Analytical solutions to hyperbolic heat conductive models using Green's function method

Ya Jun YU*

* Department of Engineering Mechanics, Northwestern Polytechnical University, Xi'an 710129, PR China

E-mail: yuyj@nwpu.edu.cn

Received: 5 February 2018; Revised: 4 April 2018; Accepted: 19 April 2018

Abstract

In this work, the existing theoretical heat conductive models such as: Cattaneo-Vernotte model, simplified thermomass model, and single-phase-lag two-step model are summarized, and then a general model of hyperbolic heat conduction (HHC) is presented with boundary conditions prescribed as: (1) temperature at the boundary; (2) heat flux (not the temperature gradient) at the boundary. The convective boundary condition is not considered because it is impossible to produce a fluid motion at such time scale, e.g. picoseconds. In the context of HHC, the reciprocity relation of Green's function is systematically proven, based on which Green's function solution equation of the general model is completely derived. Meanwhile, Green's functions are deduced under several different sets of boundary conditions. Thus, solution to HHC based on Green's function is obtained, and it consists of three parts, i.e. boundary conditions, initial conditions and heat generations. The priority of the present solution is that the time-dependent boundary condition and heat generation may be dealt with, and that it is easy to extend the solution to multi-dimensional case. The accuracy of the solution is verified through numerical examples, and Laplace transform method is adopted to avoid the dispersions around the front of temperature wave for jump-type boundary conditions, thus the solution is further perfected.

Keywords : Analytical solution, Hyperbolic heat conduction, Green's function, Laplace transform method

1. Introduction

Classical heat conduction, as a mode of heat transfer, is driven by temperature gradient in solids or liquids. The classical parabolic heat conduction based on Fourier's law is sufficiently accurate for most common engineering situations. However, parabolic heat conduction breaks down in extreme situations, such as very short duration, high heat fluxes, and very low temperatures (Frankel et al., 1987; Xiong, et al., 2017), because it contradicts physical facts (Ma and He, 2017; Mitra et al., 1995; Peshkov, 1994; Ren et al., 2012; Xiong, et al., 2015) and admits an infinite speed (He et al., 2004; Xiong, et al., 2011, 2012). To eliminate these paradoxes, various theories of non-classical heat conduction (or heat wave, heat conduction with second sound) have been proposed.

To permit a finite speed of heat conduction, Cattaneo (1958) and Vernotte (1958) proposed a new form of the heat conduction law by introducing a relaxation term and resulting in a hyperbolic differential equation (C-V model). The C-V model abandons the assumption of classical heat conduction law that the heat flux and the temperature gradient across a material volume occur at the same instant of time. For specific initial boundary value problems, the C-V model introduces a sharp wave front in the history of heat wave propagation. Recently, the theoretical basis of hyperbolic heat conduction is systematically discussed (Li and Cao, 2018; Sobolev, 2018).

On a separate front, the thermomass model (Guo and Hou, 2010) was introduced to further study heat conduction mechanisms at microscale or even nanoscale, with the constitutive equation derived based on the phonon gas equation of motion. Cao and Guo (2007) described heat transfer in solids due to phonon gas motion using Einstein's mass-energy relationship, confirming that the classical heat conduction broke down in the cases of ultrahigh heat fluxes and ultralow temperature.

With wide application of short-pulse energy deposition in micromachining, investigations interest on microscale heat conduction has been aroused. At microscale, the temperature response may be dramatically affected by the

individual heat carriers, such as phonons, electrons, and photons. It is widely known that the lattice energy is due to the thermal vibration of the solid, while the lattice vibration energy is quantized and called as a phonon in the language of quantum. In metals, heat conduction is carried both by phonons and free electrons, while in semiconductors heat energy is transferred mainly by the phonons. A two-step model was proposed by distinguishing the electron temperature from the lattice temperature (Anisimov et al., 1974; Qui and Tien, 1992; Qui and Tien, 1993): the electrons in metals are firstly excited by electrons-phonon interaction into higher energy level; the excited electrons produce a hot free electron gas, then the electron gas diffuses through the metal lattice and generates phonons by collision with the lattice (Hays-Stang and Haji-Sheikh, 1994).

It has been well established that Green's function is an effective way to deal with initial boundary partial differential equations, and hence it has been widely applied to solve heat conduction problems. For typical example, Cole et al. (2010) derived Green's functions of heat conduction associated with various initial boundary conditions in rectangular, cylindrical and spherical coordinates systems. Kim and Noda (2001) solved three-dimensional (3D) heat conduction equation of functionally graded materials by adopting a Green's function approach based on the laminate theory. Gray et al. (2003) derived free space Green's functions for exponentially graded materials, and an explicit Green's approach for heat conduction was also proposed based on Green's function calculated in the Laplace domain (Loureiro et al., 2009; Loureiro and Mansur, 2009). Majumdar and Xia (2007) analyzed the laser heating of materials using Green's function method. In addition, Green's function was also applied in inverse transient heat conduction problems (Fernandes et al., 2010).

In the context of HHC, Haji-Sheikh and Beck (1994) solved the temperature solution for thermal wave equation. And later, Haji-Sheikh et al. (2002) identified the anomalies of the hyperbolic heat conduction and suggested appropriate remedies. Around 2010, Chen solved hyperbolic heat conduction problems using a hybrid Green's function method with zero initial conditions and the heat generation was also neglected (Chen, 2009, 2010; Chen and Chen, 2010). Recently, Haji-Sheikh et al. (2013) once again solved the temperature solutions in thin films using thermal wave Green's function solution equation. Chen's work (Chen, 2009) was limited to zero initial conditions and the Green's function was obtained in the Laplace domain with the energy generation neglected. In the recent paper of Haji-Sheikh et al. (2013), the Green's function in infinite solids is firstly obtained. However, for thermal wave theory, the disturbed region is zero to the front of temperature wave. If the thermal signal does not arrive at the boundary, it may be considered as an infinite problem. So HHC in finite medium is adequate to express practical responses. In this work, HHC in finite medium is considered making use of the Green's function method: (1) the energy generation term and the initial conditions are selected arbitrarily. (2) several different sets of boundary conditions are considered to apply Green's function method in hyperbolic heat conductive problems to the fullest extent. For second kind boundary condition, the heat flux is directly given instead of temperature gradient type boundary conditions. (3) the Green's function solution equation for finite medium is derived in time domain under different boundary conditions. In addition, in the formulation, the reciprocity relation of Green's function is firstly proven in the context of HHC, and then Green's function solution equation is obtained. As a result, the formulation is impeccable. By substituting boundary conditions, initial conditions, heat generation, and Green's function into the Green's function solution equation, the solution to the considered problem is thus obtained. The solution procedure is validated by numerical implementations. Finally, the results for boundary conditions part is improved by adopting Laplace transform method.

2. Theoretical models

As previously mentioned, hyperbolic heat conduction (or heat conduction with second sound) was formulated to eliminate the paradox of infinite speed of energy transfer inherent in the classical diffusion heat conduction model, and to describe more accurately heat transfer in low temperature or high power. Three theoretical models of hyperbolic heat conduction are presented in this section.

2.1 Cattaneo-Vernotte model

The simplest and earliest linear hyperbolic heat conduction theory is the Cattaneo-Vernotte model. To account for the existence of second sound detected in liquid helium (Peshkov, 1994), Cattaneo (1958) and Vernotte (1958) proposed a modified heat conduction equation. For homogeneous and isotropic materials, it reads as:

$$q + \tau_{cv} \frac{\partial q}{\partial t} = -k \frac{\partial \theta}{\partial x} \quad (1)$$

Combining with the equation of energy conservation:

$$\rho c_E \frac{\partial \theta}{\partial t} = Q - \frac{\partial q}{\partial x} \quad (2)$$

one obtains the governing equation for temperature, as:

$$\tau_{cv} \rho c_E \frac{\partial^2 \theta}{\partial t^2} + \rho c_E \frac{\partial \theta}{\partial t} = k \frac{\partial^2 \theta}{\partial x^2} + \left(Q + \tau_{cv} \frac{\partial Q}{\partial t} \right) \quad (3)$$

Equation (3) indicates the C-V model and takes the form of wave, where q , τ_{cv} , k , θ , ρ , c_E , Q denote heat flux, relaxation time, thermal conductivity, temperature, density, specific heat and heat generation, respectively.

2.2 Simplified thermomass model

Since 1917, the concept of inertia for heat has been suggested (Nernst, 1917; Onsager, 1931), while a quantitative description of the inertia has seldom been reported except those (Tolman, 1930; Eckart, 1940; Landau and Lifshitz, 1959). Recognizing that heat conduction in dielectrics is attributed to the motion of “phonon gas”, Cao and Guo (2007) introduced the concept of “thermomass” according to Einstein’s mass-energy relation, and proposed the state equation of phonon gas based on the Debye state equation for solids (Tien and Lienhard, 1979). Then, an equation of motion was established using Newtonian dynamics with consideration of the driving, inertia and resistant forces. A non-Fourier heat conduction was also formulated, as:

$$\left(1 - \frac{q^2}{\rho^2 c_E^3 \theta^3} \right) k \frac{\partial \theta}{\partial x} + q = 0 \quad (4)$$

The law indicates that Fourier’s law breaks down under such conditions: ultrahigh heat fluxes and ultralow temperatures. Later, Guo and Hou (2010) derived the equation for one-dimensional heat conduction as:

$$\tau_{TM} \frac{\partial q}{\partial t} - l_{TM} \rho c_E \frac{\partial \theta}{\partial t} + l_{TM} \frac{\partial q}{\partial x} - b k \frac{\partial \theta}{\partial x} + k \frac{\partial \theta}{\partial x} + q = 0 \quad (5)$$

where τ_{TM} , l_{TM} and b are relaxation time, characteristic length parameter and square of thermal Mach number of the phonon gas, respectively. If the thermal disturbance is small, one obtains the simplified thermomass model by neglecting the terms with b and l_{TM} , as:

$$\tau_{TM} \frac{\partial q}{\partial t} + k \frac{\partial \theta}{\partial x} + q = 0 \quad \text{or} \quad q + \tau_{TM} \frac{\partial q}{\partial t} = -k \frac{\partial \theta}{\partial x} \quad (6)$$

Combining with the continuity equation of phonon gas, one may obtain a thermal wave equation with the heat generation included, as:

$$\tau_{TM} \rho c_E \frac{\partial^2 \theta}{\partial t^2} + \rho c_E \frac{\partial \theta}{\partial t} = k \frac{\partial^2 \theta}{\partial x^2} + \left(Q + \tau_{TM} \frac{\partial Q}{\partial t} \right) \quad (7)$$

Apparently, Eq. (5) is a new extension of Fourier's law, while they are essentially different: the former is the motion equation of phonon gas, while the latter is an empirical formula. Detailed explanation for the physical meaning of τ_{TM} is also presented: the characteristic time describes a lagging response, in time, between the drift velocity of the phonon gas and the driving force of the phonon gas (Guo and Hou, 2010). It is noted that this new model may not reduce to Fourier's law even for steady heat conduction.

2.3 Single-phase-lag two-step model

To develop the single-phase-lag two-step model, energy input of a substance is presumed to be absorbed by electron and lattice, and the electrons gas temperature is different from the lattice temperature, and both are related via (Anisimov et al., 1974; Qui and Tien, 1992; Qui and Tien, 1993):

$$C_l \frac{\partial \theta_l}{\partial t} = \Gamma [\theta_e - \theta_l] \quad (8)$$

where C_l and Γ denote heat capacity of lattice and electron-phonon coupling factor; θ_e and θ_l are temperature of electron and lattice, respectively. With the contribution of the energy storage of electron gas and lattice included, the equation of energy balance may be written as:

$$C_e \frac{\partial \theta_e}{\partial t} + C_l \frac{\partial \theta_l}{\partial t} = Q - \frac{\partial q}{\partial x} \quad (9)$$

where C_e is heat capacity of electron. It follows from (8) and (9) that:

$$C \frac{\partial \theta_l}{\partial t} + C \tau_e \frac{\partial^2 \theta_l}{\partial t^2} = Q - \frac{\partial q}{\partial x} \quad (10)$$

where $C = C_e + C_l$ and $\tau_e = C_e C_l / \Gamma C$. Considering the time-lag between the temperature gradient and the resulting heat flux, one has (for brevity, θ_l will be denoted as θ) (Tzou, 1989, 1992):

$$q(t + \tau_q) = -k \frac{\partial \theta(t)}{\partial x} \quad (11)$$

where τ_q is relaxation time. Expanding (11) in the Taylor series and retaining the first two terms yields:

$$q + \tau_q \frac{\partial q}{\partial t} = -k \frac{\partial \theta}{\partial x} \quad (12)$$

By eliminating q between Eqs. (10) and (12), one may obtains, as:

$$C \frac{\partial \theta}{\partial t} + C(\tau_e + \tau_q) \frac{\partial^2 \theta}{\partial t^2} + C \tau_e \tau_q \frac{\partial^3 \theta}{\partial t^3} = k \frac{\partial^2 \theta}{\partial x^2} + \left(Q + \tau_q \frac{\partial Q}{\partial t} \right) \quad (13)$$

where the parameters τ_e and τ_q are quite small (Hays-Stang and Haji-Sheikh, 1999), as a result, the third term on the left side of (13) can be discarded, yielding:

$$C(\tau_e + \tau_q) \frac{\partial^2 \theta}{\partial t^2} + C \frac{\partial \theta}{\partial t} = k \frac{\partial^2 \theta}{\partial x^2} + \left(Q + \tau_q \frac{\partial Q}{\partial t} \right) \quad (14)$$

Note that no reference is made to the type of material studied. That is, by selecting appropriate values of τ_e and τ_q , Eq. (14) holds for dielectric materials, conductors or semiconductors (Hays-Stang and Haji-Sheikh, 1999).

3. General hyperbolic heat conduction model

A unified form of the three hyperbolic heat conduction models is presented (Lam, 2013), as:

$$A \frac{\partial^2 \theta}{\partial t^2} + \frac{\partial \theta}{\partial t} = B^2 \frac{\partial^2 \theta}{\partial x^2} + f_Q(x, t) \quad (15)$$

$$f_Q(x, t) = D \cdot Q + E \cdot \frac{\partial Q}{\partial t} \quad (16)$$

The C-V model, the simplified thermomass model, and the single-phase-lag two-step model can be readily obtained with a proper selection of the coefficients A , B^2 , D , E , as demonstrated in Table 1.

Table 1. Coefficients for general hyperbolic heat conduction

Thermal model	Coefficient			
	A	B^2	D	E
Cattaneo-Vernotte model	τ_{CV}	α	$1/\rho c_E$	$\tau_{CV}/\rho c_E$
Simplified thermomass model	τ_{TM}	α	$1/\rho c_E$	$\tau_{TM}/\rho c_E$
Single-phase-lag two-step model	$\tau_e + \tau_q$	k/C	$1/C$	τ_q/C

To completely describe hyperbolic heat conduction problems, the boundary conditions at $\tilde{x}_\alpha = 0$ and l , where α (e.g., *right*, *left*) represents the right and left boundary, may be prescribed as:

The first kind:

$$\theta(\tilde{x}_\alpha, t) = f_\alpha(t) \quad (17)$$

The second kind:

$$q(\tilde{x}_\alpha, t) = g_\alpha(t) \quad (18)$$

Some works incautiously apply the form $k \frac{\partial \theta(\tilde{x}_\alpha, t)}{\partial \mathbf{n}_x} = g_\alpha(t)$ as boundary conditions for HHC (Gheitaghy and Talaei,

2013; Fong and Lam, 2014, Lam, 2010; Lam and Fong, 2012; Monteiro et al., 2009). Although the nonhomogeneous boundary condition can be solved by performing a functional transformation or using integral transform method, the meaning of the prescribed function $g_\alpha(t)$ is, however, obscure. So in this work, the heat flux at the boundary is prescribed. In Eqs. (17) and (18), the functions $f_\alpha(t)$ and $g_\alpha(t)$ should be specified. To denote the second kind boundary condition (18) by temperature, we resort to the modified Fourier's law, *e.g.* Eqs. (1) (6) and (12), which are commonly expressed as

$$q + \tau \frac{\partial q}{\partial t} = -k \frac{\partial \theta}{\partial x} \quad (19)$$

where τ are τ_{CV} , τ_{TM} , τ_q , respectively. With aids of (18), it is obtained that

$$k \frac{\partial \theta(\tilde{x}_\alpha, t)}{\partial x} = f_\alpha(t) \text{ and } f_\alpha(t) = -(g_\alpha + \tau \dot{g}_\alpha) \quad (20)$$

In the present study, a variety of boundary conditions are considered by combining (17) and (20), and denoted by $XijBt$ ($i, j=1, 2$) with Bt indicating that f_α are function of time.

The initial conditions are prescribed as:

$$\theta(x, 0) = \varphi(x) \quad (21)$$

$$\frac{\partial \theta(x, 0)}{\partial t} = \psi(x) \quad (22)$$

4. Analytical solutions based on Green's function

In this section, solutions to 1D hyperbolic heat conduction problems are obtained for selected boundary conditions at $\tilde{x}_\alpha = 0$ and l , and the functions f_α , φ and ψ are not specified to obtain a general solution.

4.1 Proof of reciprocity relation of Green's function

The Green's function of temperature governed by (15) and the corresponding initial boundary conditions is the solution to the following auxiliary equation:

$$A \frac{\partial^2 G}{\partial t^2} + \frac{\partial G}{\partial t} = B^2 \frac{\partial^2 G}{\partial x^2} + \delta(x, t | x', \tau) \quad (23)$$

with homogeneous boundary conditions and zero initial conditions (i.e., $f_\alpha = 0$ and $\varphi = \psi = 0$), Eq. (23) may be written alternatively, as:

$$A \frac{\partial^2 G}{\partial t^2}(x, t | \hat{x}, \hat{t}) + \frac{\partial G}{\partial t}(x, t | \hat{x}, \hat{t}) = B^2 \frac{\partial^2 G}{\partial x^2}(x, t | \hat{x}, \hat{t}) + \delta(x, t | \hat{x}, \hat{t}) \quad (24)$$

$$A \frac{\partial^2 G'}{\partial t^2}(x, -t | \hat{x}, -\hat{t}) - \frac{\partial G'}{\partial t}(x, -t | \hat{x}, -\hat{t}) = B^2 \frac{\partial^2 G'}{\partial x^2}(x, -t | \hat{x}, -\hat{t}) + \delta(x, -t | \hat{x}, -\hat{t}) \quad (25)$$

where the Green's function in (25) is denoted by G' to distinguish from that in (24). Multiplying (24) by $G'(x, -t|\hat{x}, -\hat{t})$ and (25) by $G(x, t|\hat{x}, \hat{t})$, and subtracting, then integrating over region 0 to l and over time from 0 to t' , one obtains:

$$\begin{aligned} & \int_0^{t'} \int_0^l \left[B^2 \left(G' \frac{\partial^2 G}{\partial x^2} - G \frac{\partial^2 G'}{\partial x^2} \right) - \left(G' \frac{\partial G}{\partial t} + G \frac{\partial G'}{\partial t} \right) - A \left(G' \frac{\partial^2 G}{\partial t^2} - G \frac{\partial^2 G'}{\partial t^2} \right) \right] dx dt \\ & = G'(x, -t|\hat{x}, -\hat{t}) - G(x, \hat{t}|\hat{x}, \hat{t}) \end{aligned} \quad (26)$$

where t' is greater than either $\hat{t}$ or $\hat{t}'$. Considering that:

$$G' \frac{\partial^2 G}{\partial x^2} - G \frac{\partial^2 G'}{\partial x^2} = \frac{\partial}{\partial x} \left(G' \frac{\partial G}{\partial x} - G \frac{\partial G'}{\partial x} \right) \quad (27)$$

$$G' \frac{\partial^2 G}{\partial t^2} - G \frac{\partial^2 G'}{\partial t^2} = \frac{\partial}{\partial t} \left(G' \frac{\partial G}{\partial t} - G \frac{\partial G'}{\partial t} \right) \quad (28)$$

one may rewrite (26) as:

$$\begin{aligned} & \int_0^{t'} \left[B^2 \left(G' \frac{\partial G}{\partial x} - G \frac{\partial G'}{\partial x} \right) \right]_0^l dt - \int_0^{t'} \left[(G'G)_0 \right]' dx - \int_0^{t'} \left[A \left(G' \frac{\partial G}{\partial t} - G \frac{\partial G'}{\partial t} \right) \right]_0^{t'} dx \\ & = G'(x, -t|\hat{x}, -\hat{t}) - G(x, \hat{t}|\hat{x}, \hat{t}) \end{aligned} \quad (29)$$

Here, the first term vanishes because both Green's functions satisfy the same boundary conditions; the second and third terms can also be simplified considering the zero initial condition both Green's functions share $G(x, t|\hat{x}, 0) = G'(x, -t|\hat{x}, 0) = 0$ and the causality relation of Green's function $G(x, t|\hat{x}, t') = G'(x, -t|\hat{x}, t') = 0$.

Accordingly, one has:

$$G(\hat{x}, \hat{t}|\hat{x}, \hat{t}) = G'(x, -t|\hat{x}, -\hat{t}) \quad (30)$$

which is the reciprocity relation of Green's function is obtained. In alternative form, (A.9) may be expressed as:

$$G(x, t|x', \tau) = G'(x', -\tau|x, -t) \quad (31)$$

4.2 Derivation of Green's function solution equation

This subsection is devoted to the 1D Green's function solution equation for the above-mentioned general hyperbolic heat conduction model referred to the formulation for heat conduction (Cole et al., 2010).

The Green's function of temperature given by Eq. (15) and the corresponding initial boundary conditions is the solution to the following auxiliary equation:

$$A \frac{\partial^2 G}{\partial t^2} + \frac{\partial G}{\partial t} = B^2 \frac{\partial^2 G}{\partial x^2} + \delta(x, t|x', \tau) \quad (32)$$

with the homogeneous boundary conditions and zero initial conditions:

$$\begin{aligned} X_{ij} \quad (i=1,2; j=1,2); \\ G(x,0)=0; \quad \frac{\partial G(x,0)}{\partial t}=0 \end{aligned} \quad (33)$$

In view of the reciprocity relation (31), which is proved in the context of hyperbolic heat conduction in the previous subsection, the auxiliary equation (32) may be rewritten as:

$$A \frac{\partial^2 G}{\partial \tau^2} - \frac{\partial G}{\partial \tau} = B^2 \frac{\partial^2 G}{\partial x'^2} + \delta(x', t | x, \tau) \quad (34)$$

The original hyperbolic heat conduction (15) is written in terms of x' and τ , yielding:

$$A \frac{\partial^2 \theta}{\partial \tau^2} + \frac{\partial \theta}{\partial \tau} = B^2 \frac{\partial^2 \theta}{\partial x'^2} + f(x', \tau) \quad (35)$$

Multiplying (35) by $G(x, t | x', \tau)$ and (34) by $\theta(x', \tau)$, and subtracting, then integrating over region 0 to l and over time from 0 to $t + \varepsilon$, one arrives at:

$$\begin{aligned} \int_0^{t+\varepsilon} \int_0^l \left[A \left(G \frac{\partial^2 \theta}{\partial \tau^2} - \theta \frac{\partial^2 G}{\partial \tau^2} \right) + \frac{\partial(\theta G)}{\partial \tau} \right] dx' d\tau \\ = \int_0^{t+\varepsilon} \int_0^l \left[B^2 \left(G \frac{\partial^2 \theta}{\partial x'^2} - \theta \frac{\partial^2 G}{\partial x'^2} \right) \right] dx' d\tau + \int_0^{t+\varepsilon} \int_0^l G f_\theta(x', \tau) dx' d\tau - \theta(x, t) \end{aligned} \quad (36)$$

It follows from (36) that:

$$\begin{aligned} \theta(x, t) = B^2 \int_0^{t+\varepsilon} \left[\left(G \frac{\partial \theta}{\partial x'} - \theta \frac{\partial G}{\partial x'} \right) \right]_0^l d\tau + \int_0^{t+\varepsilon} \int_0^l G f_\theta(x', \tau) dx' d\tau \\ - A \int_0^l \left[\left(G \frac{\partial \theta}{\partial \tau} - \theta \frac{\partial G}{\partial \tau} \right) \right]_0^{t+\varepsilon} dx' - \int_0^l \left[(\theta G) \right]_0^{t+\varepsilon} dx' \end{aligned} \quad (37)$$

where

$$\int_0^l \left[(\theta G) \right]_0^{t+\varepsilon} dx' = - \int_0^l \left[\theta(x', 0) G(x, t | x', 0) \right] dx' = - \int_0^l \left[\varphi(x') G(x, t | x', 0) \right] dx' \quad (38)$$

$$\int_0^l \left[\left(G \frac{\partial \theta}{\partial \tau} - \theta \frac{\partial G}{\partial \tau} \right) \right]_0^{t+\varepsilon} dx' = \int_0^l \left[\varphi(x') \frac{\partial G(x, t | x', 0)}{\partial \tau} - \psi(x') G(x, t | x', 0) \right] dx' \quad (39)$$

considering the causality relation $G(x, t | x', t + \varepsilon) = 0$ and the initial conditions (21) and (22):

$$\theta(x', 0) = \varphi(x'), \quad \dot{\theta}(x', 0) = \psi(x').$$

Finally, by taking the limit of (37) as $\varepsilon \rightarrow 0$, the Green's function solution equation has the form:

$$\begin{aligned}
 \theta(x, t) = & \underbrace{B^2 \int_0^t \left[G \frac{\partial \theta}{\partial x'} - \theta \frac{\partial G}{\partial x'} \right]_0^l d\tau}_{\text{for boundary conditions}} \\
 & + \underbrace{\int_0^t \left[\varphi(x') G(x, t | x', 0) \right] dx' - A \int_0^t \left[\varphi(x') \frac{\partial G(x, t | x', 0)}{\partial \tau} - \psi(x') G(x, t | x', 0) \right] dx'}_{\text{for initial conditions}} \\
 & + \underbrace{\int_0^t \int_0^l G(x, t | x', \tau) f_Q(x', \tau) dx' d\tau}_{\text{for energy generation}}
 \end{aligned} \quad (40)$$

We have therefore proved that the solution to hyperbolic heat conduction problems, described by Eq. (15) and the corresponding initial boundary conditions, can be expressed in terms of the boundary conditions, the initial conditions, the energy generation and the Green's function.

For a concise solution structure, the term of Eq. (40) $\int_0^t \left[G \frac{\partial \theta}{\partial x'} - \theta \frac{\partial G}{\partial x'} \right]_0^l d\tau$ should be simplified according to

the boundary conditions at $\tilde{x}'_\alpha = 0, l$. For the first kind boundary conditions:

$$\theta(\tilde{x}'_\alpha, t) = f_\alpha(t) \quad (41)$$

$$G(\tilde{x}'_\alpha, t) = 0 \quad (42)$$

Then, one has:

$$\begin{aligned}
 & \int_0^t \left(G(\tilde{x}'_\alpha, t) \frac{\partial \theta(\tilde{x}'_\alpha, \tau)}{\partial x'} - \theta(\tilde{x}'_\alpha, \tau) \frac{\partial G(\tilde{x}'_\alpha, \tau)}{\partial x'} \right) d\tau \\
 & = - \int_0^t \theta(\tilde{x}'_\alpha, \tau) \frac{\partial G(\tilde{x}'_\alpha, \tau)}{\partial x'} d\tau = - \int_0^t f_\alpha(\tau) \frac{\partial G(\tilde{x}'_\alpha, \tau)}{\partial x'} d\tau
 \end{aligned} \quad (43)$$

The results for $\int_0^t \left[G \frac{\partial \theta}{\partial x'} - \theta \frac{\partial G}{\partial x'} \right]_0^l d\tau$ are summarized in Table 2.

Table 2. Calculation for $\int_0^t \left(G \frac{\partial \theta}{\partial x'} - \theta \frac{\partial G}{\partial x'} \right)_0^l d\tau$ under $XijBt$.

BC	$\int_0^t \left(G \frac{\partial \theta}{\partial x'} - \theta \frac{\partial G}{\partial x'} \right)_0^l d\tau$
<i>X11</i>	$-\int_0^t f_{right}(\tau) \frac{\partial G(x'=l, \tau)}{\partial x'} d\tau + \int_0^t f_{left}(\tau) \frac{\partial G(x'=0, \tau)}{\partial x'} d\tau$
<i>X12</i>	$\int_0^t \frac{1}{k} f_{right}(\tau) G(x'=l, \tau) d\tau + \int_0^t f_{left}(\tau) \frac{\partial G(x'=0, \tau)}{\partial x'} d\tau$
<i>X21</i>	$-\int_0^t f_{right}(\tau) \frac{\partial G(x'=l, \tau)}{\partial x'} d\tau - \int_0^t \frac{1}{k} f_{left}(\tau) G(x'=0, \tau) d\tau$

X22	$\int_0^l \frac{1}{k} f_{right}(\tau) G(x'=l, \tau) d\tau - \int_0^l \frac{1}{k} f_{left}(\tau) G(x'=0, \tau) d\tau$
-----	--

Hereto, the formulation of the Green's function remains the only unsolved issue, which is addressed next.

4.3 Green's function

As previously mentioned, the Green's function is the solution to the auxiliary initial boundary value problem, which is similar to the initial boundary problem for hyperbolic heat conduction with two notable differences: (1) the generation function in the auxiliary equation is the Dirac delta function, as shown by Eq. (32); (2) the initial boundary conditions are homogeneous, expressed by Eq. (33). In the monograph (Cole et al., 2010), several methods for obtaining Green's functions are presented. Here, method of separation of variables is adopted.

According to Wang et al. (2008), the solution of the initial boundary value problem (32) and (33) is identical to that of the following problem:

$$A\ddot{G} + \dot{G} = B^2 \frac{\partial^2 G}{\partial x^2} \quad (44)$$

$$X_{ij} \quad (i=1,2; j=1,2) \quad (45)$$

$$G(x,0)=0 \quad (46)$$

$$\frac{\partial G(x,0)}{\partial t} = \frac{\delta(x-x')}{A} \quad (47)$$

The Green's function can be obtained by using the method of separation of variables, as:

$$G = T(t) X(x) \quad (48)$$

with $X(x)$ satisfying:

$$\frac{\partial^2 X(x)}{\partial x^2} + \beta_m^2 X(x) = 0 \quad (49)$$

subjected to boundary conditions (45) and

$$AT'' + T' + (\beta_m B)^2 T = 0 \quad (50)$$

as well as initial conditions (46) and (47). It follows from (49) that:

$$X(x) = c \sin \beta_m x + d \cos \beta_m x \quad (51)$$

For illustration, consider the boundary condition of type X11. In this case, one has:

$$G(0,t) = G(l,t) = 0 \quad (52)$$

Then

$$X(0) = X(l) = 0 \quad (53)$$

From (51) and (53), one has:

$$\sin \beta_m l = 0 \quad (54)$$

yielding:

$$\beta_m = \frac{m\pi}{l} \quad (m = 1, 2, \dots) \quad (55)$$

Consequently, the solution to (49) is:

$$X_m(x) = c \sin \beta_m x, \quad \beta_m = \frac{m\pi}{l} \quad (m = 1, 2, \dots) \quad (56)$$

Here, β_m^2 , $F_{eiegn} = \sin \beta_m x$ and $M_m = \int_0^l F_{eiegn} F_{eiegn} dx = \int_0^l \sin \beta_m x \sin \beta_m x dx = \frac{l}{2}$ are called the Eigenvalues, Eigen functions and normal square, respectively.

The general solution of (50) is:

$$T_m = e^{-\frac{t-\tau}{2A}} \left[a_m \cos \gamma_m (t-\tau) + b_m \sin \gamma_m (t-\tau) \right] \quad (57)$$

where

$$\gamma_m = \frac{1}{2A} \sqrt{4A(\beta_m B)^2 - 1} \quad (58)$$

As illustration, for boundary condition X11, one obtains from (56) and (57) that:

$$G = \sum_{m=1}^{\infty} e^{-\frac{t-\tau}{2A}} \left[a'_m \cos \gamma_m (t-\tau) + b'_m \sin \gamma_m (t-\tau) \right] \sin \beta_m x \quad (59)$$

where $a'_m = ca_m$ and $b'_m = cb_m$. In view of (46), one gets:

$$G = \sum_{m=1}^{\infty} b'_m e^{-\frac{t-\tau}{2A}} \sin \gamma_m (t-\tau) \sin \beta_m x \quad (60)$$

To satisfy (47), one has:

$$\sum_{m=1}^{\infty} b'_m \gamma_m \sin \beta_m x = \frac{\delta(x-x')}{A} \quad (61)$$

which requires that:

$$b'_m = \frac{1}{\gamma_m M_m} \int_0^l \frac{\delta(x-x')}{A} \sin \beta_m x dx = \frac{1}{A \gamma_m M_m} \sin \beta_m x' \quad (62)$$

Equation (60) can be rewritten as:

$$G = \sum_{m=1}^{\infty} e^{-\frac{t-\tau}{2A}} \frac{1}{A \gamma_m M_m} \sin \beta_m x' \sin \beta_m x \sin \gamma_m (t-\tau) \quad (63)$$

which is the Green's function for 1D problems with $X11$ boundary conditions. Similarly, corresponding results for boundary conditions Xij ($i, j = 1, 2$) can be obtained, as summarized in Table 3.

Table 3. Green's function for one-dimensional problems under boundary

conditions Xij ($i, j = 1, 2$) ($\gamma_m = \frac{1}{2A} \sqrt{4A(\beta_m B)^2 - 1}$).

BC	Green's function
$X11$	$G = \sum_{m=1}^{\infty} e^{-\frac{t-\tau}{2A}} \frac{1}{A\gamma_m M_m} \sin \beta_m x' \sin \beta_m x \sin \gamma_m (t - \tau); \beta_m = m\pi/l; M_m = l/2$
$X12$	$G = \sum_{m=0}^{\infty} e^{-\frac{t-\tau}{2A}} \frac{1}{A\gamma_m M_m} \sin \beta_m x' \sin \beta_m x \sin \gamma_m (t - \tau); \beta_m = (2m+1)\pi/2l; M_m = l/2$
$X21$	$G = \sum_{m=0}^{\infty} e^{-\frac{t-\tau}{2A}} \frac{1}{A\gamma_m M_m} \cos \beta_m x' \cos \beta_m x \sin \gamma_m (t - \tau); \beta_m = (2m+1)\pi/2l; M_m = l/2$
$X22$	$G = \sum_{m=0}^{\infty} e^{-\frac{t-\tau}{2A}} \frac{1}{A\gamma_m M_m} \cos \beta_m x' \cos \beta_m x \sin \gamma_m (t - \tau); \beta_m = m\pi/l; M_m = l, l/2$

5. Numerical examples

For a specific problem, the solution requires three main steps:

- 1) determine the geometry, initial boundary conditions and material constants;
- 2) obtain the Green's function according to Table 3;
- 3) obtain the temperature solution from the Green's function solution equation Eq. (40), in which the boundary conditions part is referred to Table 2.

To validate the present solution to hyperbolic heat conduction problems, selected numerical examples are presented below in the case of C-V model. For convenience, the governing equations (1) and (2) of the C-V model are non-dimensionalized, as:

$$q^* + \frac{1}{2} \frac{\partial q^*}{\partial t^*} = -\frac{1}{2} \frac{\partial \theta^*}{\partial x^*} \quad (64)$$

and

$$\frac{\partial \theta^*}{\partial t^*} = Q^* - \frac{\partial q^*}{\partial x^*} \quad (65)$$

by introducing the following dimensionless quantities: $\theta^* = \frac{\theta}{\theta_{con}}, q^* = \frac{q\sqrt{\alpha\tau}}{k\theta_{con}}, Q^* = \frac{2\tau Q}{c}, x^* = \frac{x}{2\sqrt{\alpha\tau}}, t^* = \frac{t}{2\tau}$,

where θ_{con} is a constant temperature value, $\alpha = k/c$ and $c = \rho c_E$. In the following, the asterisk will be neglected for brevity. Combing (64) and (65), one has:

$$\frac{\partial^2 \theta}{\partial t^2} + 2 \frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial x^2} + f(x, t) \text{ and } f(x, t) = \frac{\partial Q}{\partial t} + 2Q \quad (66)$$

The solution to (66) has previously been obtained by Lam and Fong (2011a and 2011b) using the solution structure

theorem. For the present study, the solution is obtained by using the method of Green's function, with $A = 0.5$, $B^2 = 0.5$ and $f_Q(x, t) = 0.5f(x, t)$.

5.1 Example 1

A slab is subjected to a spatially decaying heat generation $f_Q(x, t) = 0.5f(x, t) = 50e^{-5x}$ with both sides thermally insulated, as: $X22 \left(\frac{\partial \theta(0, t)}{\partial x} = 0, \frac{\partial \theta(1, t)}{\partial x} = 0 \right)$, and the initial conditions are given as:

$$\theta(x, 0) = \varphi(x) = \sin\left(\frac{\pi}{2}x\right); \quad \dot{\theta}(x, 0) = \psi(x) = 50e^{-5x}.$$

According to Table 3, Green's function in such case is:

$$G = e^{\tau-t} \frac{2}{\sqrt{-1}} \sin \sqrt{-1}(t-\tau) + \sum_{m=1}^{\infty} e^{\tau-t} \frac{4}{\sqrt{(m\pi)^2 - 1}} \cos m\pi x' \cos m\pi x \sin \sqrt{(m\pi)^2 - 1}(t-\tau) \quad (67)$$

Thus, the solution may be expressed, as:

$$\begin{aligned} \theta(x, t) = & \int_0^1 G(x, t|x', 0) \varphi(x') dx' + \int_0^1 AG(x, t|x', 0) \psi(x') dx' \quad (\text{for initial conditions}) \\ & - \int_0^1 A \frac{\partial G(x, t|x', 0)}{\partial \tau} \varphi(x') dx' \quad (\text{for initial conditions}) \\ & + \int_0^t \int_0^1 G(x, t|x', \tau) f_Q(x', \tau) dx' d\tau \quad (\text{for energy generation}) \end{aligned} \quad (68)$$

The results calculated by different value of m is presented in Fig. 1 to show the convergence of the temperature for a specific point and instant ($x = 0.5$, $t = 0.5$) (Torabi and Saedodin, 2011). The predicted temperature distributions are presented in Fig. 2, and compared with existing results of Lam and Fong (2011a). Excellent agreement is achieved, indicating the validity of using the Green's function method to solve hyperbolic heat conduction problems, and the correctness of the present solution.

Fig. 1. Non-dimensional temperature at $x = 0.5$, $t = 0.5$ calculated with different m .

Fig. 2. Distribution of non-dimensional temperature at selected time instants ($m = 50$).

5.2 Example 2

The slab is heated by a time-varying and spatially-decaying heat generation, as:

$$f_Q(x, t) = \frac{1}{2} f(x, t) = 50e^{-5x} \left(\frac{1}{2} e^{-t} + \frac{3}{2} e^{-5t} \right) \text{ with boundary conditions: } X21 \left(\frac{\partial \theta(0, t)}{\partial x} = 0, \theta(1, t) = 0 \right). \text{ And the}$$

initial conditions is given: $\theta(x, 0) = \varphi(x) = \sin(\pi x)$; $\dot{\theta}(x, 0) = \psi(x) = 0$.

Green's function of this issue is expressed:

$$G = \sum_{m=0}^{\infty} e^{\tau-t} \frac{4}{\sqrt{\left(\frac{1}{2}(2m+1)\pi\right)^2 - 1}} \cos \frac{1}{2}(2m+1)\pi x' \cos \frac{1}{2}(2m+1)\pi x \sin \sqrt{\left(\frac{1}{2}(2m+1)\pi\right)^2 - 1} (t - \tau) \quad (69)$$

and the solution has the from:

$$\begin{aligned} \theta(x, t) = & \int_0^1 G(x, t | x', 0) \varphi(x') dx' - \int_0^1 A \frac{\partial G(x, t | x', 0)}{\partial \tau} \varphi(x') dx' \quad (\text{for initial conditions}) \\ & + \int_0^t \int_0^1 G(x, t | x', \tau) f_Q(x', \tau) dx' d\tau \quad (\text{for energy generation}) \end{aligned} \quad (70)$$

The convergence of the temperature for a specific point and instant ($x = 0.5$, $t = 0.5$) is referred to Fig. 1. The predicted temperature distributions are shown in Fig. 3. The predictions calculated from (70) agree excellently well with those obtained using the solution structure theorem (Lam and Fong, 2011b).

(a)

(b)

Fig. 3. Non-dimensional temperature at different time instants ($m = 50$).

Thus far, the terms associated with initial conditions and energy generation of the solution (40) as well as the Green's function shown in Table 3 is verified. To check the correctness of the terms associated with boundary conditions, a further case study is needed.

5.3 Example 3

A slab is heated at the left surface while the opposite side is kept to zero temperature, i.e. $X11 (\theta(0,t)=1, \theta(1,t)=0)$. Heat generation and initial conditions are not considered.

Green's function in this case is:

$$G = \sum_{m=1}^{\infty} e^{-\tau-t} \frac{4}{\sqrt{(m\pi)^2 - 1}} \sin m\pi x' \sin m\pi x \sin \sqrt{(m\pi)^2 - 1} (t - \tau) \quad (71)$$

and the solution is obtained, as:

$$\theta(x, t) = \int_0^t \frac{1}{2} \frac{\partial G(x' = 0, \tau)}{\partial x'} d\tau \quad (72)$$

Fig. 4. Non-dimensional temperature at $x = 0.025, 0.1, 0.25, 0.4, 0.475$ using different m .

Fig. 5. Distribution of non-dimensional temperature at selected time instants.

In Fig. 4, non-dimensional temperatures at five specific points ($x = 0.025, 0.1, 0.25, 0.4, 0.475$) are plotted at the instant $t = 0.5$ to show the convergence. It is obtained that the convergence of results at $x = 0.1, 0.25, 0.4$ is better than that of results at $x = 0.025, 0.475$, which is rational because there exist jumps at $x = 0$ and 0.5 in the instant $t = 0.5$ (see Fig. 5). To get an accurate result, m is adopted as 1000 in following calculation. According to the property

of wave function, the velocity of heat conduction in Eq. (66) is 1, so the wave front should be $x = 0.25, 0.5$ and 0.75 at time $t = 0.25, 0.5$ and 0.75 , respectively, which means that the temperature vanishes once $x > 0.25, 0.5$ and 0.75 . The predicted temperature distribution along the x -axis is shown in Fig. 5 for $t = 0.25, 0.5$ and 0.75 , from which it is seen that sharp jumps do exist at $x = 0.25, 0.5$ and 0.75 , and the temperature is essentially undisturbed on the right hand side of each jump. In other words, the predictions match well with analytical results, indicating that the present method based on Green's function is efficient. However, it is found that there exists dispersions (or wiggling) at the front of temperature wave, which may be also observed from Fig. 4. The drawback can be explained by the fact that the solution from (71) and (72) are expressed in form of trigonometric series. Meanwhile, it is expected that the jumps at $x = 0.25, 0.5$ and 0.75 could be described by selecting more trigonometric series terms. To overcome this defect, for hyperbolic heat conduction with zero heat generation and initial conditions, we adopt the Laplace transform method. And two kind of boundary conditions are considered. Here, the general model without heat generation is applied, i.e. Eqs. (15) and (19). The Laplace transform of (15) is

$$B^2 \frac{\partial^2 \bar{\theta}(x, s)}{\partial x^2} - s(As + 1)\bar{\theta}(x, s) = 0 \quad (73)$$

with the aid of zero initial conditions and zero heat generation, where s is the Laplace transform variable, and $\bar{\theta}$ is the temperature in Laplace domain. The solution to (73) may be obtained

$$\bar{\theta}(x, s) = \bar{\theta}_1 \exp(-k_1 x) + \bar{\theta}_2 \exp(k_1 x) \quad (74)$$

where $k_1 = \sqrt{\frac{s(As + 1)}{B^2}}$. And $\bar{\theta}_i$ ($i = 1, 2$) are to be determined using the boundary conditions. The boundary conditions at $\tilde{x}_\alpha = 0, l$ ($\alpha = \text{left, right}$) in Laplace transform is:

The first kind:

$$\theta(\tilde{x}_\alpha, s) = f_\alpha(s) \quad (75)$$

The second kind:

$$\theta(\tilde{x}_\alpha, s) = f_\alpha(s) \quad (76)$$

The Laplace transform of (19) has the form:

$$(1 + \tau s)q(x, s) = -k \frac{\partial \theta(x, s)}{\partial x} \quad (77)$$

Thus, the second kind of boundary condition is rewritten, as:

$$k \frac{\partial \theta(\tilde{x}_\alpha, s)}{\partial x} = -(1 + \tau s)g_\alpha(s) \quad (78)$$

Based on the boundary conditions, e.g. (75) and (78), $\bar{\theta}_i$ ($i = 1, 2$) in Eq. (74) may be obtained. Thus far, the solutions in Laplace domain have been completed. For Example 3, the boundary condition at the left side may be expressed in Laplace domain as $\bar{\theta}(0, t) = 1/s$, so the analytical solution to HHC in Laplace domain is $\bar{\theta}(x, s) = \exp(-k_1 x)/s$. To obtain the temporal solutions, we adopt an algorithm NILT proposed by Brancik (1999), which is based on fast Fourier transformation, and is improved using a quotient-difference algorithm. Its efficiency and feasibility is evaluated, even in fractional calculus (Sheng et al., 2011). Using this procedure, the Example 3 is reconsidered, and the result is shown in Fig. 6. The temperature's distributions are same as that from Green's function, while the dispersions around the front of the temperature wave are avoided.

Fig. 6. Distribution of non-dimensional temperature at selected time instants using Laplace transform method.

6. Conclusion

The reciprocity relation of Green's function is firstly proven in the context of a general hyperbolic heat conductive model, such as Cattaneo-Vernotte model, the simplified thermomass model, and the single-phase-lag two-step model. And then, the corresponding Green's function solution equation is derived using Green's functions method. Subsequently, explicit Green's functions are formulated under several different sets of boundary conditions. For validation, selected numerical studies are carried out and excellent agreement is achieved with existing results using the solution structure theorem and analytical results. In addition, to avoid the dispersions around the front of temperature wave, a procedure based on Laplace transform method is proposed. More importantly, the newly proposed solution may be applied to similar problems, such as non-Fick transient mass transfer and diffusion.

Acknowledgements

This study was funded by National Key R&D Problem of China (No. 2017YFB1102801), National Natural Science Foundation of China (Nos. 11572237), and Fundamental Research Funds for the Central Universities (3102017OQD072).

References

- Anisimov, S.I., Kapeliovich, B.L., Perelman, T.L., Electron emission from metal surfaces exposed to ultra-short laser pulses, *Sov. Phys. JETP*, Vol. 39 (1974), pp. 375-377.
- Brancik, L., Programs for fast numerical inversion of Laplace transforms in MATLAB language environment, *Proceedings of the 7th Conference MATLAB'99*, Czech Republic, Prague, (1999), pp. 27-39.
- Cao, B.Y., Guo, Z.Y., Equation of motion of a phonon gas and non-Fourier heat conduction, *J. Appl. Phys.*, Vol. 102, No. 5 (2007), pp. 053503.
- Cattaneo, C., A form of heat equation which eliminates the paradox of instantaneous propagation, *CR Acad Sci.*, Vol. 247 (1958), pp. 431-433.
- Chen, T.M., A hybrid Green's function method for the hyperbolic heat conduction problems, *Int. J. Heat Mass Transfer*, Vol. 52 (2009), pp. 4273-4278.
- Chen, T.M., Numerical solution for the hyperbolic heat conduction problems in the radial-spherical coordinate system using a hybrid Green's function method, *Int. J. Heat Mass Transfer*, Vol. 53 (2010), pp. 1319-1325.

- Chen, T.M., Chen, C.C., Numerical solution for the hyperbolic heat conduction problems in the radial-spherical coordinate system using a hybrid Green's Function problems in the radial-spherical coordinate system using a hybrid Green's Function, *Int. J. Therm. Sci.*, Vol. 49 (2010), pp. 1193-1196.
- Cole, K.D., Haji-Sheikh, A., Beck, J.V., Litkouhi, B., Heat conduction using Green's Functions, CRC Press, New York (2010).
- Eckart, C., The thermodynamics of irreversible processes III relativistic theory of the simple fluid, *Phys. Rev.*, Vol. 58, No. 10 (1940), pp. 919-924.
- Fernandes, A.P., Sousa, P.F.B., Borges, V.L., Guimaraes, G., Use of 3D-transient analytical solution based on Green's function to reduce computational time in inverse heat conduction problems, *Applied Mathematical Modelling*, Vol. 34 (2010), pp. 4040-4049.
- Fong, E., Lam, T.T., Asymmetrical collision of thermal waves in thin films: An analytical solution, *International Journal of Thermal Sciences*, Vol. 77 (2014), pp. 55-65.
- Frankel, J.I., Vick, B., Ozisik, M.N., General formulation and analysis of hyperbolic heat conduction in composite media, *Int. J. Heat Mass Transfer*, Vol. 30, No. 7 (1987), pp. 1293-1305.
- He T.H., Tian, X.G., Shen, Y.P., A two-dimensional generalized thermal shock problem for a half-space in electromagneto-thermoelasticity, *International Journal of Engineering Science*, Vol. 42 (2004), pp. 809-823.
- Gheitaqhy, A.M., Talaei, M.R., Solving hyperbolic heat conduction using electrical simulation, *Journal of Mechanical Science and Technology*, Vol. 27, No. 12 (2013), pp. 3885-3891.
- Gray, L.J., Kaplan, T., Richardson, J.D., Paulino, G.H., Green's functions and boundary integral analysis for exponentially graded materials: heat conduction, *J. Appl. Mech. Trans. ASME*, Vol. 70 (2003), pp. 543-549.
- Guo, Z.Y., Hou, Q.W., Thermal wave based on the thermomass model, *J. Heat Transfer, Trans. ASME*, Vol. 132 (2010), pp. 653-666.
- Haji-Sheikh, A., Beck, J.V., Green's function solution for thermal wave equation in finite bodies, *Int. J. Heat Mass Transfer*, Vol. 37, No. 17 (1994), pp. 2615-2626.
- Haji-Sheikh, A., Minkowycz, W.J., Sparrow, E.M., Certain anomalies in the analysis of hyperbolic heat conduction, *J. Heat Transfer, Trans. ASME*, Vol. 124, No. (2002), pp. 307-319.
- Haji-Sheikh, A., Monte, F.D., Beck, J.V., Temperature solutions in thin films using thermal wave Green's function solution equation, *International Journal of Heat and Mass Transfer*, Vol. 62 (2013), pp. 78-86.
- Hays-Stang, K.J., Haji-Sheikh, A., A unified solution for heat conduction in thin films, *Int. J. Heat Mass Transfer*, Vol. 42 (1999), pp. 455-465.
- Li, S.N., Cao, B.Y., Entropic constitutive relation and modeling for Fourier and hyperbolic heat conductions, *Entropy*, Vol. 19 (2018), pp. 644-656.
- Kim, K.S., Noda, N., Green's function approach to three-dimensional heat conduction equation of functionally graded materials, *J. Therm. Stresses*, Vol. 24 (2001), pp. 457-477.
- Landau, L.D., Lifshitz, E.M., *Fluid Mechanics*, Pergamon, London (1959).
- Lam, T.T., A unified solution of several heat conduction models, *Int. J. Heat Mass Transfer*, Vol. 56 (2013), pp. 653-666.
- Lam, T.T., Thermal propagation in solids due to surface laser pulsation and oscillation, *International Journal of Thermal Sciences*, Vol. 49, No. 9 (2010), pp. 1639-1648.
- Lam, T.T., Fong, E., Application of solution structure theorems to Cattaneo-Vernotte heat conduction equation with non-homogeneous boundary conditions, *Heat and Mass Transfer*, Vol. 49, No. 4 (2012), pp. 509-519.
- Lam, T.T., Fong, E., Application of solution structure theorem to non-Fourier heat conduction problems: Analytical approach, *Int. J. Heat Mass Transfer*, Vol. 54 (2011a), pp. 4796-4806.
- Lam, T.T., Fong, E., Heat diffusion vs. wave propagation in solids subjected to exponentially-decaying heat source: Analytical solution, *Int. J. Therm. Sci.*, Vol. 50 (2011b), pp. 2104-2116.
- Loureiro, F.S., Mansur, W.J., Vasconcellos, C.A.B., A hybrid time/Laplace integration method based on numerical Green's functions in conduction heat transfer, *Comput. Methods Appl. Mech. Engrg.*, Vol. 198 (2009), pp. 2662-2672.
- Loureiro, F.S., Mansur, W.J., A new family of time integration methods for heat conduction problems using numerical green's functions, *Comput. Mech.*, Vol. 44 (2009), pp. 519-531.
- Ma, Y.B., He, T.H., The transient response of a functionally graded piezoelectric rod subjected to a moving heat source

- under fractional order theory of thermoelasticity, *Mechanics of Advanced Materials and Structures*, Vol. 24, No. 9 (2017), pp. 789-796.
- Majumdar, P., Xia, H., A Green's function model for the analysis of laser heating of materials, *Applied Mathematical Modelling*, Vol. 31 (2007), pp. 1186-1200.
- Mitra, K., Kumar, A., Vedavarz, A., Moallemi, M.K., Experimental evidence of hyperbolic heat conduction in processed meat, *J. Heat Transfer, Trans. ASME*, Vol. 117, No. 3 (1995), pp. 568-573.
- Monteiro, E.R., Macedo, E.N., Quaresma, J.N.N., Cotta, R.M., Integral transform solution for hyperbolic heat conduction in a finite slab, *International Communications in Heat and Mass Transfer*, Vol. 36, No. 4 (2009), pp. 297-303.
- Nernst, W., *Die Theoretischen Grundlagen Des Neuen Wärmesatzes*, Knapp, Halle (1917).
- Onsager, L., Reciprocal relations in irreversible processes. I, *Phys. Rev.*, Vol. 37, No. 4 (1931), pp. 405-426.
- Peshkov, V., Second sound in helium II, *J. Phys.*, Vol. 8 (1994), pp. 381-382.
- Qui, T.Q., Tien, C.L., Short-pulse laser heating on metals, *Int. J. Heat Mass Transfer*, Vol. 35 (1992), pp. 719-726.
- Qui, T.Q., Tien, C.L., Heat transfer mechanisms during short pulse laser heating on metals, *J. Heat Transfer, Trans. ASME*, Vol. 115 (1993), pp. 835-841.
- Ren, Y., Chen, J.K., Zhang, Y., Modeling of ultrafast phase changes in metal films induced by an ultrashort laser pulse using a semi-classical two-temperature model, *International Journal of Heat and Mass Transfer*, Vol. 55 (2012), pp. 1620-1627.
- Sheng, H., Li, Y., and Chen, Y., Application of numerical inverse Laplace transform algorithms in fractional calculus, *Journal of the Franklin Institute*, Vol. 348, No. 2 (2011), pp. 315-330.
- Sobolev, S.L., Hyperbolic heat conduction, effective temperature, and third law for nonequilibrium systems with heat flux, *Physical Review E*, Vol. 97 (2018), pp. 022122.
- Tien, C.L., Lienhard, J.H., *Statistical Thermodynamics*, Hemisphere, Washington, DC (1979).
- Tolman, R.C., On the weight of heat and thermal equilibrium in general relativity, *Phys. Rev.*, Vol. 35, No. 8 (1930), pp. 904-924.
- Torabi, M., Saedodin, S., Analytical and numerical solutions of hyperbolic heat conduction in cylindrical coordinates, *J. Thermophys. Heat Tran.*, Vol. 25, No. 2 (2011), pp. 239-253.
- Tzou, D.Y., On the thermal shock wave induced by a moving heat source, *J. Heat Transfer ASME*, Vol. 111 (1989), pp. 232-238.
- Tzou, D., Thermal shock phenomena under high-rate response in solids, *Annual Review of Heat Transfer*, Vol. 4, No. 4 (1992), pp. 111-185.
- Vernotte, P., Paradoxes in the continuous theory of the heat conduction, *CR Acad Sci.*, Vol. 246 (1958), pp. 3154-3155.
- Wang, L.Q., Zhou, X.S., Wei, X.H., *Heat conduction: mathematical models and analytical solutions*, Springer-Verlag, Heidelberg, Germany (2008).
- Xiong, Q.L., Tian, X.G., Response of a semi-infinite microstretch homogeneous isotropic body under thermal shock, *Journal of Applied Mechanics*, Vol. 78 (2011), pp. 044503.
- Xiong, Q.L., Tian, X.G., Lu, T.J., Atomistic modeling of electron relaxation effect on femtosecond laser-induced thermoelastic response of gold films, *Journal of Applied Physics*, Vol. 112 (2012), pp. 024313.
- Xiong, Q.L., Li, Z.H., Tian, X.G., Ultrafast thermomechanical responses of a copper film under femtosecond laser trains: a molecular dynamics study, *Proc. R. Soc. A*, Vol. 471 (2015), pp. 20150614.
- Xiong, Q.L., Li, Z.H., Kitamura, T., Effect of crystal orientation on femtosecond laser-induced thermomechanical responses and spallation behaviors of copper films, *Scientific Reports*, Vol. 7 (2017), pp. 9218.