Analytical solutions to hyperbolic heat conductive models using Green’s function method

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Abstract
In this work, the existing theoretical heat conductive models such as: Cattaneo-Vernotte model, simplified thermomass model, and single-phase-lag two-step model are summarized, and then a general model of hyperbolic heat conduction (HHC) is presented with boundary conditions prescribed as: (1) temperature at the boundary; (2) heat flux (not the temperature gradient) at the boundary. The convective boundary condition is not considered because it is impossible to produce a fluid motion at such time scale, e.g. picoseconds. In the context of HHC, the reciprocity relation of Green’s function is systematically proven, based on which Green’s function solution equation of the general model is completely derived. Meanwhile, Green’s functions are deduced under several different sets of boundary conditions. Thus, solution to HHC based on Green’s function is obtained, and it consists of three parts, i.e. boundary conditions, initial conditions and heat generations. The priority of the present solution is that the time-dependent boundary condition and heat generation may be dealt with, and that it is easy to extend the solution to multi-dimensional case. The accuracy of the solution is verified through numerical examples, and Laplace transform method is adopted to avoid the dispersions around the front of temperature wave for jump-type boundary conditions, thus the solution is further perfected.

Keywords: Analytical solution, Hyperbolic heat conduction, Green’s function, Laplace transform method

1. Introduction

Classical heat conduction, as a mode of heat transfer, is driven by temperature gradient in solids or liquids. The classical parabolic heat conduction based on Fourier’s law is sufficiently accurate for most common engineering situations. However, parabolic heat conduction breaks down in extreme situations, such as very short duration, high heat fluxes, and very low temperatures (Frankel et al., 1987; Xiong, et al., 2017), because it contradicts physical facts (Ma and He, 2017; Mitra et al., 1995; Peshkov, 1994; Ren et al., 2012; Xiong, et al., 2015) and admits an infinite speed (He et al., 2004; Xiong, et al., 2011, 2012). To eliminate these paradoxes, various theories of non-classical heat conduction (or heat wave, heat conduction with second sound) have been proposed.

To permit a finite speed of heat conduction, Cattaneo (1958) and Vernotte (1958) proposed a new form of the heat conduction law by introducing a relaxation term and resulting in a hyperbolic differential equation (C-V model). The C-V model abandons the assumption of classical heat conduction law that the heat flux and the temperature gradient across a material volume occur at the same instant of time. For specific initial boundary value problems, the C-V model introduces a sharp wave front in the history of heat wave propagation. Recently, the theoretical basis of hyperbolic heat conduction is systematically discussed (Li and Cao, 2018; Sobolev, 2018).

On a separate front, the thermomass model (Guo and Hou, 2010) was introduced to further study heat conduction mechanisms at microscale or even nanoscale, with the constitutive equation derived based on the phonon gas equation of motion. Cao and Guo (2007) described heat transfer in solids due to phonon gas motion using Einstein’s mass-energy relationship, confirming that the classical heat conduction broke down in the cases of ultrahigh heat fluxes and ultralow temperature.

With wide application of short-pulse energy deposition in micromachining, investigations interest on microscale heat conduction has been aroused. At microscale, the temperature response may be dramatically affected by the
individual heat carriers, such as phonons, electrons, and photons. It is widely known that the lattice energy is due to the thermal vibration of the solid, while the lattice vibration energy is quantized and called as a phonon in the language of quantum. In metals, heat conduction is carried both by phonons and free electrons, while in semiconductors heat energy is transferred mainly by the phonons. A two-step model was proposed by distinguishing the electron temperature from the lattice temperature (Anisimov et al., 1974; Qui and Tien, 1992; Qui and Tien, 1993): the electrons in metals are firstly excited by electrons-phonon interaction into higher energy level; the excited electrons produce a hot free electron gas, then the electron gas diffuses through the metal lattice and generates phonons by collision with the lattice (Hays-Stang and Haji-Sheikh, 1994).

It has been well established that Green’s function is an effective way to deal with initial boundary partial differential equations, and hence it has been widely applied to solve heat conduction problems. For typical example, Cole et al. (2010) derived Green’s functions of heat conduction associated with various initial boundary conditions in rectangular, cylindrical and spherical coordinates systems. Kim and Noda (2001) solved three-dimensional (3D) heat conduction equation of functionally graded materials by adopting a Green’s function approach based on the laminate theory. Gray et al. (2003) derived free space Green’s functions for exponentially graded materials, and an explicit Green’s approach for heat conduction was also proposed based on Green’s function calculated in the Laplace domain (Loureiro et al., 2009; Loureiro and Mansur, 2009). Majumdar and Xia (2007) analyzed the laser heating of materials using Green’s function method. In addition, Green’s function was also applied in inverse transient heat conduction problems (Fernandes et al., 2010).

In the context of HHC, Haji-Sheikh and Beck (1994) solved the temperature solution for thermal wave equation. And later, Haji-Sheikh et al. (2002) identified the anomalies of the hyperbolic heat conduction and suggested appropriate remedies. Around 2010, Chen solved hyperbolic heat conduction problems using a hybrid Green’s function method with zero initial conditions and the heat generation was also neglected (Chen, 2009, 2010; Chen and Chen, 2010). Recently, Haji-Sheikh et al. (2013) once again solved the temperature solutions in thin films using thermal wave Green’s function solution equation. Chen’s work (Chen, 2009) was limited to zero initial conditions and the Green’s function was obtained in the Laplace domain with the energy generation neglected. In the recent paper of Haji-Sheikh et al. (2013), the Green’s function in infinite solids is firstly obtained. However, for thermal wave theory, the disturbed region is zero to the front of temperature wave. If the thermal signal does not arrive at the boundary, it may be considered as an infinite problem. So HHC in finite medium is adequate to express practical responses. In this work, HHC in finite medium is considered making use of the Green’s function method: (1) the energy generation term and the initial conditions are selected arbitrarily. (2) several different sets of boundary conditions are considered to apply Green’s function method in hyperbolic heat conductive problems to the fullest extent. For second kind boundary condition, the heat flux is directly given instead of temperature gradient type boundary conditions. (3) the Green’s function solution equation for finite medium is derived in time domain under different boundary conditions. In addition, in the formulation, the reciprocity relation of Green’s function is firstly proven in the context of HHC, and then Green’s function solution equation is obtained. As a result, the formulation is impeccable. By substituting boundary conditions, initial conditions, heat generation, and Green’s function into the Green’s function solution equation, the solution to the considered problem is thus obtained. The solution procedure is validated by numerical implementations. Finally, the results for boundary conditions part is improved by adopting Laplace transform method.

2. Theoretical models

As previously mentioned, hyperbolic heat conduction (or heat conduction with second sound) was formulated to eliminate the paradox of infinite speed of energy transfer inherent in the classical diffusion heat conduction model, and to describe more accurately heat transfer in low temperature or high power. Three theoretical models of hyperbolic heat conduction are presented in this section.

2.1 Cattaneo-Vernotte model

The simplest and earliest linear hyperbolic heat conduction theory is the Cattaneo-Vernotte model. To account for the existence of second sound detected in liquid helium (Peshkov, 1994), Cattaneo (1958) and Vernotte (1958) proposed a modified heat conduction equation. For homogeneous and isotropic materials, it reads as:
\[ q + \tau_{CV} \frac{\partial q}{\partial t} = -k \frac{\partial \theta}{\partial x} \]  

(1)

Combining with the equation of energy conservation:

\[ \rho c_p \frac{\partial \theta}{\partial t} = Q - \frac{\partial q}{\partial x} \]  

(2)

one obtains the governing equation for temperature, as:

\[ \tau_{CV} \rho c_p \frac{\partial^2 \theta}{\partial t^2} + \rho c_p \frac{\partial \theta}{\partial t} = k \frac{\partial^2 \theta}{\partial x^2} + \left( Q + \tau_{CV} \frac{\partial Q}{\partial t} \right) \]  

(3)

Equation (3) indicates the C-V model and takes the form of wave, where \( q, \tau_{CV}, k, \theta, \rho, c_p, Q \) denote heat flux, relaxation time, thermal conductivity, temperature, density, specific heat and heat generation, respectively.

### 2.2 Simplified thermomass model

Since 1917, the concept of inertia for heat has been suggested (Nernst, 1917; Onsager, 1931), while a quantitative description of the inertia has seldom been reported except those (Tolman, 1930; Eckart, 1940; Landau and Lifshitz, 1959). Recognizing that heat conduction in dielectrics is attributed to the motion of “phonon gas”, Cao and Guo (2007) introduced the concept of “thermomass” according to Einstein’s mass-energy relation, and proposed the state equation of phonon gas based on the Debye state equation for solids (Tien and Lienhard, 1979). Then, an equation of motion was established using Newtonian dynamics with consideration of the driving, inertia and resistant forces. A non-Fourier heat conduction was also formulated, as:

\[ \left( 1 - \frac{q^2}{\rho c_p \varepsilon \theta} \right) k \frac{\partial \theta}{\partial x} + q = 0 \]  

(4)

The law indicates that Fourier’s law breaks down under such conditions: ultrahigh heat fluxes and ultralow temperatures. Later, Guo and Hou (2010) derived the equation for one-dimensional heat conduction as:

\[ \tau_{TM} \frac{\partial q}{\partial t} - l_{TM} \rho c_p \frac{\partial \theta}{\partial t} + l_{TM} \frac{\partial q}{\partial x} = b k \frac{\partial \theta}{\partial x} + k \frac{\partial \theta}{\partial x} + q = 0 \]  

(5)

where \( \tau_{TM}, l_{TM} \) and \( b \) are relaxation time, characteristic length parameter and square of thermal Mach number of the phonon gas, respectively. If the thermal disturbance is small, one obtains the simplified thermomass model by neglecting the terms with \( b \) and \( l_{TM} \), as:

\[ \tau_{TM} \frac{\partial q}{\partial t} + k \frac{\partial \theta}{\partial x} + q = 0 \]  

or

\[ q + \tau_{TM} \frac{\partial q}{\partial t} = -k \frac{\partial \theta}{\partial x} \]  

(6)

Combining with the continuity equation of phonon gas, one may obtain a thermal wave equation with the heat generation included, as:

\[ \tau_{TM} \rho c_p \frac{\partial^2 \theta}{\partial t^2} + \rho c_p \frac{\partial \theta}{\partial t} = k \frac{\partial^2 \theta}{\partial x^2} + \left( Q + \tau_{TM} \frac{\partial Q}{\partial t} \right) \]  

(7)
Apparently, Eq. (5) is a new extension of Fourier’s law, while they are essentially different: the former is the motion equation of phonon gas, while the latter is an empirical formula. Detailed explanation for the physical meaning of $\tau_{TM}$ is also presented: the characteristic time describes a lagging response, in time, between the drift velocity of the phonon gas and the driving force of the phonon gas (Guo and Hou, 2010). It is noted that this new model may not reduce to Fourier’s law even for steady heat conduction.

2.3 Single-phase-lag two-step model

To develop the single-phase-lag two-step model, energy input of a substance is presumed to be absorbed by electron and lattice, and the electrons gas temperature is different from the lattice temperature, and both are related via (Anisimov et al., 1974; Qui and Tien, 1992; Qui and Tien, 1993):

$$C_l \frac{\partial \theta}{\partial t} + C_e \frac{\partial \theta}{\partial t} = \Gamma \left[ \theta_e - \theta_l \right]$$  \hspace{1cm} (8)

where $C_l$ and $\Gamma$ denote heat capacity of lattice and electron-phonon coupling factor; $\theta_e$ and $\theta_l$ are temperature of electron and lattice, respectively. With the contribution of the energy storage of electron gas and lattice included, the equation of energy balance may be written as:

$$C_e \frac{\partial \theta_e}{\partial t} + C_e \frac{\partial \theta_l}{\partial t} = Q - \frac{\partial q}{\partial x}$$  \hspace{1cm} (9)

where $C_e$ is heat capacity of electron. It follows from (8) and (9) that:

$$C_l \frac{\partial \theta}{\partial t} + C_e \frac{\partial \theta_e}{\partial t} + \tau_e \frac{\partial^2 \theta}{\partial t^2} = \Gamma \left[ \theta_e - \theta_l \right]$$  \hspace{1cm} (10)

where $C = C_e + C_l$ and $\tau_e = C_eC_l/\Gamma C$. Considering the time-lag between the temperature gradient and the resulting heat flux, one has (for brevity, $\theta_l$ will be denoted as $\theta$) (Tzou, 1989, 1992):

$$q(t + \tau_q) = -k \frac{\partial \theta(t)}{\partial x}$$  \hspace{1cm} (11)

where $\tau_q$ is relaxation time. Expanding (11) in the Taylor series and retaining the first two terms yields:

$$q + \tau_q \frac{\partial q}{\partial t} = -k \frac{\partial \theta(t)}{\partial x}$$  \hspace{1cm} (12)

By eliminating $q$ between Eqs. (10) and (12), one may obtains, as:

$$C_l \frac{\partial \theta}{\partial t} + C_l \frac{\partial \theta_e}{\partial t} + \tau_e \frac{\partial^2 \theta_e}{\partial t^2} + C_e \frac{\partial \theta_l}{\partial t} + \tau_e \frac{\partial^2 \theta_l}{\partial t^2} = k \frac{\partial^2 \theta}{\partial x^2} + \left( Q + \tau_q \frac{\partial Q}{\partial t} \right)$$  \hspace{1cm} (13)
where the parameters $\tau_e$ and $\tau_q$ are quite small (Hays-Stang and Haji-Sheikh, 1999), as a result, the third term on the left side of (13) can be discarded, yielding:

$$C\left(\tau_e + \tau_q\right)\frac{\partial^2 \theta}{\partial t^2} + C\frac{\partial \theta}{\partial t} = k\frac{\partial^2 \theta}{\partial x^2} + \left(Q + \tau_q \frac{\partial Q}{\partial t}\right)$$

(14)

Note that no reference is made to the type of material studied. That is, by selecting appropriate values of $\tau_e$ and $\tau_q$, Eq. (14) holds for dielectric materials, conductors or semiconductors (Hays-Stang and Haji-Sheikh, 1999).

3. General hyperbolic heat conduction model

A unified form of the three hyperbolic heat conduction models is presented (Lam, 2013), as:

$$A\frac{\partial^2 \theta}{\partial t^2} + \frac{\partial \theta}{\partial t} = B^2\frac{\partial^2 \theta}{\partial x^2} + f_\theta(x,t)$$

(15)

$$f_\theta(x,t) = D \cdot Q + E \cdot \frac{\partial Q}{\partial t}$$

(16)

The C-V model, the simplified thermomass model, and the single-phase-lag two-step model can be readily obtained with a proper selection of the coefficients $A, B^2, D, E$, as demonstrated in Table 1.

<table>
<thead>
<tr>
<th>Thermal model</th>
<th>Coefficient</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cattaneo-Vernotte model</td>
<td>$\tau_{CV}$ $\alpha$ $1/\rho c_E$ $\tau_{CV}/\rho c_E$</td>
</tr>
<tr>
<td>Simplified thermomass model</td>
<td>$\tau_{TM}$ $\alpha$ $1/\rho c_E$ $\tau_{TM}/\rho c_E$</td>
</tr>
<tr>
<td>Single-phase-lag two-step model</td>
<td>$\tau_e + \tau_q$ $k/C$ $1/C$ $\tau_q/C$</td>
</tr>
</tbody>
</table>

To completely describe hyperbolic heat conduction problems, the boundary conditions at $x_a = 0$ and $l$, where $\alpha$ (e.g., right, left) represents the right and left boundary, may be prescribed as:

The first kind:

$$\theta(x_a,t) = f_{\alpha}(t)$$

(17)

The second kind:

$$q(x_a,t) = g_{\alpha}(t)$$

(18)

Some works incautiously apply the form $k \frac{\partial \theta(x_a,t)}{\partial n_x} = g_{\alpha}(t)$ as boundary conditions for HHC (Gheitaghy and Talaee,
2013; Fong and Lam, 2014, Lam, 2010; Lam and Fong, 2012; Monteiro et al., 2009). Although the nonhomogeneous boundary condition can be solved by performing a functional transformation or using integral transform method, the meaning of the prescribed function \( g_a(t) \) is, however, obscure. So in this work, the heat flux at the boundary is prescribed. In Eqs. (17) and (18), the functions \( f_a(t) \) and \( g_a(t) \) should be specified. To denote the second kind boundary condition (18) by temperature, we resort to the modified Fourier’s law, e.g. Eqs. (1) (6) and (12), which are commonly expressed as

\[
q + \tau \frac{\partial q}{\partial t} = -k \frac{\partial \theta}{\partial x}
\]

where \( \tau \) are \( \tau_1, \tau_2, \tau_q \), respectively. With aids of (18), it is obtained that

\[
k \frac{\partial \theta (\tilde{x}, t)}{\partial x} = f_a(t) \text{ and } g_a(t) = -(g_a + \tau g_a)
\]

In the present study, a variety of boundary conditions are considered by combing (17) and (20), and denoted by \( X_{ij}Bt \) \((i, j = 1, 2)\) with \( Bt \) indicating that \( f_a \) are function of time.

The initial conditions are prescribed as:

\[
\theta (x, 0) = \varphi (x)
\]

\[
\frac{\partial \theta (x, 0)}{\partial t} = \psi (x)
\]

4. Analytical solutions based on Green’s function

In this section, solutions to 1D hyperbolic heat conduction problems are obtained for selected boundary conditions at \( \tilde{x} = 0 \) and \( l \), and the functions \( f_a, \varphi \) and \( \psi \) are not specified to obtain a general solution.

4.1 Proof of reciprocity relation of Green’s function

The Green’s function of temperature governed by (15) and the corresponding initial boundary conditions is the solution to the following auxiliary equation:

\[
A \frac{\partial^2 G}{\partial t^2} + \frac{\partial G}{\partial t} = B^2 \frac{\partial^2 G}{\partial x^2} + \delta (x, l|\tilde{x}, \tau)
\]

with homogeneous boundary conditions and zero initial conditions(i.e., \( f_a = 0 \) and \( \varphi = \psi = 0 \)), Eq. (23) may be written alternatively, as:

\[
A \frac{\partial^2 G}{\partial t^2} (x, l|\tilde{x}, \tilde{t}) + \frac{\partial G}{\partial t} (x, l|\tilde{x}, \tilde{t}) = B^2 \frac{\partial^2 G}{\partial x^2} (x, l|\tilde{x}, \tilde{t}) + \delta (x, l|\tilde{x}, \tilde{t})
\]

\[
A \frac{\partial^2 G'}{\partial t^2} (x, -l|\tilde{x}, \tilde{t}) - \frac{\partial G'}{\partial t} (x, -l|\tilde{x}, \tilde{t}) = B^2 \frac{\partial^2 G'}{\partial x^2} (x, -l|\tilde{x}, \tilde{t}) + \delta (x, -l|\tilde{x}, \tilde{t})
\]
where the Green’s function in (25) is denoted by \( G' \) to distinguish from that in (24). Multiplying (24) by \( G' \left( x, x' \right) \) and (25) by \( G \left( x, x' \right) \), and subtracting, then integrating over region 0 to \( l \) and over time from 0 to \( t' \), one obtains:

\[
\int_0^l \left[ B \left( G' \frac{\partial^2 G}{\partial x^2} - G \frac{\partial^2 G'}{\partial x^2} \right) - \left( G' \frac{\partial G}{\partial t} + G \frac{\partial G'}{\partial t} \right) - A \left( G' \frac{\partial^2 G}{\partial t^2} - G \frac{\partial^2 G'}{\partial t^2} \right) \right] dx \, dt = G'\left( \hat{x}, -\hat{i} \right) - G\left( \hat{x}, \hat{i} \right)
\]

where \( t' \) is greater than either \( \hat{i} \) or \( \hat{i} \). Considering that:

\[
G' \frac{\partial^2 G}{\partial x^2} - G \frac{\partial^2 G'}{\partial x^2} = \frac{\partial}{\partial x} \left( G' \frac{\partial G}{\partial x} - G \frac{\partial G'}{\partial x} \right)
\]

\[
G' \frac{\partial^2 G}{\partial t^2} - G \frac{\partial^2 G'}{\partial t^2} = \frac{\partial}{\partial t} \left( G' \frac{\partial G}{\partial t} - G \frac{\partial G'}{\partial t} \right)
\]

one may rewrite (26) as:

\[
\int_0^l \left[ B \left( G' \frac{\partial^2 G}{\partial x^2} - G \frac{\partial^2 G'}{\partial x^2} \right) \right] dx - \int_0^l \left[ \left( G' \frac{\partial G}{\partial t} + G \frac{\partial G'}{\partial t} \right) \right] dx - \int_0^l \left[ A \left( G' \frac{\partial^2 G}{\partial t^2} - G \frac{\partial^2 G'}{\partial t^2} \right) \right] dx = G'\left( \hat{x}, -\hat{i} \right) - G\left( \hat{x}, \hat{i} \right)
\]

(29)

Here, the first term vanishes because both Green’s functions satisfy the same boundary conditions; the second and third terms can also be simplified considering the zero initial condition both Green’s functions share \( G\left( x, x' \right), 0 = G'\left( x, x' \right), 0 = 0 \) and the causality relation of Green’s function \( G\left( x, x' \right) = G'\left( x, x' \right) \). Accordingly, one has:

\[
G\left( \hat{x}, \hat{i} \right) = G'\left( \hat{x}, -\hat{i} \right)
\]

(30)

which is the reciprocity relation of Green’s function is obtained. In alternative form, (A.9) may be expressed as:

\[
G\left( x, x' \right) = G'\left( x', -x \right)
\]

(31)

### 4.2 Derivation of Green’s function solution equation

This subsection is devoted to the 1D Green’s function solution equation for the above-mentioned general hyperbolic heat conduction model referred to the formulation for heat conduction (Cole et al., 2010).

The Green’s function of temperature given by Eq. (15) and the corresponding initial boundary conditions is the solution to the following auxiliary equation:

\[
A \frac{\partial^2 G}{\partial t^2} + \frac{\partial G}{\partial t} = B^2 \frac{\partial^2 G}{\partial x^2} + \delta\left( x, t | x', \tau \right)
\]

(32)

with the homogeneous boundary conditions and zero initial conditions:
In view of the reciprocity relation (31), which is proved in the context of hyperbolic heat conduction in the previous subsection, the auxiliary equation (32) may be rewritten as:

$$\frac{\partial^2 G}{\partial \tau^2} - \frac{\partial G}{\partial \tau} = B^2 \frac{\partial^2 G}{\partial x^2} + \delta(x', \tau)$$  \hspace{1cm} (34)$$

The original hyperbolic heat conduction (15) is written in terms of \( x' \) and \( \tau \), yielding:

$$\frac{\partial^2 \theta}{\partial \tau^2} + \frac{\partial \theta}{\partial \tau} = B^2 \frac{\partial^2 \theta}{\partial x^2} + f(x', \tau)$$  \hspace{1cm} (35)$$

Multiplying (35) by \( G(x', \tau) \) and (34) by \( \theta(x', \tau) \), and subtracting, then integrating over region 0 to \( l \) and over time from 0 to \( t + \varepsilon \), one arrives at:

$$\int_0^{t+\varepsilon} \int_0^l \left( \frac{G \partial^2 \theta}{\partial \tau^2} - \theta \frac{\partial G}{\partial \tau} \right) dx'd\tau = \int_0^{t+\varepsilon} \int_0^l B^2 \left( \frac{\partial^2 \theta}{\partial x^2} - \theta \frac{\partial G}{\partial x} \right) dx'd\tau + \int_0^{t+\varepsilon} \int_0^l \theta G \theta dx'd\tau - \theta(x, t)$$  \hspace{1cm} (36)$$

It follows from (36) that:

$$\theta(x, t) = B^2 \int_0^{t+\varepsilon} \int_0^l \left( \frac{G \partial \theta}{\partial \tau} - \theta \frac{\partial G}{\partial x} \right) dx'd\tau + \int_0^{t+\varepsilon} \int_0^l \theta G \theta dx'd\tau - \int_0^{t+\varepsilon} \int_0^l \left( \frac{\partial G}{\partial \tau} \right) dx'$$  \hspace{1cm} (37)$$

where

$$\int_0^l \left( \theta G \right) dx' = \int_0^l \left[ \theta(x, \cdot, 0) G(x, t | x', 0) \right] dx' = \int_0^l \left[ \frac{\partial \theta}{\partial \tau} G(x, t | x', 0) \right] dx'$$  \hspace{1cm} (38)$$

$$\int_0^l \left( G \frac{\partial \theta}{\partial \tau} - \theta G \frac{\partial}{\partial x} \right) dx' = \int_0^l \left[ \phi(x') \frac{\partial \theta}{\partial \tau} G(x, t | x', 0) \right] dx' - \psi(x') G(x, t | x', 0)$$  \hspace{1cm} (39)$$

considering the causality relation \( G(x, t | x', t + \varepsilon) = 0 \) and the initial conditions (21) and (22):

$$\theta(x', 0) = \phi(x'), \quad \dot{\theta}(x', 0) = \psi(x').$$

Finally, by taking the limit of (37) as \( \varepsilon \to 0 \), the Green’s function solution equation has the form:
\[
\theta(x,t) = B^2 \int_0^1 \left[ \left( \frac{\partial \theta}{\partial x'} - \theta \frac{\partial G}{\partial x'} \right) \right]_0^l d\tau
\]

for boundary conditions

\[
+ \int_0^1 \left[ \varphi(x') G(x,t|x',0) \right] dx' - A \int_0^1 \left[ \varphi(x') \frac{\partial G(x,t|x',0)}{\partial \tau} - \varphi(x') G(x,t|x',0) \right] dx'
\]

for initial conditions

\[
+ \int_0^1 \int G(x,t|x',\tau) f_0(x',\tau) dx' d\tau
\]

for energy generation

(40)

We have therefore proved that the solution to hyperbolic heat conduction problems, described by Eq. (15) and the corresponding initial boundary conditions, can be expressed in terms of the boundary conditions, the initial conditions, the energy generation and the Green’s function.

For a concise solution structure, the term of Eq. (40) \( \int_0^1 \left[ \left( \frac{\partial \theta}{\partial x'} - \theta \frac{\partial G}{\partial x'} \right) \right]_0^l d\tau \) should be simplified according to the boundary conditions at \( x'_a = 0, l \). For the first kind boundary conditions:

\[
\theta(x'_a,t) = f_a(t)
\]

(41)

\[
G(x'_a, t) = 0
\]

(42)

Then, one has:

\[
\int_0^1 \left( G(x'_a, t) \frac{\partial \theta}{\partial x'} - \theta(x'_a, t) \frac{\partial G}{\partial x'} \right) d\tau
\]

\[
= -\int_0^1 \theta(x'_a, t) \frac{\partial G}{\partial x'} d\tau = -\int_0^1 f \left( x'_a, t \right) \frac{\partial G}{\partial x'} d\tau
\]

(43)

The results for \( \int_0^1 \left[ \left( \frac{\partial \theta}{\partial x'} - \theta \frac{\partial G}{\partial x'} \right) \right]_0^l d\tau \) are summarized in Table 2.

<table>
<thead>
<tr>
<th>BC</th>
<th>( \int_0^1 \left[ \left( \frac{\partial \theta}{\partial x'} - \theta \frac{\partial G}{\partial x'} \right) \right]_0^l d\tau )</th>
</tr>
</thead>
<tbody>
<tr>
<td>X11</td>
<td>(-\int_0^1 f_{\text{right}}(\tau) \frac{\partial G(x'=l,\tau)}{\partial x'} d\tau + \int_0^1 f_{\text{left}}(\tau) \frac{\partial G(x'=0,\tau)}{\partial x'} d\tau)</td>
</tr>
<tr>
<td>X12</td>
<td>(\int_0^1 \frac{1}{k} f_{\text{right}}(\tau) G(x'=l,\tau) d\tau + \int_0^1 f_{\text{left}}(\tau) \frac{\partial G(x'=0,\tau)}{\partial x'} d\tau)</td>
</tr>
<tr>
<td>X21</td>
<td>(-\int_0^1 f_{\text{right}}(\tau) \frac{\partial G(x'=l,\tau)}{\partial x'} d\tau - \int_0^1 \frac{1}{k} f_{\text{left}}(\tau) G(x'=0,\tau) d\tau)</td>
</tr>
</tbody>
</table>

Table 2. Calculation for \( \int_0^1 \left[ \left( \frac{\partial \theta}{\partial x'} - \theta \frac{\partial G}{\partial x'} \right) \right]_0^l d\tau \) under XijBt.
4.3 Green’s function

As previously mentioned, the Green’s function is the solution to the auxiliary initial boundary value problem, which is similar to the initial boundary problem for hyperbolic heat conduction with two notable differences: (1) the generation function in the auxiliary equation is the Dirac delta function, as shown by Eq. (32); (2) the initial boundary conditions are homogeneous, expressed by Eq. (33). In the monograph (Cole et al., 2010), several methods for obtaining Green’s functions are presented. Here, method of separation of variables is adopted.

According to Wang et al. (2008), the solution of the initial boundary value problem (32) and (33) is identical to that of the following problem:

\[ A\dot{G} + \dot{G} + B \frac{\partial^2 G}{\partial x^2} = 0 \]  

\( X_{ij} \quad (i = 1,2; j = 1,2) \)  

\[ G(x,0) = 0 \]  

\[ \frac{\partial G(x,0)}{\partial t} = \frac{\delta(x-x')}{A} \]

The Green’s function can be obtained by using the method of separation of variables, as:

\[ G = T(t)X(x) \]

with \( X(x) \) satisfying:

\[ \frac{\partial^2 X(x)}{\partial x^2} + \beta_n^2 X(x) = 0 \]

subjected to boundary conditions (45) and

\[ AT'' + T' + (\beta_n B)^2 T = 0 \]

as well as initial conditions (46) and (47). It follows from (49) that:

\[ X(x) = c \sin \beta_n x + d \cos \beta_n x \]

For illustration, consider the boundary condition of type \( X'1 \). In this case, one has:

\[ G(0,t) = G(l,t) = 0 \]

Then
\[ X(0) = X(t) = 0 \] (53)

From (51) and (53), one has:
\[ \sin \beta_m l = 0 \] (54)
yielding:
\[ \beta_m = \frac{m \pi}{l} \quad (m = 1, 2, \ldots) \] (55)

Consequently, the solution to (49) is:
\[ X_m(x) = c \sin \beta_m x, \quad \beta_m = \frac{m \pi}{l} \quad (m = 1, 2, \ldots) \] (56)

Here, \( \beta_m^2 \), \( F_{\text{eigen}} = \sin \beta_m x \) and \( M_m = \int_0^l F_{\text{eigen}} F_{\text{eigen}} dx = \int_0^l \sin \beta_m x \sin \beta_m x dx = \frac{l}{2} \) are called the Eigenvalues, Eigen functions and normal square, respectively.

The general solution of (50) is:
\[ T_m = e^{-\frac{t-x}{\gamma}} \left[ a_m \cos \gamma_m (t - \tau) + b_m \sin \gamma_m (t - \tau) \right] \] (57)

where
\[ \gamma_m = \frac{1}{2A} \sqrt{4A \left( \beta_m B \right)^2 - 1} \] (58)

As illustration, for boundary condition \( X_{11} \), one obtains from (56) and (57) that:
\[ G = \sum_{m=1}^{\infty} e^{-\frac{t-x}{\gamma}} \left[ a'_m \cos \gamma_m (t - \tau) + b'_m \sin \gamma_m (t - \tau) \right] \sin \beta_m x \] (59)

where \( a'_m = c a_m \) and \( b'_m = c b_m \). In view of (46), one gets:
\[ G = \sum_{m=1}^{\infty} b'_m e^{-\frac{t-x}{\gamma}} \sin \gamma_m (t - \tau) \sin \beta_m x \] (60)

To satisfy (47), one has:
\[ \sum_{m=1}^{\infty} b'_m \gamma_m \sin \beta_m x = \frac{\delta(x-x')}{A} \] (61)

which requires that:
\[ b'_m = \frac{1}{\gamma'_m M_m} \int_0^l \frac{\delta(x-x')}{A} \sin \beta_m x dx = \frac{1}{A \gamma'_m M_m} \sin \beta_m x' \] (62)

Equation (60) can be rewritten as:
\[ G = \sum_{m=1}^{\infty} e^{-\frac{t-x}{\gamma}} \frac{1}{A \gamma'_m M_m} \sin \beta_m x' \sin \beta_m x \sin \gamma_m (t - \tau) \] (63)
which is the Green’s function for 1D problems with $X_{11}$ boundary conditions. Similarly, corresponding results for boundary conditions $X_{ij}$ ($i, j = 1,2$) can be obtained, as summarized in Table 3.

Table 3. Green’s function for one-dimensional problems under boundary conditions $X_{ij}$ ($i, j = 1,2$) ($\gamma_m = \frac{1}{2A} \sqrt{4A(\beta_x B^2 - 1)}$).

<table>
<thead>
<tr>
<th>BC</th>
<th>Green’s function</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_{11}$</td>
<td>$G = \sum_{n=0}^{\infty} e^{-\frac{\pi^2}{A} t} \frac{1}{\alpha_n A M_n} \sin \beta_m x \sin \beta_m t \sin \gamma_n (t - \tau) ; \quad \beta_m = m\pi/l ; \quad M_m = l/2$</td>
</tr>
<tr>
<td>$X_{12}$</td>
<td>$G = \sum_{n=0}^{\infty} e^{-\frac{\pi^2}{A} t} \frac{1}{\alpha_n A M_n} \sin \beta_m x \sin \beta_m t \sin \gamma_n (t - \tau) ; \quad \beta_m = (2m+1)\pi/2l ; \quad M_m = l/2$</td>
</tr>
<tr>
<td>$X_{21}$</td>
<td>$G = \sum_{n=0}^{\infty} e^{-\frac{\pi^2}{A} t} \frac{1}{\alpha_n A M_n} \cos \beta_m x \cos \beta_m t \sin \gamma_n (t - \tau) ; \quad \beta_m = (2m+1)\pi/2l ; \quad M_m = l/2$</td>
</tr>
<tr>
<td>$X_{22}$</td>
<td>$G = \sum_{n=0}^{\infty} e^{-\frac{\pi^2}{A} t} \frac{1}{\alpha_n A M_n} \cos \beta_m x \cos \beta_m t \sin \gamma_n (t - \tau) ; \quad \beta_m = m\pi/l ; \quad M_m = l/2$</td>
</tr>
</tbody>
</table>

5. Numerical examples

For a specific problem, the solution requires three main steps:
1) determine the geometry, initial boundary conditions and material constants;
2) obtain the Green’s function according to Table 3;
3) obtain the temperature solution from the Green’s function solution equation Eq. (40), in which the boundary conditions part is referred to Table 2.

To validate the present solution to hyperbolic heat conduction problems, selected numerical examples are presented below in the case of C-V model. For convenience, the governing equations (1) and (2) of the C-V model are non-dimensionalized, as:

$$q^* + \frac{1}{2} \frac{\partial q^*}{\partial t} = -\frac{1}{2} \frac{\partial \theta^*}{\partial x^*}$$

and

$$\frac{\partial \theta^*}{\partial t^*} = Q^* - \frac{\partial q^*}{\partial x^*}$$

by introducing the following dimensionless quantities: $\theta^* = \frac{\theta}{\theta_{\text{con}}}$, $q^* = \frac{q \sqrt{\alpha \tau}}{k \theta_{\text{con}}}$, $Q^* = \frac{2\tau Q}{c}$, $\alpha^* = \frac{x}{2\sqrt{\alpha \tau}}$, $t^* = \frac{t}{\tau_{\text{con}}}$, where $\theta_{\text{con}}$ is a constant temperature value, $\alpha = k/c$ and $c = \rho c_p$. In the following, the asterisk will be neglected for brevity. Combing (64) and (65), one has:

$$\frac{\partial^2 \theta}{\partial t^2} + \frac{\partial q}{\partial t} = \frac{\partial^2 \theta}{\partial x^2} + f(x,t) \quad \text{and} \quad f(x,t) = \frac{\partial Q}{\partial t} + 2Q$$

The solution to (66) has previously been obtained by Lam and Fong (2011a and 2011b) using the solution structure...
For the present study, the solution is obtained by using the method of Green’s function, with 
\[ A = 0.5, \quad B^2 = 0.5 \quad \text{and} \quad f_Q(x,t) = 0.5f(x,t). \]

### 5.1 Example 1

A slab is subjected to a spatially decaying heat generation 
\[ f_Q(x,t) = 0.5f(x,t) = 50e^{-5t}, \]
with both sides thermally insulated, as:
\[ X \begin{align*} 22 \left( \frac{\partial \theta(0,t)}{\partial x} = 0, \quad \frac{\partial \theta(1,t)}{\partial x} = 0 \right), \end{align*} \]
and the initial conditions are given as:
\[ \theta(x,0) = \phi(x) = \sin \frac{\pi}{2} x; \quad \vartheta(x,0) = \psi(x) = 50e^{-5x}. \]

According to Table 3, Green’s function in such case is:
\[
G = e^{-t} \frac{2}{\sqrt{-1}} \sin \sqrt{-1}(t - \tau) + \sum_{m=1}^{\infty} e^{-t} \frac{4}{\sqrt{(m\pi)^2 - 1}} \cos m\pi x' \cos m\pi x \sin \sqrt{(m\pi)^2 - 1}(t - \tau)
\]  

Thus, the solution may be expressed, as:
\[
\theta(x,t) = \int_0^1 G(x,t|x',0)\phi(x')dx' + \int_0^1 AG \left( x, t \left| x', 0 \right. \right) \psi(x')dx' \quad \text{(for initial conditions)}
\]
\[
-\int_0^1 A \frac{\partial G \left( x, t \left| x', 0 \right. \right)}{\partial \tau} \phi(x')dx' \quad \text{(for initial conditions)}
\]
\[
+\int_0^1 \int_0^1 G(x,t|x',\tau)f_Q(x',\tau) dx'd\tau \quad \text{(for energy generation)}
\]

The results calculated by different value of \( m \) is presented in Fig. 1 to show the convergence of the temperature for a specific point and instant \((x = 0.5, t = 0.5)\) (Torabi and Saedodin, 2011). The predicted temperature distributions are presented in Fig. 2, and compared with existing results of Lam and Fong (2011a). Excellent agreement is achieved, indicating the validity of using the Green’s function method to solve hyperbolic heat conduction problems, and the correctness of the present solution.
5.2 Example 2

The slab is heated by a time-varying and spatially-decaying heat generation, as:

\[ f_0(x,t) = \frac{1}{2} f(x,t) = 50e^{-5t} \left( \frac{1}{2} e^{-x} + \frac{3}{2} e^{-3x} \right) \]

with boundary conditions:

\[ X 21 \left( \frac{\partial \theta(0,t)}{\partial x} = 0, \ \theta(1,t) = 0 \right) \]

And the initial conditions is given:

\[ \theta(x,0) = \varphi(x) = \sin(\pi x); \ \dot{\theta}(x,0) = \psi(x) = 0. \]

Green’s function of this issue is expressed:

\[
G = \sum_{m=0}^\infty e^{-\tau} \cos \left( \frac{\pi}{2} (2m+1) \right) \cos \left( \frac{\pi}{2} (2m+1) x \right) \sin \left( \frac{\pi}{2} (2m+1) x \right) -1(t-\tau)
\]

and the solution has the from:

\[
\theta(x,t) = \int_0^t \int_0^t G(x,t|x',0) \varphi(x') dx' - \int_0^t A \left( \frac{\partial G(x,t|x',0)}{\partial \tau} \right) \psi(x') dx' \quad \text{(for initial conditions)}
\]

\[
+ \int_0^t \int_0^t G(x,t|x',\tau) f_0(x',\tau) dx' d\tau \quad \text{(for energy generation)}
\]

The convergence of the temperature for a specific point and instant \( (x = 0.5, t = 0.5) \) is referred to Fig. 1. The predicted temperature distributions are shown in Fig. 3. The predictions calculated from (70) agree excellently well with those obtained using the solution structure theorem (Lam and Fong, 2011b).
Fig. 3. Non-dimensional temperature at different time instants ($m = 50$).

Thus far, the terms associated with initial conditions and energy generation of the solution (40) as well as the Green’s function shown in Table 3 is verified. To check the correctness of the terms associated with boundary conditions, a further case study is needed.

5.3 Example 3

A slab is heated at the left surface while the opposite side is kept to zero temperature, i.e. $X11 (\theta(0,t) = 1, \theta(1,t) = 0)$. Heat generation and initial conditions are not considered.

Green's function in this case is:
\[ G = \sum_{m=1}^{\infty} \frac{e^{-\frac{t}{\tau}}}{\sqrt{(m\pi)^2 - 1}} \sin m\pi x' \sin m\pi x \sqrt{(m\pi)^2 - 1} (t - \tau) \] 

(71)

and the solution is obtained, as:

\[ \theta(x,t) = \int_{0}^{t} \frac{1}{2} \frac{\partial G(x',\tau)}{\partial x'} d\tau \]

(72)

Fig. 4. Non-dimensional temperature at \( x = 0.025, 0.1, 0.25, 0.4, 0.475 \) using different \( m \).

In Fig. 4, non-dimensional temperatures at five specific points \( x = 0.025, 0.1, 0.25, 0.4, 0.475 \) are plotted at the instant \( t = 0.5 \) to show the convergence. It is obtained that the convergence of results at \( x = 0.1, 0.25, 0.4 \) is better than that of results at \( x = 0.025, 0.475 \), which is rational because there exist jumps at \( x = 0 \) and \( 0.5 \) in the instant \( t = 0.5 \) (see Fig. 5). To get an accurate result, \( m \) is adopted as 1000 in following calculation. According to the property
of wave function, the velocity of heat conduction in Eq. (66) is 1, so the wave front should be \( x = 0.25, 0.5 \) and 0.75 at time \( t = 0.25, 0.5 \) and 0.75, respectively, which means that the temperature vanishes once \( x > 0.25, 0.5 \) and 0.75.

The predicted temperature distribution along the \( x \)-axis is shown in Fig. 5 for \( t = 0.25, 0.5 \) and 0.75, from which it is seen that sharp jumps do exist at \( x = 0.25, 0.5 \) and 0.75, and the temperature is essentially undisturbed on the right hand side of each jump. In other words, the predictions match well with analytical results, indicating that the present method based on Green’s function is efficient. However, it is found that there exists dispersions (or wiggling) at the front of temperature wave, which may be also observed from Fig. 4. The drawback can be explained by the fact that the solution from (71) and (72) are expressed in form of trigonometric series. Meanwhile, it is expected that the jumps at \( x = 0.25, 0.5 \) and 0.75 could be described by selecting more trigonometric series terms. To overcome this defect, for hyperbolic heat conduction with zero heat generation and initial conditions, we adopt the Laplace transform method.

And two kind of boundary conditions are considered. Here, the general model without heat generation is applied, i.e. Eqs. (15) and (19). The Laplace transform of (15) is

\[
B^2 \frac{\partial^2 \theta(x,s)}{\partial x^2} - s(As + 1) \theta(x,s) = 0
\]

with the aid of zero initial conditions and zero heat generation, where \( s \) is the Laplace transform variable, and \( \theta \) is the temperature in Laplace domain. The solution to (73) may be obtained

\[
\theta(x,s) = \partial_{i} \exp(-k_{i}x) + \partial_{j} \exp(k_{j}x)
\]

where \( k_{i} = \sqrt{\frac{s(As + 1)}{B^2}} \). And \( \partial_{i} \) \((i = 1, 2)\) are to be determined using the boundary conditions. The boundary conditions at \( \tilde{x}_{a} = 0, 1 \) \((\alpha = \text{left, right})\) in Laplace transform is:

The first kind:

\[
\theta(\tilde{x}_{a}, s) = f_{a}(s)
\]

The second kind:

\[
\theta(\tilde{x}_{a}, s) = f_{a}^r(s)
\]

The Laplace transform of (19) has the form:

\[
(1 + \tau s) q(x,s) = -k \frac{\partial \theta(x,s)}{\partial x}
\]

Thus, the second kind of boundary condition is rewritten, as:

\[
k \frac{\partial \theta(\tilde{x}_{a}, s)}{\partial x} = -(1 + \tau s) g_{a}(s)
\]

Based on the boundary conditions, e.g. (75) and (78), \( \partial_{i} \) \((i = 1, 2)\) in Eq. (74) may be obtained. Thus far, the solutions in Laplace domain have been completed. For Example 3, the boundary condition at the left side may be expressed in Laplace domain as \( \theta(0, t) = 1/s \), so the analytical solution to HHC in Laplace domain is \( \theta(x,s) = \exp(-k_{j}x)/s \). To obtain the temporal solutions, we adopt an algorithm NILT proposed by Brancik (1999), which is based on fast Fourier transformation, and is improved using a quotient-difference algorithm. Its efficiency and feasibility is evaluated, even in fractional calculus (Sheng et al., 2011). Using this procedure, the Example 3 is reconsidered, and the result is shown in Fig. 6. The temperature’s distributions are same as that from Green’s function, while the dispersions around the front of the temperature wave are avoided.
6. Conclusion

The reciprocity relation of Green’s function is firstly proven in the context of a general hyperbolic heat conductive model, such as Cattaneo-Vernotte model, the simplified thermomass model, and the single-phase-lag two-step model. And then, the corresponding Green's function solution equation is derived using Green’s functions method. Subsequently, explicit Green’s functions are formulated under several different sets of boundary conditions. For validation, selected numerical studies are carried out and excellent agreement is achieved with existing results using the solution structure theorem and analytical results. In addition, to avoid the dispersions around the front of temperature wave, a procedure based on Laplace transform method is proposed. More importantly, the newly proposed solution may be applied to similar problems, such as non-Fick transient mass transfer and diffusion.

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