Linear Instability Analysis of Lean Premixed Combustion*

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Abstract

Combustion oscillation of a self excited combustion-acoustic phenomenon occurs inside the gas turbine combustor. In general, excessive pressure fluctuation of combustion oscillation may impair the gas turbine engine operation and may result in hardware damages. Therefore, combustion oscillation has been one of the problems of the gas turbine development. In this paper, for predicting combustion-acoustic instability on the designing stage, we have developed one-dimensional linear instability analysis method and verified the analysis accuracy by premixed combustor laboratory tests. The phase lag between combustion dynamics and acoustic system is considered to affect whether combustion acts to destabilize the whole system. Heat release fluctuation is considered to be influenced from fuel/air ratio fluctuation, which generates at fuel nozzle position and flows to combustion region. Its advection delay time is one of the important parameters in the instability. Further, even if heat source itself is steady, the heat release can fluctuate because gas flow rate fluctuates acoustically, and as results, combustion-acoustic instable can occur. At this mechanism, the flame position and the reaction delay time have important roles. By theoretical discussions and laboratory verifications, we conclude that analysis models of this paper have captured the basic framework needed to predict combustion-acoustic instability.

Key words: Gas Turbine, Combustor, Combustion Oscillation, Acoustics, Analysis

1. Introduction

Combustion oscillation of a self excited combustion-acoustic phenomenon occurs inside the gas turbine combustor in operating at higher temperatures with reducing NO_x and other gaseous emissions. In general, excessive pressure fluctuation of combustion oscillation may impair the gas turbine engine operation and may result in hardware damages. Therefore, combustion oscillation has been one of the problems of the gas turbine development [1]-[2]. The mechanism of combustion-acoustic instability is well researched, and many analytical and experimental researches have been discussed for many types of combustors [3]-[16]. According to Rayleigh [3] referenced by many researchers, the combustion-acoustic instability criterion is described as the following inequality

$$\int_{t_0} \int_V p' q' dV dt > \int_{t_0} \int_V \Phi dV dt, \quad (1)$$

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where p' , q' , V , Φ , t and t_0 are perturbations of pressure and heat release density, combustor volume, acoustic energy dissipation, time, and period of oscillation, respectively. The left hand side of Rayleigh's criterion (1) means energy amplification rate of dynamic variables p' and q' which are correlated with each other, and the right hand side means damping rate of them. Combustion-acoustic instability occurs when Rayleigh's criterion (1) is satisfied. Damping rate increase of the right hand side of Rayleigh's criterion (1) can be considered an instability suppression method. The design by this suppression method is relatively easy, so that the detailed combustion predictions are not needed [2]. However, especially at low frequencies, there is a problem that the dimension of suppression device is to be large. So, in this paper, we focused on amplification rate decrease of the left hand side of Rayleigh's criterion (1) at low frequencies due to tuning the interaction between the combustion dynamics and the acoustic dynamics. Also, to achieve this, one-dimensional combustion-acoustic linear instability analysis method and its verification results are discussed regarding the cylindrical premixed combustor model having variable location flame holder and fuel nozzle shown in Figure 1.

In Section 2, equations to analyze combustion-acoustic instability are formulated. Those consist of two correlated mechanisms, acoustic system driven by heat release fluctuation, and heat release fluctuation driven by acoustic system. The acoustic system equations are formulated from basic equations of the fluid dynamics and are considered that heat release can fluctuate because gas flow rate fluctuates acoustically, even if the heat source is steady. In formulating the heat release fluctuation equations, the advection delay time between the fuel nozzle position and the flame position is considered, by referenced the prior reports [7]-[11]. Also, as a more important parameter, the reaction delay time between fuel increase and combustion is considered.

In Section 3, combustion-acoustic linear instability is discussed by solving the equations which are formulated at Section 2. By integrating the wave equations which have been formulated in the previous section, sequentially from the downstream end toward upstream, the acoustic reflection coefficient of flame can be evaluated. This acoustic reflection coefficient can theoretically exceed the one, and at this time, the system may become unstable by the reflection greater than the incident waves. Analysis results of a simple problem in absence of the advection delay time for a validation are consistent with the result of well-known Rijke tube problem [3].

In Section 4, the instability analysis method was verified by premixed combustion laboratory tests. While changing the flame position, or the distance from the fuel nozzle position to the flame position, the experimental results were compared with analysis results. The combustion-acoustic instability was very sensitive to conditions and parameters, and therefore, analysis parameters were forced to be slightly adjusted. Although there were such slight adjustments, the whole trends of analysis results and test results were consistent with each other. Conclusions are summarized in the final section.

Fig. 1 Schematic of the cylindrical premixed combustor model.

2. Formulations of combustion-acoustic system

In this section, combustion-acoustic system equations are formulated. Those consist of two correlated mechanisms, acoustic system driven by heat release fluctuation, and heat release fluctuation driven by acoustic system.

2.1. Formulation of acoustic system driven by heat release fluctuation

In this section, the acoustic aspects formulation of combustion-acoustic system is discussed. Basic equations of the fluid dynamics can be described as

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} = 0, \quad (2)$$

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \Psi, \quad (3)$$

$$\rho T \frac{DS}{Dt} = q, \quad (4)$$

$$dp - c^2 d\rho = (\gamma - 1) \rho T dS, \quad (5)$$

$$p = \rho R T, \quad (6)$$

where ρ , p , $\mathbf{u}$, q , S , T , γ , R , c and $D/Dt (\equiv \partial/\partial t + \mathbf{u} \cdot \nabla)$ are density, pressure, velocity, heat release density, entropy, temperature, heat capacity ratio, gas constant, speed of sound, Lagrange differential, respectively. Also, Ψ represents viscosity and external force terms. Consider the equation of energy (4), the Lagrange differential of the equation of state (5) can be written as

$$\frac{D^2 p}{Dt^2} - \frac{D}{Dt} \left(c^2 \frac{D\rho}{Dt} \right) = (\gamma - 1) \frac{Dq}{Dt}. \quad (7)$$

Further, the left hand side of this equation can be written as

$$\begin{aligned} \frac{D^2 p}{Dt^2} &= \left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) p \\ &= \frac{\partial^2 p}{\partial t^2} + 2\mathbf{u} \cdot \nabla \frac{\partial p}{\partial t} + \mathbf{u} \cdot (\mathbf{u} \cdot \nabla) \nabla p + \frac{D\mathbf{u}}{Dt} \cdot \nabla p, \end{aligned} \quad (8)$$

$$\begin{aligned} -\frac{D}{Dt} \left(c^2 \frac{D\rho}{Dt} \right) &= \rho c^2 \nabla \cdot \frac{D\mathbf{u}}{Dt} - \frac{Dc^2}{Dt} \frac{D\rho}{Dt} - 2\rho c^2 (\nabla \cdot \mathbf{u})^2 \\ &\quad - \rho c^2 \nabla \cdot [(\mathbf{u} \cdot \nabla) \mathbf{u} - \mathbf{u} (\nabla \cdot \mathbf{u})], \end{aligned} \quad (9)$$

where the equation of continuity (2) has been substituted into the equation (9). Consider the equation of motion (3),

$$\frac{D\mathbf{u}}{Dt} \cdot \nabla p + \rho c^2 \nabla \cdot \frac{D\mathbf{u}}{Dt} = -c^2 \nabla^2 p + (\gamma - 1) \rho T \nabla S \cdot \frac{D\mathbf{u}}{Dt} + c^2 \nabla \cdot \Psi \quad (10)$$

can be written. Substituting the equations (8), (9) and (10) into the equation of state (7), the nonlinear wave equation equivalent to basic equations (2)-(6) can be derived as

$$\begin{aligned} \frac{\partial^2 p}{\partial t^2} - c^2 \nabla^2 p &= (\gamma - 1) \frac{Dq}{Dt} - (\gamma - 1) \rho T \nabla S \cdot \frac{D\mathbf{u}}{Dt} + \frac{Dc^2}{Dt} \frac{D\rho}{Dt} \\ &\quad - 2\mathbf{u} \cdot \nabla \frac{\partial p}{\partial t} - \mathbf{u} \cdot (\mathbf{u} \cdot \nabla) \nabla p \\ &\quad + 2\rho c^2 (\nabla \cdot \mathbf{u})^2 + \rho c^2 \nabla \cdot [(\mathbf{u} \cdot \nabla) \mathbf{u} - \mathbf{u}(\nabla \cdot \mathbf{u})] \\ &\quad - c^2 \nabla \cdot \Psi . \end{aligned} \quad (11)$$

When nonlinear terms, viscosity terms, external force terms, and advection terms which are written like as $(\bar{\mathbf{u}} \cdot \nabla) X'$ with respect to a perturbation value X' can be neglected, the linear acoustic wave equation driven by heat release fluctuation can be obtained as follows, where $(\bar{\cdot})$ and $(\cdot)'$ are the mean value and the perturbation value, respectively:

$$\frac{\partial^2 p'}{\partial t^2} - c^2 \nabla^2 p' \cong (\gamma - 1) \frac{\partial q'}{\partial t} - (\gamma - 1) \bar{\rho} \bar{T} \nabla \bar{S} \cdot \frac{\partial \mathbf{u}'}{\partial t} . \quad (12)$$

The physical characteristic of the source term of linear wave equation (12) is evaluated as follows. As can be seen from the equation of state (7), entropy fluctuation has an important role in combustion-acoustic system. Linearizing the equation of energy (4), linear entropy fluctuation can be written as

$$\bar{\rho} \bar{T} \left(\frac{\partial S'}{\partial t} + \bar{\mathbf{u}} \cdot \nabla S' \right) = q' - \bar{\rho} \bar{T} \nabla \bar{S} \cdot \mathbf{u}' . \quad (13)$$

The right hand side of this equation is a excitation source of the linear entropy fluctuation flowing along with the mean flow $\bar{\mathbf{u}}$. The first term means a heterogeneous excitation due to an unsteady heat source. The second term means that heat release can fluctuate by flow velocity fluctuation despite a steady heat source $q' = 0$. The summation of these heat release fluctuation should be considered as the excitation source of linear acoustic system equations, as can be seen in the right hand side of the equation (12).

In the cylindrical structure like combustors, sound waves at low frequencies can be considered as axial one-dimensional. Such cases, Fourier transform defined as

$$p'(t, x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{p}'(\omega, x) e^{i\omega t} d\omega \quad (14)$$

of the linear acoustic wave equation (12) can be written as

$$\frac{\partial^2 \hat{p}'}{\partial x^2} + k^2 \hat{p}' = -i k \frac{\gamma - 1}{c} \hat{Q}' , \quad (15)$$

$$\hat{Q}' \equiv \hat{q}' - \bar{\rho} \bar{T} \frac{\partial \bar{S}}{\partial x} \hat{u}' = \hat{q}' - \frac{\bar{q}}{\bar{u}} \hat{u}' , \quad (16)$$

where i , $k \equiv \omega/c$, $\omega \equiv 2\pi f$, and f are the imaginary unit, wave number, angular frequency, and frequency of acoustic oscillation, respectively. The defined new variable $\hat{Q}'$ means the linear heat release fluctuation that should be considered as the excitation source driving the acoustic aspects of combustion-acoustic system.

2.2. Formulation of heat release fluctuation driven by acoustic system

In this section, the combustion dynamic aspects formulation, which is one-dimensional by cross-sectional averaging, is discussed. Referenced Williams [4], in this paper, heat release sources which consist of combustion reaction, heat conduction, and mixing diffusion are considered. The cross-sectional average of the equation of energy can be written as

$$\rho T \left(\frac{\partial S}{\partial t} + u \frac{\partial S}{\partial x} \right) = q, \quad q = q_C + q_W + q_D \cong q_C, \quad (17)$$

$$q_C \equiv \sum_l \frac{1}{A} \int_A h_l w_l dA, \quad (18)$$

$$q_W \equiv -\frac{1}{A} \int_A \nabla \cdot \mathbf{q} dA = -\frac{1}{A} \frac{\partial A}{\partial x} \mathbf{q} \cdot \mathbf{x} - \frac{1}{A} \oint_{\partial A} \mathbf{q} \cdot \mathbf{n} dC \cong 0, \quad (19)$$

$$q_D \equiv \sum_l \left(\frac{1}{A} \frac{\partial A}{\partial x} \rho h_l Y_l \mathbf{v}_l \cdot \mathbf{x} + \frac{1}{A} \oint_{\partial A} \rho h_l Y_l \mathbf{v}_l \cdot \mathbf{n} dC \right) \cong 0, \quad (20)$$

where h_l , w_l , Y_l , $\mathbf{v}_l$, A , ∂A , dC , $\mathbf{q}$, $\mathbf{n}$ and $\mathbf{x}$ are enthalpy of chemical element l , fuel consumption rate by reaction, concentration, mixing diffusion rate, cross-sectional area, boundary of the cross-section, circumferential length unit on the boundary of the cross section, heat flux, the normal outward unit vector, and the axial unit vector respectively.

Cross-sectional averaged heat release by heat conduction q_W can be evaluated as the heat loss at the wall with Gauss integral formula. Here, the adiabatic combustor wall and the uniform cross-section are assumed, and thus, the heat release by heat conduction is considered as zero. Also, cross-sectional averaged heat release by mixing diffusion q_D can be neglected by cross-sectional averaging. Hence as one-dimensional heat source of the adiabatic wall combustor, considering combustion reaction q_C only seems to be sufficient.

Consider the comprehensive reaction model which the gas composition is assumed unburnt fuel, unburnt air, and combusted synthetic gases. As the cause of heat release fluctuation by combustion $q' (\cong q'_C)$, we focus on reaction rate fluctuation w' . And, as the cause of reaction rate fluctuation w' , we focus on fuel/air ratio fluctuation Y' . And then, assume the quasi-static relation among these variables which can be written as

$$q' = \frac{\bar{q}}{\bar{w}} w', \quad w' = N \frac{\bar{w}}{\bar{Y}} Y', \quad N \equiv \frac{\bar{Y}}{\bar{w}} \frac{\partial w}{\partial Y}. \quad (21)$$

Fuel/air ratio fluctuation is generated by fuel/air supply fluctuation at the fuel nozzle position, and flows to the combustion region by advection, and is consumed by combustion reaction [4]. This fuel mass balance mechanism can be described as

$$\rho \left(\frac{\partial Y}{\partial t} + u \frac{\partial Y}{\partial x} \right) = m \delta(x - x_N) - w, \quad (22)$$

where x_N and $m (\equiv \rho_N u_N A_N / A)$ are the fuel nozzle position and fuel mass flux, and where, ρ_N , u_N , A_N , and A are fuel density, fuel injection velocity, cross-sectional area of the nozzle hole, and cross-sectional area of the combustor, respectively. The function $\delta(\cdot)$ is Dirac's delta function. For the simplicity, concentration mixing diffusion effect has been neglected at the equation (22). Fourier transform of the lowest order perturbation of the equation (22) can be written as

$$i\omega\bar{\rho}\hat{Y}' + \bar{\rho}\bar{u}\frac{\partial\hat{Y}'}{\partial x} = \bar{m}\hat{\phi}'\delta(x-x_N) - \bar{w}\left(\frac{\hat{w}'}{\bar{w}} - \frac{\hat{u}'}{\bar{u}}\right), \quad \hat{\phi}' \equiv \left(\frac{\hat{u}'_N}{\bar{u}_N} - \frac{\hat{u}'}{\bar{u}}\right)_{x=x_N}, \quad (23)$$

where the mean flow value $\bar{\rho}\bar{u}\partial\bar{Y}/\partial x = \bar{m}\delta(x-x_N) - \bar{w}$ has been considered. The defined new variable $\hat{\phi}'$ is the normalized fuel/air ratio fluctuation at the nozzle position.

At the upstream of flame, the second term of the right hand side of the equation (23) is vanished. Solving this equation, fuel/air ratio fluctuation at the upstream can be obtained as

$$\hat{Y}' = \lambda\bar{Y}\hat{\phi}'\exp\left[-i\frac{\omega}{\bar{u}}(x-x_N)\right], \quad (24)$$

where the defined new variable λ is introduced for correcting the concentration mixing diffusion effect which has neglected at the equation (22).

At the combustion region, fuel consumption with reaction corresponding to the second term of the right hand side of the equation (23) should be considered. Integrating the equation (23) over a interval including the flame position $x = x_F$, the fuel mass balance at the combustion region can be written as

$$\int_{x_F-\varepsilon}^{x_F+\varepsilon}\hat{w}'dx + i\omega\int_{x_F-\varepsilon}^{x_F+\varepsilon}\bar{\rho}\hat{Y}'dx = \bar{\rho}\bar{u}\hat{Y}'\Big|_{x=x_F-\varepsilon} - \bar{\rho}\bar{u}\hat{Y}'\Big|_{x=x_F+\varepsilon} + \int_{x_F-\varepsilon}^{x_F+\varepsilon}\frac{\bar{w}}{\bar{u}}\hat{u}'dx. \quad (25)$$

The right hand side of this equation means fuel mass inflowing from the external region to this integral region. The left hand side means fuel mass increase in the integral region due to the inflowing above. Each term of this equation means, from left to right respectively, fuel consumption by combustion reaction, fuel mass increase by unburnt fuel, fuel mass inflowing from the upstream, fuel mass flowing out to the downstream, and fuel mass increase by synthetic gas flow rate increase. Here, we assume low unburnt combustion, and the second term of the right hand side of the equation (25) is assumed to be zero. As the consistent assumption, all of fuel can be assumed to be combusted completely in the combustion region, and the approximation of the second term of the left hand side can be assumed as $\bar{\rho}\hat{Y}' \cong \bar{\rho}\bar{Y}w'/N\bar{w}$. Further, we assume that variables are uniform in the combustion region, and hence the equation (25) can be integrated as follows:

$$\hat{q}' + i\omega\sigma\hat{q}' = \frac{\bar{\rho}\bar{u}\bar{Y}}{2\varepsilon\bar{w}}\bar{q}\lambda\hat{\phi}'\exp(-i\omega\tau) + \bar{q}\frac{\hat{u}'}{\bar{u}}, \quad \sigma \equiv \frac{\bar{\rho}\bar{Y}}{N\bar{w}}, \quad \tau \equiv \frac{x_F - x_N}{\bar{u}}, \quad (26)$$

where the quasi-static relation of the equation (21) has been considered. The time parameter τ means the delay time of fuel/air ratio fluctuation flowing along with the advection $\bar{u}$ from the fuel nozzle position $x = x_N$ to the flame position $x = x_F$. The time parameter σ is related to the reaction delay time between the heat release fluctuation and the fuel mass fluctuation. By integrating the mean flow components of the equation (22), the coefficient of the first term of the right hand side of the equation (26) can be found as $\bar{\rho}\bar{u}\bar{Y}/2\varepsilon\bar{w} = 1$. Hence, the heat release fluctuation can be estimated as

$$\frac{\hat{q}'}{\bar{q}} = \frac{1}{1+i\omega\sigma}\frac{\hat{u}'}{\bar{u}} + \frac{1}{1+i\omega\sigma}\lambda\hat{\phi}'\exp(-i\omega\tau). \quad (27)$$

Acoustic variables $\hat{\phi}'$ which are defined at the fuel nozzle position can be converted to variables reference the flame position. Solving the wave equation (15) under no heat release condition $\hat{Q}' = 0$, the acoustic transfer relation can be described as

$$\begin{pmatrix} \hat{p}' \\ \bar{\rho} c \hat{u}' \end{pmatrix}_{x=x_N} = \begin{pmatrix} \cos kl & i \sin kl \\ i \sin kl & \cos kl \end{pmatrix} \begin{pmatrix} \hat{p}' \\ \bar{\rho} c \hat{u}' \end{pmatrix}_{x=x_F}, \quad kl \equiv \frac{\omega}{c} (x_F - x_N), \quad (28)$$

where kl means the acoustic phase delay, which is less than the advection phase delay $\omega\tau$ depending on Mach number $M (= \bar{u}/c)$. Substituting the acoustic transfer relation (28) and the heat release fluctuation (27) into the equation (16), the excitation source driving the acoustic aspects of combustion-acoustic system and the normalized heat release transfer functions G and H can be obtained as

$$\hat{Q}' = \hat{q}' - \frac{\bar{q}}{\bar{u}} \hat{u}' = G \frac{\bar{q}}{\bar{u}} \hat{u}' + H \frac{\bar{q}}{\bar{u}} \frac{\hat{p}'}{\bar{\rho} c}, \quad (29)$$

$$G \equiv \frac{-i\omega\sigma}{1+i\omega\sigma} - \lambda \frac{\exp(-i\omega\tau)}{1+i\omega\sigma} \left(\cos M\omega\tau - i \frac{\bar{\rho} c}{Z_N} \frac{\bar{u}}{\bar{u}_N} \sin M\omega\tau \right), \quad (30)$$

$$H \equiv -\lambda \frac{\exp(-i\omega\tau)}{1+i\omega\sigma} \left(i \sin M\omega\tau - \frac{\bar{\rho} c}{Z_N} \frac{\bar{u}}{\bar{u}_N} \cos M\omega\tau \right), \quad (31)$$

where $Z_N \equiv \hat{p}'/\hat{u}'_N$ is the acoustic impedance of the fuel nozzle.

Summarizing formulations of Section 2.1 and 2.2, analysis models of one-dimensional combustion-acoustic system and the interactions of models can be described as Figure 2.

Fig. 2 Diagram of one-dimensional combustion-acoustic analysis model.

3. Linear instability analysis of combustion-acoustic system

In this section, combustion-acoustic instability of one-dimensional model shown as Figure 3 is discussed by solving the equations (15) and (29). Here, the heat release density distribution is assumed uniform over the combustion region like as

$$\bar{q} = q_0 \delta_*(x - x_F, 2\varepsilon), \quad q_0 \equiv \frac{1}{A} \int_V \bar{q} dV, \quad (32)$$

$$\delta_*(x - \xi_1, \xi_2) \equiv \frac{1}{\xi_2} [h(x - \xi_1 + \xi_2/2) - h(x - \xi_1 - \xi_2/2)], \quad (33)$$

where $h(\cdot)$ means Heaviside's step function. The defined new function $\delta_*(x - \xi_1, \xi_2)$ is like a step function, and its integral over the non-zero range is always the one. When the width ξ_2 is the limit of zero, this function becomes Dirac's delta function.

Fig. 3 Schematic of one-dimensional combustion-acoustic analysis model.

Assuming large impedance nozzles $1/Z_N \cong 0$ and the small Mach number mean flow $M \cong 0$, the analysis models can be written as

$$\frac{\partial^2 \hat{p}'}{\partial x^2} + k^2 \hat{p}' = -ik \frac{\gamma - 1}{c} \hat{Q}', \quad (34)$$

$$\hat{Q}' = G \frac{\bar{q}}{\bar{u}} \hat{u}' + H \frac{\bar{q}}{\bar{u}} \frac{\hat{p}'}{\bar{\rho} c}, \quad i\omega \hat{u}' = -\frac{1}{\bar{\rho}} \frac{\partial \hat{p}'}{\partial x}, \quad (35)$$

$$G = \frac{-i\omega\sigma}{1+i\omega\sigma} - \lambda \frac{\exp(-i\omega\tau)}{1+i\omega\sigma}, \quad H = 0, \quad (36)$$

$$\sigma \equiv \frac{\bar{\rho} \bar{Y}}{N \bar{w}}, \quad N \equiv \frac{\bar{Y}}{\bar{w}} \frac{\partial w}{\partial Y}, \quad \tau \equiv \frac{x_F - x_N}{\bar{u}}, \quad (37)$$

and

$$\frac{\partial^2 \hat{p}'}{\partial x^2} + k^2 \hat{p}' = -\eta \frac{\partial \hat{p}'}{\partial x} \delta_*(x - x_F, 2\varepsilon), \quad \eta \equiv -\frac{\gamma - 1}{\bar{\rho} \bar{u} c^2} q_0 G, \quad (38)$$

where the equation (38) is the combination of the equations (34), (35), (36), and (32).

3.1. Integration to formulate the acoustic reflection coefficient of flame

The equation (38) can be solved by integrating sequentially from the downstream end. At the most downstream open end $x = L$, we regard the boundary conditions as the following acoustic impedance $Z_3 = \hat{p}'/\hat{u}'$ and the acoustic reflection coefficient R_3 by referenced the evaluation of Morse and Ingard [17],

$$R_3 = \frac{Z_3 - \bar{\rho}_H c_H}{Z_3 + \bar{\rho}_H c_H}, \quad Z_3 = \bar{\rho}_H c_H \left[1 - \frac{J_1(2k_H a)}{k_H a} + i \frac{H_1(2k_H a)}{k_H a} \right], \quad (39)$$

where $J_1(\cdot)$, $H_1(\cdot)$, and a are Bessel function, Struve function, and radius of open end, respectively. Variables with subscript H means those are belong to high temperature T_H region.

Subsequently, we consider the integral from the downstream end $x = L$ to the downstream of the combustion region $x = x_F + \varepsilon$. By integrating the equation (38), the

acoustic transfer relation between the ends of this interval can be obtained. Substituting the boundary condition (39) into this acoustic transfer relation, the acoustic reflection coefficient R_2 at the downstream of the combustion region can be obtained as

$$R_2 = R_3 \exp(-2i k L_D), \quad Z_2 = \bar{\rho}_H c_H \frac{Z_3 + i \bar{\rho}_H c_H \tan k_H L_D}{\bar{\rho}_H c_H + i Z_3 \tan k_H L_D}. \quad (40)$$

Subsequently, we consider the integral from the downstream $x = x_F + \varepsilon$ to the upstream $x = x_F - \varepsilon$ of the combustion region. The acoustic reflection coefficient R_1 at the upstream of the combustion region can be obtained by the same way as the downstream region. Further, the thickness of the combustion region can be considered to be enough thin $\varepsilon \rightarrow 0$ than the acoustic wave length, and hence the acoustic reflection coefficient of flame R_1 can be obtained asymptotically as

$$R_1 = \frac{R_2 - \tanh \mu}{1 - R_2 \tanh \mu}, \quad \mu \equiv \frac{1}{2} \left(\eta - \ln \frac{c_H}{c_L} \right), \quad (41)$$

where variables with subscript L means those are belong to low temperature T_L region. The equation (41) is the heat release fluctuation model so called which is reformulated as the boundary condition seen from the upstream of combustion region.

3.2. Instability of one-dimensional combustion-acoustic system

The acoustic wave equation (38) can be integrated easily in the upstream of combustion region, because of no combustion in this region. The acoustic transfer relation and the boundary conditions as

$$\begin{pmatrix} \hat{p}' \\ \bar{\rho} c \hat{u}' \end{pmatrix}_{x=0} = \begin{pmatrix} \cos k L_U & i \sin k L_U \\ i \sin k L_U & \cos k L_U \end{pmatrix} \begin{pmatrix} \hat{p}' \\ \bar{\rho} c \hat{u}' \end{pmatrix}_{x=x_F - \varepsilon}, \quad (42)$$

$$\hat{p}'|_{x=0} = -Z_0 \hat{u}'|_{x=0}, \quad \hat{p}'|_{x=x_F - \varepsilon} = Z_1 \hat{u}'|_{x=x_F - \varepsilon} \quad (43)$$

can be considered, where Z_0 , Z_1 , and L_U are the acoustic impedance at the upstream end and the downstream end, and the length of this interval. Substituting the boundary conditions (43) into the acoustic transfer relation (42), the eigen equation can be obtained as

$$\exp(2i k L_U) = R_0 R_1, \quad R_0 = \frac{Z_0 - \bar{\rho}_L c_L}{Z_0 + \bar{\rho}_L c_L}, \quad R_1 = \frac{Z_1 - \bar{\rho}_L c_L}{Z_1 + \bar{\rho}_L c_L} \quad (44)$$

where the acoustic reflection coefficients R_1 has been evaluated in the previous section. Solving the eigen equation (44), the complex eigen values k can be obtained. The real part of the eigen value means the resonant frequency f of the system, and the ratio of the real part and the imaginary part means the damping ratio ζ of the system. Those are

$$f = \frac{c}{2\pi} \Re(k) = \frac{c_L}{4\pi L_U} \arg(R_0 R_1), \quad (45)$$

$$\zeta = \frac{\Im(k)}{\Re(k)} = -\frac{c_L}{4\pi L_U f} \ln |R_0| |R_1|, \quad (46)$$

where $\Re(\cdot)$ and $\Im(\cdot)$ mean the real part and the imaginary part, respectively. Here, we

should note that the acoustic reflection coefficient is generally a function of the frequency f , therefore it is necessary for the resonant frequency estimation to solve the equation (45) with respect to the frequency f .

The system becomes unstable when damping ratio is negative. According to the equation (46), that is when the inequality $|R_0||R_1| > 1$ is satisfied. Substituting the acoustic reflection coefficient of flame R_1 of the equation (41) into the inequality $|R_0||R_1| > 1$, the instability criterion of one-dimensional combustion-acoustic system as

$$\begin{aligned} & 2(1 - |R_0|^2) \Re(R_2) \Re(\tanh \mu) - 2(1 + |R_0|^2) \Im(R_2) \Im(\tanh \mu) \\ & > (|R_0|^2 - |R_2|^2)(1 - |\tanh \mu|^2) + (1 - |R_0|^2)(1 + |R_2|^2) \end{aligned} \quad (47)$$

can be obtained. In assuming the approximation $|R_0| \cong 1$ to the inequality (47), the criterion of whether $|R_1| > 1$ can be obtained. Hence, it is found that the absolute value of the acoustic reflection coefficient of flame $|R_1|$ can exceed the one at the combustion-acoustic system, although it is usually less than the one at the general no combustion simple acoustic system. Especially when the approximation $|R_0| \cong 1$ can be considered as valid, the instability of combustion-acoustic system is governed whether the absolute value of the acoustic reflection coefficient of flame $|R_1|$ exceeds the one.

3.3. Validation at a simple problem

Let us discuss a simple problem to validate the instability criterion (47). By assuming slight temperature difference $c_H/c_L \cong 1$, low heat release $\mu \cong \eta/2 \ll 1$, and the ideal end $R_3 = R_0 \cong -1$, the criterion can be approximated as $2 \Im(R_2) \Im(\eta) < |R_2|^2 - 1$. Also under these assumptions, the acoustic reflection coefficient of behind the flame can be approximated as $R_2 = -\exp(-2in\pi L_D/L)$, where n is the acoustic mode number. Assume the influence of fuel/air ratio fluctuation as negligible $\lambda = 0$, the heat release transfer function η can be approximated as $\eta \cong i(\gamma - 1)q_0\omega\sigma/(1 + i\omega\sigma)/\bar{\rho}\bar{u}c^2$, and therefore its imaginary part is $\Im(\eta) \cong (\gamma - 1)q_0\omega\sigma/\bar{\rho}\bar{u}c^2$. Consider that the coefficient $(\gamma - 1)\omega\sigma/\bar{\rho}c^2$ is positive, the instability criterion (47) can be approximated as follows:

$$(q_0/\bar{u}) \sin(2n\pi L_D/L) < 0. \quad (48)$$

Red region in Figure 4 shows the combustion region where the system becomes unstable according to the approximated instability criterion (48), in case of the exothermic reaction, $q_0 > 0$. The dotted lines in Figure 4 represent the acoustic mode shapes. The results are consistent with the result of the well-known Rijke tube problem [3].

If the reaction delay time σ is not considered, such simple instable can't even occur, so that the imaginary part of the heat release transfer function $\Im(\eta)$ comes zero. So, the reaction delay time σ seems having an important role in combustion-acoustic instability.

Fig. 4 The range of the combustion region where the system becomes unstable in a simple case.

4. Verification at laboratory combustion tests

For verify the analysis method, two types tests by the cylindrical premixed combustor were carried out. One confirmed the instability in the absence of fuel/air ratio fluctuation. The other one confirmed the instability involving the fuel/air ratio fluctuation while changing the position of the flame holder and the fuel nozzle.

4.1. Test facility features and test conditions of common to test cases

Figure 5 shows configurations of test facility which consist of combustor, plenum chamber, burner with fuel supply system, air supply system, and measurement system. Combustor is a pipe made with quartz which the inner diameter and the length are 45mm and 1,000mm, respectively. The downstream end is opened to the atmosphere. The other end is connected to a plenum chamber which the dimensions are width 400mm, depth 400mm, and height 700mm. Combustion oscillation was measured by microphone at the horizontal 400mm away position of the same height as the top end. Burner and air system are different by test case, and will be explained in a later section. Tests were carried out under room temperature $T_L = 288\text{K}$ and atmospheric pressure $\bar{p} = 10^5 \text{ Pa}$ condition.

Fig. 5 Schematic of laboratory combustion test facility.

4.2. Laboratory combustion test in absence of the advection delay time

Bunsen burner supplies air using Venturi effect. Air is supplied in accordance with fuel flow rate, and air supply fluctuation can be considered to be in conjunction with fuel supply fluctuation. Hence, the fuel/air ratio fluctuation of Bunsen burner is expected to be small as can be seen from the formulation of fuel/air ratio fluctuation $\hat{\phi}'$ in the equation (23). For confirming the combustion-acoustic instability analysis without the influence of fuel/air ratio fluctuation $\lambda = 0$, Bunsen burner was applied to the test.

Configurations of test combustor and burner are shown in Figure 6(a). Test burner was Bunsen burner which was attached an extension brass pipe and inserted from the bottom of the combustor. Tests were carried out while moving the flame position from 80mm to 680mm of which the origin is the upstream end of combustor. The burner was remodeled to be able to move this distance as shown in Figure 6(a). Typical measured spectrums of combustion oscillation are shown in Figure 6(b). The peak level of the first acoustic mode was tracked while changing the flame position, and its results are shown as circles in Figure 7, which levels were normalized by the maximum peak level measured at the flame position 230mm. Combustion conditions and analysis parameters are summarized in Table 1.

Fig. 6 Configurations of test combustor and test burner.

Fig. 7 Instability of the 1st. acoustic mode while changing the flame position of Bunsen burner.

Table 1 Combustion conditions of Bunsen burner and analysis parameters

Fuel type and fuel calorific value	Town gas of 41.4 MJ/m ³
Fuel flow rate	2.17×10 ⁻⁵ m ³ / sec
Heat release density	q ₀ = 5.64×10 ⁵ W / m ²
Speed of sound	c _L = 341 m / sec , c _H = 440 m / sec
Gas velocity	u _L = 2.42 m / sec , u _H = 4.04 m / sec
Reaction delay time	σ = 2.07×10 ⁻¹ msec
Additional acoustic resistance of end	Z _{U,R} / ρ̄c = 5.0×10 ⁻³ , Z _{D,R} / ρ̄c = 5.1×10 ⁻²

The bold solid line in Figure 7 is the analysis result which was evaluated with the analysis models (41) and (46). The response magnification was evaluated by the reciprocal of the damping ratio (46), and then, normalized by the maximum response magnification evaluated at the flame position 230mm. Taking account of $\lambda = 0$, the normalized heat release transfer functions applied to analysis were

$$G = \frac{-i\omega\sigma}{1+i\omega\sigma}, \quad (49)$$

where the reaction delay time σ was estimated as the advection transit time to pass through the flame, where the flame thickness was assumed as 0.5mm. The upstream boundary condition was estimated as $Z_0 = Z_{U,R} - i\bar{\rho}c(A/A_C)\cot kL_C$, by solving the wave equation, where A , A_C , L_C and $Z_{U,R}$ are the cross-sectional area of combustor and plenum chamber, the vertical length of plenum chamber, and the acoustic resistance $\Re(Z_0)$, respectively. The downstream boundary condition was based on the theoretical model (39) and was adjusted by the additional resistance $Z_{D,R}$. These acoustic resistances were adjusted to be analysis results consistent with experimental results.

The thin solid lines in Figure 7 show analytical parameter study results under the same conditions as the bold solid line except the additional resistance $Z_{D,R}/\bar{\rho}c$ which were assumed as 5.9×10^{-2} (15% larger) and 4.3×10^{-2} (15% smaller), respectively. In smaller damping case of $Z_{D,R}/\bar{\rho}c = 4.3 \times 10^{-2}$, the system damping ratio (46) was evaluated as negative near the flame position 230mm, so, the response magnification could not be defined there. As can be seen from Figure 7, the instability is found to be sensitive to slight parameter differences. Although there were such parameter slight adjustments, the whole trends of analysis and experimental results were consistent with each other.

4.3. Laboratory combustion test while changing the advection delay time

To confirm the combustion-acoustic instability analysis with the influence of the fuel/air advection delay time $\lambda \neq 0$, a flame holder and a fuel nozzle whose positions were changeable respectively were applied. Configurations of facility are shown in Figure 8(a). The flame holder shown in Figure 8(b) was a wedge shape and was able to supply town gas to hold the piloted diffusion flame. The fuel nozzle was a pipe of the inner diameter 8mm, and had four apertures of diameter 1.5mm to inject methane from the tip. Fuel and air were forcibly supplied by the gas cylinder pressure and blower, respectively. And they were mixed while advection to the flame holder.

Fig. 8 Test facility and flame holder.

Combustion conditions and analysis parameters are summarized in Table 2. The measured peak levels of the first acoustic mode of combustion oscillation were tracked while

changing the flame and nozzle positions, and the results are shown as circles in Figure 9 and 10, which levels were normalized by the maximum peak level measured when the fuel nozzle is positioned at the upstream end of combustor and the flame holder is positioned at 220mm from the upstream end of combustor. The bold solid lines in Figure 9 and 10 are the analysis results evaluated in the mostly same way as the previous section, except for considering the influence of the fuel/air advection delay time $\lambda \neq 0$. The normalized heat release transfer functions applied to the analysis were as follows, where the mixing diffusion factor was assumed $\lambda = 0.15$:

$$G = \frac{-i\omega\sigma}{1+i\omega\sigma} - \lambda \frac{\exp(-i\omega\tau)}{1+i\omega\sigma}, \quad (50)$$

Table 2 Combustion conditions and analysis parameters

Fuel type and fuel calorific value	Methane of 40.0 MJ/m ³ (Main fuel) Town gas of 41.4 MJ/m ³ (Piloted fuel)
Piloted fuel mass ratio	0.05 kg / kg
Fuel flow rate	4.17 × 10 ⁻⁵ m ³ / sec
Heat release density	$q_0 = 1.05 \times 10^6$ W / m ²
Speed of sound	$c_L = 341$ m / sec, $c_H = 440$ m / sec
Gas velocity	$u_L = 4.50$ m / sec, $u_H = 7.52$ m / sec
Reaction delay time	$\sigma = 1.11 \times 10^{-1}$ msec
Additional acoustic resistance of end	$Z_{U,R} / \bar{\rho}c = 5.0 \times 10^{-3}$, $Z_{D,R} / \bar{\rho}c = 5.1 \times 10^{-2}$

Fig. 9 Instability of the 1st. acoustic mode while changing flame position and fixed nozzle position.

Figure 9 shows the test results while changing flame holder position x_F from 190mm position to 690mm position of which the origin is the upstream end of combustor. In this test case, fuel nozzle position was fixed at the upstream end of combustor $x_N = 0$, hence, the flame position can be interpreted as the fuel/air advection distance. As can be seen in Figure 9, the combustion-acoustic instability is found to be sensitive to the fuel/air advection distance. Further, the level of instability is seemed to be relatively large when the flame is positioned at the upstream region. These trends are similar to the previous section,

and seem that influenced from the reaction delay time effect which is determined without regarding to the fuel nozzle position. The thin solid line in Figure 9 represents the analysis results assumed the absence of advection delay time $\tau = 0$. The thin solid line is normalized by the maximum value of itself. The bold solid lines in Figure 9 and 10 represent the analysis results with considered the advection delay time $\tau = (x_F - x_N)/\bar{u}$, which is also normalized by the maximum value of the thin solid line of Figure 9. According to the comparison of Figure 7 and 9, the dynamic ranges of normalized response magnification, which is the ratio of the maximum value and the minimum value, are different by about 10 times. It can be considered to be almost caused that heat release density is larger at Figure 9 than Figure 7 as can be seen from Table 1 and 2. According to the form of the equation (38), the influence of heat source to combustion-acoustic system is described as the coefficient $\eta = -(\gamma - 1)q_0 G / \bar{\rho} \bar{u} c^2$ and the flame position x_F . Also, the reason why the flame position of Figure 9 where the response magnification comes to maximum is shifted to upper stream than that of Figure 7 would be considered to be caused by the influence of combustion conditions as well.

Fig. 10 Instability of the 1st. acoustic mode while changing nozzle position and fixed flame position.

Figure 10 shows the test results while changing fuel nozzle position, which were normalized by the same value as Figure 9. In this case, flame holder was fixed at the position $x_F = 220\text{mm}$, where experimental results of Figure 9 was the most unstable. Figure 10 was sensitive to parameter differences, and its reason can be considered that the system tended to be sensitive to parameter differences when the flame position was near 220mm as can be seen from Figure 9. If the flame positioned at more downstream where the system was stable are assumed, the analysis results come to be more dull to parameter differences. Analysis and experimental results of respective of Figure 7, 9 and 10 were well consistent with each other with respect to the whole trend and sensitivity to conditions.

5. Conclusions

The linear analysis models to predict the instability of one-dimensional combustion-acoustic system was formulated, and verified by two types premixed combustion laboratory tests. One of them verified the instability in the absence of fuel/air ratio fluctuation. The other verified the instability which was influenced from the advection delay time of the

fuel/air ratio fluctuation.

The phase lag between the heat release fluctuations and the acoustic system affects whether the combustion dynamics acts to destabilize the whole system. Especially, the reaction delay time of the heat release and the acoustic properties of behind the flame have important roles in the instability, and which determine the rough outline of the instability. Further, this instability is very sensitive to such conditions and parameters as the advection delay time. Therefore, not only whole system analysis methods, precise analysis parameter evaluation methods are also required, and those would be achieved by the development of large numerical simulation technique. In this paper, analysis parameters were forced to be slightly adjusted, because simple methods were applied for parameter evaluations. Although there were such slight adjustments, the whole trends of analysis results and test results were consistent with each other. By these verification results and theoretical discussions, we conclude that the analysis models of this paper have captured the basic framework of combustion-acoustic system, and have proceeded to the next development.

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