Figs. 2 and 3 (a), (b). The central points of the spiral move nearer to the pipe wall remarkably for smaller n . In the case of White-Metzner fluid the value of ϕ_b is larger than that of ϕ_a as shown in Fig. 2, and the elastic force increases the flow rate of the secondary flow similarly to the earlier results^[4,6,7]. From Fig. 3(c), (d), the smaller n is and the larger s is, the larger ϕ_b is and the larger the degree of increase in the flow rate of the secondary flow. For fixed n , the radial position x_1 giving the maximum value of ϕ_b is somewhat larger than that of ϕ_a , and the elastic force shifts the central position of the spiral to the pipe wall (Figs. 2 and 3 (b), (c), (d)). But the viscosity index n is the maximum factor

Fig. 2 Distributions of coefficients $\phi_a(n, x_1)$ and $\phi_b(n, s, x_1)$

effective to shift the central point toward the pipe wall. From eq. (41) the flow rate of the secondary flow increases for larger Re and We because both ϕ_a and ϕ_b are plus.

Equation (33) for the axial flow velocity v_3 can be rewritten as

$$v_3 = v_3^{(0)}(n, x_1) + \frac{1}{R} \left[v_a(n, x_1) + v_b(n, x_1) R e^2 + v_c(n, s, x_1) R e W e + v_d(n, s, x_1) W e^2 \right] \sin x_2 \quad (42)$$

Figure 4 shows the zeroth approximate solution $v_3^{(0)} = 1 - x_1^{(n+1)/n}$ and the coefficients v_a, v_b, v_c, v_d of the first approximate solution. Figure 5 shows the distributions of the axial flow velocity $v_3^{(0)}$ in a straight pipe and the axial flow velocity v_3 on the central plane $P_i P_o$ for $R = 3$. The distribution for Power law fluid is in Fig. 5(a) and that for White-Metzner fluid being in Fig. 5(b). The symbol $\bigcirc$ shows the position where the axial flow velocity is the maximum. The axial flow velocity of Power law fluid is represented by $v_3^{(0)}$, v_a and v_b alone. As well-known, $v_3^{(0)}$ shows a gently-sloping distribution in the vicinity of the pipe center for small n (Fig. 4(a)). The coefficient v_a for $n = 1$ shows a minus value. Hence, in the case of small Re the maximum axial flow velocity is near z axis (near point P_i) (Fig. 5(a)). With the decrease of n it moves nearer to point P_o . As v_b is always plus and the centrifugal force increases with the increment of Re , the position of the maximum axial flow velocity shifts toward point P_o . The coefficients v_c and v_d in Fig. 4(b), (c) represent the influence of the elastic force, and their maximum values are large when n is small and s is large. The

Fig. 3 Values of stream function showing secondary flow (values of $\phi R \times 10^3$ in Figs. (a), (b) and $-\phi R \times 10^3$ in Figs. (c), (d))

position exists where the values of v_c and v_d are minus for small s . The position of the maximum axial flow velocity (Fig. 5(b)) moves toward point P_i and that degree is larger for larger We .

Fig. 6 shows the stream line on the central plane. The arrow there represents the flow direction. Eq. (39) for the stream line on the central plane does not contain R denoting the reciprocal of the radius-ratio, but in Fig. 6 the stream line for $R = 3$ is drawn. Eq. (39) can be rewritten as

$$x_3 = \frac{1}{\sin x_2} \int_0^{x_1} \{ \phi_a(n, x_1)/x_1 \} R e + \{ \phi_b(n, s, x_1)/x_1 \} W e^2 dx_1 \quad (43)$$

As ϕ_a/x_1 and ϕ_b/x_1 are plus, for larger Re and We the outward shift of the stream line on the central plane starting at point $x_1 = 0$ and $x_3 = 0$ with increasing x_3 is large similarly to the earlier analyses^[4,6,7] and agrees with the experimental result^[7]. In the case of Power law fluid for $R = 50$, the stream lines starting at $x_1 = 0$ and $x_3 = 0$ cross at $x_3 = 53.0^\circ$, 46.2° and 46.4° for $x_1 = 0.5$ and $x_2 = \pi/2$ when

Fig. 4(a) Distributions of axial flow velocity $v_3^{(0)}(n, x_1)$ in a straight pipe and coefficients $v_a(n, x_1)$ and $v_b(n, x_1)$

Fig. 4(c) Distributions of coefficient $v_d(n, s, x_1)$

Fig. 4(b) Distributions of coefficient $v_c(n, s, x_1)$

Fig. 5(a) Distributions of axial flow velocity $v_3^{(0)}$ in a straight pipe, and axial flow velocity v_3 on the central plane for $R = 3$ in the case of Power law fluid

Fig. 5(b) Distributions of axial flow velocity $v_3^{(0)}$ in a straight pipe, and axial flow velocity v_3 on the central plane for $R = 3$ in case that $Re = 1$ and $We = 1$

Fig. 6 Stream line on the central plane when $R = 3$ and $Re = 50$

$n = 0.4, 0.7$ and 1.0 , respectively. For small n the secondary flow is large, but the distribution of the axial flow velocity is gently-sloping in the vicinity of the pipe center. Therefore, the variation of the stream line on the central plane has not a simple tendency for the variation of n . On

the other hand, the outward shift of the stream line on the central plane with increasing x_3 is large for large s in the case of White-Metzner fluid, because the secondary flow increases with s .

The values of the radial flow velocity v_1 (eq. (31)) and the circumferential flow velocity v_2 (eq. (32)) are finite independent of the values of s and n . However, the inexpedience exists that the axial flow velocity v_3 (eq. (33)) at the pipe center $x_1 = 0$ changes in relation to the circumferential angle x_2 when $s - n = 0$. The inconsistency exists in the first term of $P(x_1)$ in eq. (34) for the isotropic pressure p when $n \leq 1/4$. Namely, p is infinite at $x_1 = 0$ for $n < 1/4$, and p at $x_1 = 0$ is not isotropic for $n = 1/4$ because it changes in relation to x_2 . Moreover, the deviatoric stress (eq. (36)) at $x_1 = 0$ is infinite for $n - 1 < 0$ and/or $s - n - 1 < 0$. The viscosity of real fluid is finite at zero shear-rate, but one of Power law fluid of $n < 1$ is infinite at zero shear-rate as stated in Chapter 1. The viscosity and/or the relaxation time of White-Metzner model are infinite at $x_1 = 0$ for $n - 1 < 0$ and/or $s - n - 1 < 0$, respectively, because the second invariant of the rate-of-deformation tensor

$$M \equiv 4|\Pi \cdot \epsilon| = \left(\frac{n+1}{n}\right)^2 x_1^{\frac{2}{n}} - \frac{2}{R} \frac{n+1}{n} x_1^{\frac{1}{n}} \left(\frac{\partial v_3^{(1)}}{\partial x_1} - v_3^{(0)} \sin x_2 \right) \quad \dots\dots\dots (44)$$

vanishes at $x_1 = 0$ in the present analysis. Hence, it is thought that the inconsistency appears in the values of the axial flow velocity, the isotropic pressure and the deviatoric stress at the pipe center.

The viscosity and the normal stress coefficient of visco-elastic fluid like many polymer melts and solutions are constants independent of the shear-rate in the flow at low shear-rate, but they decrease with the shear-rate in the flow at high shear-rate. Therefore, the White-Metzner model is applied to a flow at high shear-rate where the viscosity and the normal stress coefficient can be expressed by a power law for shear-rate. As the White-Metzner model can represent the Maxwell model, the viscosity and the normal stress coefficient of which are constants independent of shear-rate, the White-Metzner model can also indicate a flow at low shear-rate. However, because the White-Metzner model composed of four material constants alone can not represent decrements of the viscosity and the normal stress coefficient from constant values with increment of shear-rate, it can not be applied to a flow including from low to high shear-rate. Therefore, the White-Metzner model is meaningful to describe a flow at higher shear-rate than lower one as well as Power law fluid. In the present pipe flow there exists a region where the shear-rate (the second invariant of the rate-of-deformation tensor) is very small, but the region is restricted to the vicinity of the pipe center for large characteristic shear-rate W/a . Hence, the present analysis can be applied

to a flow of large characteristic shear-rate except the vicinity of the pipe center. Further, it can be applied in the entire region of the pipe when the characteristic shear-rate is small (the fluid is regarded as Maxwell fluid which has constant viscosity and constant relaxation time).

5. Conclusion

The steady flow of viscoelastic fluid in a toroidal pipe of circular cross-section is analytically solved with the White-Metzner constitutive equation by a perturbation method. The results are as follows:

(1) The flow rate of a secondary flow in the cross-sectional plane increases with decreasing the viscosity index n , and with increasing the elasticity index s , the Reynolds number Re and the Weissenberg number We .

(2) The position of the maximum axial flow velocity shifts on the central plane more outward from the pipe center for smaller n and larger Re in the case of Power law fluid, and for larger s in the case of White-Metzner fluid.

(3) Owing to the centrifugal and elastic forces applied on the fluid, the stream line starting at a pipe center shifts outward on the central plane. The larger the values of s , Re and We are, the larger the amount of the outward shift is.

Hence, the effects of Re and We on the flow are shown to be similar to an available experimental result as well as earlier theoretical results. Also, it is found that the values n and s , which represent the degrees of decrement of the fluid viscosity and the first normal stress coefficient with the shear-rate respectively, have significant effects on the flow at high shear-rate to which the earlier analyses could not be applied.

The White-Metzner model can be applied to a fluid, the viscous and elastic stresses of which are obedient to a power law for shear-rate, but not exactly to a flow including from low to high shear-rate. However, in case that the ratio of the flow velocity on the pipe center to the pipe radius is large, the velocity distribution of the flow can be obtained by the present analysis except in the very small region near the pipe center.

The authors would like to thank Mr. Yoshio Wada of Fukui University for assistance in preparing the manuscript.

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