An Approach to Visco-Elastic Behaviors with a Mathematical Method

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Abstract

A new master theory on rheology is shown by using derivatives of fractional order. This theory is sufficient to describe, simply and definitely, stress-strain relations as functions of time which represent many well-known visco-elastic behaviors.

1. Derivatives of Fractional Order

To define a derivative of the r -th order ($0 \leq r \leq 1$), we have to express a multiple integral by an integral with two variables, extend it to a non-integer integral, and then differentiate it.⁽¹⁾ Namely, in the case that where function $f(t)$ converges very rapidly on zero when $t \rightarrow \infty$, its integral of the n -th order $I^n f(t)$ (n is positive integer) is given by the following formula:

$$I^n f(t) = \frac{1}{\Gamma(n)} \int_{-\infty}^t (t-\tau)^{n-1} f(\tau) d\tau \quad \dots\dots (1)$$

where $\Gamma(n)$ is a Gamma function the value of which is $(n-1)!$ when n is integer.

Thus, we can define the derivative of the r -th order $D^r f(t)$ ($0 \leq r \leq 1$) by using $1-r$ instead of n of equation (1) and $D^r = D$ $D^{r-1} = DI^{1-r}$

$$D^r f(t) = \frac{1}{\Gamma(1-r)} \frac{d}{dt} \int_{-\infty}^t (t-\tau)^{-r} f(\tau) d\tau \quad (2)$$

In this article, we also use the following formula (2-1) ~ (2-3), deduced from (2)

$$D^r f(t) = \frac{1}{\Gamma(1-r)} \int_{-\infty}^t (t-\tau)^{-r} f'(\tau) d\tau \quad \dots (2-1)$$

With $f(t) = t^n$ when $t > 0$ and $f(t) = 0$ when ≤ 0 , we have

Fig. 1.

Fig. 2.

$$D^r f(t) = \frac{\Gamma(a+1)}{\Gamma(a-r+1)} t^{a-r} \quad \dots\dots (2-2)$$

If the real part of a is positive, we have:

$$D^r e^{at} = a^r e^{at} \quad \dots\dots (2-3)$$

2. Fundamental Equation

The force of an elastic body is proportional to strain or displacement. The force of a viscous flow is proportional to the rate of strain or velocity. Instead of thinking of force or stress S in some rheological body as a series or parallel combinations of ϵ and $D\epsilon$, as is done in Maxwell's or Voigt's model, it is one of the most reasonable extensions to think of such force or stress as being proportional to a certain physical quantity $D^r \epsilon$ ($0 < r < 1$) which exists between strain and its rate $D\epsilon$. Therefore, put

$$S = r D^r \epsilon \quad \dots\dots (3)$$

where r is a constant. Now, let us apply this equation to many phenomena which represent rheological properties.

3. Creep

When a rheological body is loaded in a state called "constant stress," this state is

$$\begin{aligned} S &= 0 \quad \text{when } t \leq 0 \\ S &= S_0 \text{ (constant)} \quad \text{when } t > 0 \end{aligned}$$

From (3) and (2-2) the behavior of creep shown in Figure 1 is given by

$$\epsilon(t) = \frac{1}{r} D^{-r} S_0 = \frac{S_0}{r \Gamma(1+r)} t^r \quad \dots\dots (4)$$

This behavior has been inquired into successively by Andrade, Orowan⁽²⁾ and many other

physicists, who have shown it as the third-root formula, namely, $r=1/3$.

4. Relaxation

When a rheological body is deformed in a state called "constant strain," that state is

$$\varepsilon=0 \quad \text{when } t \leq 0$$

$$\varepsilon=\varepsilon_0 \quad (\text{constant}) \quad \text{when } t > 0$$

The behavior of relaxation shown in Figure 2 is given by

$$S(t) = r D^r \varepsilon = \frac{r \varepsilon_0}{\Gamma(1-r)} t^{-r} \quad \dots\dots (5)$$

To eliminate an anomaly such as the anomaly of the stress in (5) indicating ∞ when $t=0$, we have to give the condition as shown in Figure 3-a, namely

$$\varepsilon(t) = \varepsilon_0(t+t_0)/t_0 \quad \text{when } -t_0 \leq t \leq 0$$

$$\varepsilon(t) = \varepsilon_0 \quad \text{when } t > 0$$

The curve of relaxation shown in Figure 3-b is, then, obtainable in the following form

$$S(t) = r \frac{\varepsilon_0}{t_0} \frac{1}{\Gamma(2-r)} \left\{ (t+t_0)^{1-r} - t_0^{1-r} \right\} \quad \text{when } t \geq 0 \quad \dots\dots (6)$$

Fig. 3-a

Fig. 3-b

Fig. 4-a.

Fig. 4-b.

Since the following relations are evidently established when t is sufficiently large

$$(t+t_0)^{1-r} - t^{1-r} \left(1 + \frac{t_0}{t}\right)^{1-r} \approx t^{1-r} - (1-r)t^{-r}t_0, \quad \dots\dots (7)$$

so (6) coincides with (5).

5. Effect of Memory

When a specimen is deformed as shown in Figure 4-a, the conditions of deformation are:

$$\varepsilon = -\varepsilon_0 \quad \text{when } -t_0 < t \leq -t_1$$

$$\varepsilon = \varepsilon_1 \quad \text{when } -t_1 < t \leq 0$$

$$\varepsilon = 0 \quad \text{when } t > 0$$

Therefore, the stress, when $t \geq 0$, can be given, after some routine calculations, by

$$S(t) = \frac{r}{\Gamma(1-r)} \left\{ \frac{-\varepsilon_0}{(t+t_0)^r} + \frac{\varepsilon_0 + \varepsilon_1}{(t+t_1)^r} + \frac{-\varepsilon_1}{t^r} \right\} \quad \dots\dots (8)$$

This curve is shown in Figure 4-b.

When a specimen is loaded as shown in Figure 5-a, the specimen is under these conditions:

$$S = -S_0 \quad \text{when } -t_0 < t \leq -t_1$$

$$S = S_1 \quad \text{when } -t_1 < t \leq 0$$

$$S = 0 \quad \text{when } t > 0$$

The curves of strain shown in Figure 5-b are expressed thus:

$$\varepsilon(t) = \frac{1}{r\Gamma(1+r)} \left\{ -S_0(t+t_0)^r + (S_0 + S_1)(t+t_1)^r - S_1 t^r \right\} \quad \dots\dots (9)$$

Fig. 5-a.

Fig. 5-b.

when $t \geq 0$

Since the conditions under which the phenomenon of memory occurs are

$$\varepsilon(0) > 0$$

$$\varepsilon(t) < 0 \quad \text{when } t \text{ becomes very large,}$$

we get the following formula to express those conditions

$$\frac{t_0}{t_1} > 1 + \frac{S_1}{S_0} > \left(\frac{t_0}{t_1}\right)^r \quad \dots\dots (10)$$

6. Hysteresis

Now, to discuss the behavior of a rheological body stretched with a constant rate of strain until $\varepsilon = \varepsilon_0$, $S = S_0$ and then compressed at the same rate. The stress when $\varepsilon = vt$, is given by

$$S(t) = rD^r vt = \frac{rv}{\Gamma(2-r)} t^{1-r} \quad \dots\dots (11)$$

Substituting $t = \varepsilon/v$ into (11) gives us

$$S = \frac{rv^r}{\Gamma(2-r)} \varepsilon^{1-r} \quad \dots\dots (12)$$

In the next stage of compression we get

$$S(t) = \frac{rv}{\Gamma(2-r)} \{(t+t_0)^{1-r} - 2t^{1-r}\} \quad \dots (13)$$

$$S = \frac{rv^r}{\Gamma(2-r)} \{(2\varepsilon_0 - \varepsilon)^{1-r} - 2(\varepsilon_0 - \varepsilon)^{1-r}\} \dots (14)$$

So we can draw the curves shown in Figure 6. The values of the main points of those curves are

$$\varepsilon' = \frac{\alpha-2}{\alpha-1} \varepsilon_0, \quad \alpha = 2^{1/1-r} \quad \text{on the abscissa} \quad \dots (15)$$

Fig. 6.

$$\left. \begin{aligned} S' &= -(2-2^{1-r})S_0 \\ S_1 &= -(2 \times 2^{1-r} - 3^{1-r})S_0 \end{aligned} \right\} \text{on the ordinate} \quad \dots (16)$$

7. Dynamic Coefficients of Elasticity and Viscosity

Substituting $\varepsilon = \varepsilon_0 e^{i\omega t}$ into (3), we get

$$\begin{aligned} S &= r\varepsilon_0 i^r \omega^r e^{i\omega t} \\ &= r\omega^r \left(\cos r \frac{\pi}{2} + i \sin r \frac{\pi}{2} \right) \varepsilon \quad \dots\dots (17) \end{aligned}$$

Compare the real or imaginary parts of (17) with each term of the expression (18) of Voigt's body:

$$S = E_v \varepsilon + \eta_v D \varepsilon = (E_v + \eta_v \omega i) \varepsilon \quad \dots\dots (18)$$

and we get

$$E_v = r\omega^r \cos r \frac{\pi}{2} \quad \dots\dots (19)$$

$$\eta_v = r\omega^{r-1} \sin r \frac{\pi}{2} \quad \dots\dots (20)$$

The expressions of (19) and (20) show the experimental results (Figure 7)⁽³⁾ of the dynamic behavior of a Decalin solution of polyisobutylene.

Fig. 7.

Assuming $S = E\varepsilon + \tau D^r \varepsilon$, we get:

$$S = (E + \tau i^r \omega^r) \varepsilon \quad \dots\dots (21)$$

$$\left. \begin{aligned} E_v &= E + \tau \omega^r \cos r \frac{\pi}{2} \\ \eta_v &= \tau \omega^{r-1} \sin r \frac{\pi}{2} \end{aligned} \right\} \quad \dots\dots (22)$$

The relations shown in (22) coincide with an

Fig. 8. Visco-elasticity of Nylon-6

Fig. 9.

experiment by Fujino and his co-workers (Figure 8).⁽⁴⁾

Complex compliance J , which is the reverse number of the coefficient of ε in (21), is:

$$J(i\omega) = \frac{\Delta J}{1 + (i\omega T)^r} \quad \dots\dots (23)$$

where $\Delta J = 1/E$, $T = (\tau/E)^{1/r}$ (the most probable delayed time).

As the curve of (23) is shown in Figure 9, we can bring equation (23) into agreement with Cole-Cole's law of circular arc.⁽⁵⁾

Conclusions

The author has studied the fundamentals of a viscoelastic body. He has also investigated whether the derivative of fractional order can be applied, as a master theory, to a rheological body. As the result, he has been able to prove, by various examples, the "utility" (applicability) of the derivative of fraction order as a master theory.

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