A Short Note on the Dissipation of Seiche*

By

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§1. The frictional force for the periodic current such as the secondary oscillation, tidal currents and seiche has been heretofore put to be proportional to the velocity itself, owing chiefly to the simplicity of the mathematical treatment. However, it is taken to be proportional to the square of the velocity in other phenomena, especially for the stationary current, such as the river flow. It seems to be important to determine which is more appropriate for expressing the frictional force, and the solution of this problem leads to some clues about the production of turbulence in the periodic current of the nature. Here the author gives preliminary discussions of this problem only from the mathematical point of view.

§2. We take the simple case of one dimensional oscillation within the basin with closed ends, then the equation of motion with the friction is

$$\frac{\partial u}{\partial t} = -g \frac{\partial \zeta}{\partial x} - k|u|^n u \quad (1)$$

where $-k|u|^n u$ is the friction term and $|u|$ means the absolute value of u . The equation of continuity is

$$\frac{\partial \zeta}{\partial t} = -h \frac{\partial u}{\partial x} \quad (2)$$

where h is the depth. The friction term is here taken generally to be proportional to the $(n+1)$ th power of the velocity. When we concern only of the principal oscillation, we can obtain some simple relation of the decrease of the amplitude with time under the assumption that the frictional force is small compared with the inertia term and restoring force. Only when the friction is proportional to the linear velocity we have the exact mathematical solution

as easily seen from the above equations, since in this case we have

$$\zeta = A \cos \frac{\pi}{a} x, \quad u = -\frac{a}{\pi h} \dot{A} \sin \frac{\pi}{a} x \quad (3)$$

from the eq. (2) and the boundary condition at the ends $x=0$ and $x=a$. Then we put (3) into (1) and we have exactly

$$\frac{d^2 A}{dt^2} + gh \left(\frac{\pi}{a} \right)^2 A + k \dot{A} = 0 \quad (4)$$

and thus the solution is given easily. In general case, the normal mode of the oscillation cannot given by the simple form as (3) and the exact solution should be expressed by the sum of the series of

$A_m \cos \frac{m\pi}{a} x$, which causes inevitably higher modes of oscillation. However, when the frictional force is slight, the normal function seems to be not so different from those given by (3). Further we assume that the relation (1) is valid only within the mean condition over the whole domain and we take the normal functions given by (3), then we have

$$\frac{\partial}{\partial t} \int_0^a u dx = -g(\zeta_a - \zeta_0) - k \int_0^a |u|^n u dx$$

or

$$\begin{aligned} \frac{d^2}{dt^2} A + \omega^2 A + \frac{1}{2} k \left(\frac{a}{h\pi} \right)^n \left| \frac{dA}{dt} \right|^n \frac{dA}{dt} \\ \times \int_0^\pi \sin^{n+1} \psi d\psi \end{aligned} \quad (5)$$

where $\omega^2 = gh \left(\frac{\pi}{a} \right)^2$.

For the linear frictional force this gives the exact equation for the amplitude as seen easily, and for the square velocity case we have the frictional term as

$$\frac{1}{4} k \left(\frac{a}{h} \right) |\dot{A}| A = e |\dot{A}| A.$$

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The general solution of (5) is quite complicated but under the assumption that $\epsilon|\dot{A}|A$ is small compared with the term $\omega^2 A$, we can obtain the approximate solution of slowly varying amplitude and phase lag such as⁽¹⁾

$$A = A(t) \sin [\omega t + \varphi(t)].$$

The amplitude for the square velocity case should be given by

$$\dot{A} = -\frac{\epsilon A^2 \omega}{2\pi} \int_0^{2\pi} |\cos \psi| \cos^2 \psi d\psi$$

or

$$\dot{A} = A_0 / (1 + A_0 q t), \quad q = \frac{k}{2} \sqrt{g/h}. \quad (6)$$

The general case is also given by

$$\dot{A} = -\frac{\epsilon_n A^{n+1} \omega^n}{2\pi} \int_0^{2\pi} |\cos^n \psi| \cos^2 \psi d\psi,$$

$$\epsilon_n = \frac{k}{2} \left(\frac{a}{h\pi} \right)^n \int_0^\pi \sin^{n+1} \psi d\psi$$

or

$$A^{-n} - A_0^{-n} = n q_n t \quad (7)$$

where we put q_n as $ckg^{1/2}h^{1/2-n}$ and c is the

numerical constant easily determined. For the linear velocity case this method gives the exact decrement of the amplitude with time $A(t) = A_0 e^{-(k/2)t}$ as easily computed. The change of the phase lag with time can be also obtained from the simple calculation but for the practical purpose, the determination of the change of the phase lag or the period from the practical record is more uncertain than that of the amplitude, therefore the explicit expression of it is retained here.

The exact solution for the square velocity case by use of the Fourier expansion will also be given in another paper. The above relations (6) and (7) indicate us that the decrement of the amplitude with time is dependent on the depth h and the initial amplitude A_0 for $n \neq 0$, whereas it is only dependent on A_0 for the frictional force proportional to the velocity itself. Such conditions suggests the possibility of the check from the practical observations and experiments about the question whether the frictional force is proportional to the velocity itself or not.

(1) Maclachlan: Ordinary non-linear differential equation. pp. 87~88. 1950. London.