Harmonic Analysis of Tides by Use of the Data for a Month*

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1. Introduction

In this country, the harmonic analysis of tidal observations is now carried out by two standard methods—those of G. H. DARWIN (involving the B.A. methods published between the years 1866 and 1885, and DARWIN's modifications given in 1892) and A. T. DOODSON (1928). The former method is widely used when the period of analysis is a multiple of fortnight. But the computations are somewhat complicated in this method, since it requires the hourly heights to be re-written several times. In contrast with it, the latter method (usually called by the name 'T.I. method') uses some stencils, and is fairly less labourious than the former one. However, it is now applicable only when the period of analysis is about a year. Therefore, for the analysis of short period observations, DARWIN's or modified DARWIN's method is exclusively used in the present days.

The author here intends to apply the principles of the T. I. method for the short period analysis. The period of the analysis is about 30 days. Of course, the long period constituents and those of smaller amplitudes will be hardly given in our case, but important 13 constituents (K_1 , P_1 , O_1 , Q_1 , K_2 , S_2 , L_2 , M_2 , N_2 , ν_2 , μ_2 , MS_4 , M_4)—which can be also computed by the DARWIN's method—will be easily obtained in sufficient approximation.

The labour involved is surely smaller in our method than the previous ones. Two computers will sufficiently perform the total analysis of daily and monthly processes in 3–5 hours, and the successive computations will be made within 2–3 hours by one computer.

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2. Notations

The following notation are is used in this paper:

ζ ...the sea-level measured from a fixed datum.

σ ...the speed of a constituent, in degrees per mean solar hour.

t ...the number of hours measured from the origin of time.

R ...the amplitude of a constituent.

ε ...the phase lag of a constituent at the origin of time.

Δ ...the constant angle proper to each constituent.

a_{pq} , b_{pq} ...the non-dimensional numbers proper to each constituent, which are given from

$$a_{pq} = a_p D_q, \quad b_{pq} = b_p D_q.$$

The numerical values of a_p , b_p and D_q are already given by DOODSON (loc. cit.).

X , e ...combinations of hourly heights.

3. Principles of Analysis

We first assume that the tidal variations are given by

$$\zeta = \zeta_0 + \sum R \cos(\sigma t - \varepsilon), \quad (1)$$

where the origin of time is chosen 336 hours after the initial time of analysis (cf. § 4). Then, the results of the monthly process will be

$$\left. \begin{aligned} A_{pq} &= X_{pq} + Y_{pq} = \sum a_{pq} R \cos \delta \\ B_{pq} &= X_{pq} - Y_{pq} = \sum b_{pq} R \sin \delta \end{aligned} \right\} \quad (2)$$

where $\delta = \varepsilon - \Delta$. The numerical values of a_{pq} and b_{pq} are computed from Table VII—XIII of DOODSON's paper.

It is not difficult to obtain $[R \cos \delta]$ or $[R \sin \delta]$ for each constituent from the algebraic equations (2), since a single constituent is roughly picked up in A_{pq} and B_{pq} . For instance, if $p=q=2$, the absolute values of a_{pq} and b_{pq} for the M_2 constituent are far larger than those for other constituents.

Therefore, the exact values of $[R \cos \delta]$ and $[R \sin \delta]$ are easily obtained by the successive approximation of only one step. 7 constituents can be separated by this method.

However, three pairs of constituents (K_1, P_1 ; S_2, K_0 ; and N_2, ν_2) are hardly separated by this process because of their proximate periods. Fortunately, owing to this proximity, their mutual relations will be sufficiently given from the equilibrium theory of NEWTON.

According to the equilibrium theory, we have

$$\left. \begin{aligned} H(P_1) &= 0.331 H(K_1), & \kappa(P_1) &= \kappa(K_1), \\ H(K_2) &= 0.272 H(S_2), & \kappa(K_2) &= \kappa(S_2), \\ H(\nu_2) &= 0.194 H(N_2), & \kappa(\nu_2) &= \kappa(N_2), \end{aligned} \right\} (3)$$

$$\left. \begin{aligned} A_{2K} &= 984.3 (R \cos \delta)_K = \frac{(0.068 + 0.260 \cos \theta) A_{20} + 0.260 \sin \theta B_{20}}{1.068 + 0.520 \cos \theta} = \beta A_{20} + \beta B_{20}, \\ B_{2K} &= -984.3 (R \sin \delta)_K = \frac{(0.068 + 0.260 \cos \theta) B_{20} - 0.26 \sin \theta A_{20}}{1.068 + 0.520 \cos \theta} = \alpha B_{20} - \gamma A_{20}, \\ A_{2S} &= A_{20} - A_{2K}, & B_{2S} &= B_{20} - B_{2K} \end{aligned} \right\} (6)$$

for the first approximation. The numerical values of α , β , and γ are tabulated in Table 1 as the functions of θ (for other pairs, β is not exactly equal to γ). The relations of the form (6) are to be used in the correction terms of $[R \cos \delta]$ and $[R \sin \delta]$. For the pair (K_1, P_1), the separation by this method is not necessary, since the proportions of $(a_{1q})_p$ to $(a_{1q})_k$ and $(b_{1q})_p$ to $(b_{1q})_k$ are exactly constants in this case.

The final results of H and κ for these constituents are better to be given from the corrected $[R \cos \delta]$ and $[R \sin \delta]$ by use of the following formulas

$$\left. \begin{aligned} H(S_2) &= \frac{RF}{\sqrt{P^2 + Q^2}}, \\ \kappa(S_2) &= \delta + \tan^{-1} \frac{Q}{P}, \end{aligned} \right\} (7)$$

where

$$\left. \begin{aligned} P &= \cos(V_0 + u + \Delta)_S + 0.260 \cos(V_0 + u + \Delta)_K, \\ Q &= \sin(V_0 + u + \Delta)_S + 0.260 \cos(V_0 + u + \Delta)_K. \end{aligned} \right\} (8)$$

For the pairs (K_1, P_1) and (N_2, ν_2), the coefficient 0.260 in (8) becomes 0.33 and 0.185, respectively.

4. Preliminary Notes

The data are supposed to consist of hourly

respectively. Therefore, for instance

$$\left. \begin{aligned} (R \cos \delta)_K &= 0.272 \{ (R \cos \delta)_S \cos \theta - (R \sin \delta)_S \sin \theta \}, \\ (R \sin \delta)_K &= 0.272 \{ (R \sin \delta)_S \cos \theta + (R \cos \delta)_S \sin \theta \}, \end{aligned} \right\} (4)$$

where

$$\theta = \delta_K - \delta_S = (v_0 + u + \Delta)_S - (v_0 + u + \Delta)_K \quad (5)$$

and the suffixes S and K denote the variables with respect to S_2 and K_2 constituent, respectively.

By use of the relations (4), we have

heights for about 30 days (Strictly speaking, the period of analysis is 713 hours). These data are usually given by the unit of centimeter. Sometimes they may be tabulated to the tenth of a foot. In our case, the accuracy of 1/10 feet is at least necessary for the analysis. At any rate, these data are to be written in the standard form of the monthly report. It is to be noted that the *height at the initial time of analysis* must be written in the *column of 19 h* or 20th column. The initial time of analysis is not necessarily adjusted to the standard time 19 h, but matters will perhaps be simpler if we can do so.

Six stencils for the daily process X_1, Y_1, X_2, Y_2, X_4 , and Y_4 are next to be cut to suit the form. The form of these stencils is entirely similar to that of DOODSON's elementary method.

The breaks of observations will be interpolated by the following methods;

(i) For the breaks within a few hours, they will be interpolated by means of the graph of the hourly heights.

(ii) For longer breaks, they will be estimated by drawing the graphs of heights at intervals of 25 hours.

However, in our case, these breaks should

be only occasional and should not exceed a day at a time. If the total breaks of observations exceed two or three days, the final results will become fairly doubtful.

Day numbers are to be written from -15 to $+15$. The time corresponding to the *first column* (0 h) of *16th row* (date 0) is taken as the *origin of time*. The first row (date -15) begins from the 20th column (19 h), and the final row (date $+15$) ends at the 13th column (12th). Other rows consist of 24 columns.

5. Process of analysis

The analysis consists of 4 processes;

- (i) daily process,
- (ii) monthly process,
- (iii) computations of $R \cos \delta$ and $R \sin \delta$ and
- (iv) determination of H and κ .

1. The daily process is entirely similar to that in DOODSON's elementary method. Six sets of values X_1 , Y_1 , X_2 , Y_2 , X_4 , and Y_4 are to be computed for the dates -14 , -13 , ..., $+14$.

This process is better to be computed by two independent computers. A *soroban* is most useful for these computations.

2. The monthly process is to be given for the following functions;

X_{10} , X_{12} , X_{13} ,
 Y_{10} , Y_{12} , Y_{13} ,
 X_{20} , X_{21} , X_{22} , X_{23} , X_{24} ,
 Y_{20} , Y_{21} , Y_{22} , Y_{23} , Y_{24} ,
 X_{42} , X_{44} ,
 Y_{42} , Y_{44} .

These computations are also better to be given by two independent computers,

If we further compute the functions of the form X_{2a} , Y_{2a} in addition to these functions, the check of the results will be easily made. However, as long as the computations are carried out very carefully and checked by two computers, further check by this method will be no longer necessary.

3. We must then prepare the forms of Table 6 and Table 7. Compute the numerical values of $V_0 + u$ and F at the origin of time for each constituent, and enter the values of A from Table 2.

4. Compute θ for two pairs (S_2 , K_2) and (N_2 , ν_2), and then compute the values of α , β , and γ from Table 1. Then, it is very easy to estimate A_{2K} , B_{2K} , A_{2N} , B_{2N} , $A_{2\nu}$, and $B_{2\nu}$ by use of the values A_{20} , B_{20} , A_{23} , and B_{23} (cf. formulas (6)).

5. Prepare the tables for the computations of $R \cos \delta$ and $R \sin \delta$. The form is as follows.

Principal term		Correction terms			Sum	Divisor	$R \cos \delta$	Constituent
Symbol	Value	A_{10}	A_{12}	A_{13}				
A_{10}						859.2		K_1^*
.....								

Enter the numerical values of A_{pq} , B_{pq} in them, and compute the correction terms by multiplying the tabular values (Tables 3 and 4). The numerical values of $R \cos \delta$ and $R \sin \delta$ are finally computed in these tables.

6a. H and κ for 7 constituents (Q_1 , O_1 , L_2 , M_2 , μ_2 , MS_4 , and M_4) are computed by use of Table 5. Entering the numerical

values of $R \cos \delta$ and $R \sin \delta$ in this table, we can easily compute the final results.

6b. For other 6 constituents, the numerical values of H and κ are given from formulas (7) and the similar formulas. We shall make use of Table 6 for these computations.

7. Lastly, all computations are to be checked once more.

6. Examples of analysis and the comparison with the results given by DARWIN's method

For an example, we shall tabulate the results of analysis for the Akashi Harbour (Lat. $34^{\circ}39'N$, Long. $135^{\circ}00'E$) in Table 7. These results are obtained from the continuous records measured by the tide gauge of Richard type in August, 1955.

For the purpose of comparison, we also show the results given by DARWIN's method in this table. These two results are in good agreement with each other except for the constituents of smaller amplitudes.

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Table 1. Numerical values of α , β , and γ against for θ 's values at the intervals of 10. (This table is of course used for the determination of α , β , and γ at the intermediate values θ . However, it is better to plot these values on diagrams, and to make use these diagrams for this purpose.

θ	K_2		μ_2			
	α	$\beta=\gamma$	α	β	γ	
0	0.206	0.000	0.156	0.000	0.000	360
10	0.205	0.029	0.155	0.025	0.021	350
20	0.200	0.057	0.150	0.050	0.042	340
30	0.193	0.086	0.144	0.074	0.063	330
40	0.182	0.114	0.134	0.098	0.083	320
50	0.167	0.142	0.120	0.121	0.103	310
60	0.149	0.170	0.104	0.143	0.121	300
70	0.126	0.196	0.184	0.162	0.138	290
80	0.097	0.221	0.060	0.130	0.153	280
90	0.063	0.244	0.033	0.194	0.165	270
100	0.023	0.262	0.002	0.204	0.173	260
110	-0.024	0.275	-0.032	0.208	0.176	250
120	-0.077	0.279	-0.069	0.205	0.174	240
130	-0.136	0.272	-0.106	0.193	0.164	230
140	-0.197	0.250	-0.143	0.172	0.146	220
150	-0.255	0.211	-0.177	0.141	0.119	210
160	-0.305	0.154	-0.203	0.100	0.085	200
170	-0.339	0.081	-0.221	0.052	0.044	190
180	-0.351	0.000	-0.227	0.000	0.000	180
	α	$-\beta=-\gamma$	α	$-\beta$	$-\gamma$	θ

Table 2. Values of Δ

Q_1	207.68	N_2	80.82	K_2	106.27
O_1	216.12	μ_2	81.94	M_4	88.51
P_1	231.86	M_2	89.25	MS_4	104.25
K_1	233.14	L_2	97.69		
μ_2	73.51	S_2	105.00		

Table 3. Computation of $R \cos \delta$

Principal function	Correction terms: multiples of						Divisor	Constituent
	A_{10}	A_{12}	A_{13}					
A_{10}	—	-0.104	0.029				852.2	K_1^*
A_{12}	0.004	—	0.206				1062.9	O_1
A_{13}	-0.002	-0.074	—				1062.8	Q_1
	Correction terms: multiples of							
	A_{2K}	A_{12}	A_{22}	A_{2N}	A_{2V}	A_{24}		
A_{20}	—	-0.104	0.017	0.006	0.044	0.019	984.3	S_2^*
A_{21}	-0.049	—	-0.004	-0.054	0.071	-0.092	928.3	L_2
A_{22}	0.012	0.030	—	0.124	-0.088	0.096	1099.0	M_2
A_{23}	-0.007	-0.068	-0.018	—	—	-0.013	1084.3	N_2^*
A_{24}	0.005	0.048	0.016	-0.037	0.083	—	1103.5	μ_2
	Correction terms: multiples of							
	A_{42}	A_{44}						
A_{42}	—	0.088					509.8	MS_4
A_{44}	0.016	—					503.8	M_4

Table 4. Computation of $R \sin \delta$

Principal function	Correction terms: multiples of						Divisor	Constituent
	B_{10}	B_{12}	B_{13}					
B_{10}	—	-0.020	0.029				-661.0	K_1^*
B_{12}	0.004	—	0.206				-703.4	O_1
B_{13}	-0.002	-0.074	—				-740.3	Q_1
	Correction terms: multiples of							
	B_{2K}	B_{21}	B_{22}	B_{2N}	B_{2V}	B_{24}		
B_{20}	—	-0.104	0.017	-0.005	0.041	0.018	-984.3	S_2^*
B_{21}	0.049	—	-0.004	0.049	-0.065	-0.085	-905.6	L_2
B_{22}	-0.012	0.031	—	-0.114	0.082	0.088	-1042.1	M_2
B_{23}	0.007	-0.068	-0.018	—	—	-0.122	-999.2	N_2^*
B_{24}	-0.005	0.048	0.016	0.034	-0.077	—	-991.9	μ_2
	Correction terms: multiples of							
	B_{42}	B_{44}						
B_{42}	—	0.088					-847.8	MS_4
B_{44}	0.016	—					-805.1	M_4

Table 5. Example of computation of H and κ (i)
(Akashi, August, 1955)

	O_1	Q_1	L_2	M_2	μ_2	MS_4	M_4
$R \cos \delta$	-12.57	- 2.23	0.80	8.21	- 0.68	- 1.27	1.75
$R \sin \delta$	16.27	2.90	0.63	11.22	- 3.44	0.43	0.81
R	20.56	3.66	1.02	13.90	3.51	1.34	1.93
F	0.998	0.998	0.797	0.886	0.996	0.996	0.991
$H=RF$	20.51	3.65	0.81	13.84	3.39	1.33	1.91
δ	127.68	127.58	38.25	53.80	258.83	161.46	24.95
V_0+u	206.62	186.64	289.47	91.88	181.56	91.88	183.66
Δ	216.12	207.68	97.69	89.25	73.51	104.25	88.51
$\kappa=\text{sum}$	190.42	161.90	65.51	234.93	153.90	357.53	297.12

Table 6. Example of computation of H and κ (ii)
(Akashi, August, 1955)

Sheet 1.

	K_1	P_1	S_2	K_2	N_2	ν_2
V_0+u	241.06	127.74	0.00	302.41	71.79	201.55
Δ	233.14	231.86	105.00	106.27	80.82	81.94
$\eta=\text{sum}$	114.20	359.60	105.00	46.68	152.61	283.49
$\cos \eta$	-0.4099	1.0000	-0.2588	0.6602	-0.8879	0.2332
$\sin \eta$	0.9121	-0.0070	0.9659	0.7511	0.4599	-0.9724

Sheet 2.

Table 7. Harmonic constants of tides at
the Akashi Harbour

	K_1^*	S_2^*	N_2^*	Constituent	Method in this paper		Darwin's method	
					H	κ	H	κ
θ	—	56.32	-130.88					
P	- 0.0679	-0.0871	- 0.8448					
Q	0.9098	1.1612	0.2800					
$R \cos \delta$	-10.68	-8.22	1.84	K_1	26.52	210.40	25.62	211.57
$R \sin \delta$	21.77	6.73	3.64	O_1	20.51	190.42	20.54	189.99
R	24.24	10.62	4.08	P_1	8.78	210.40	8.54	211.57
F	0.998	1.000	0.996	Q_1	3.65	161.90	3.61	171.95
$H = \frac{RE}{\sqrt{P^2+Q^2}}$	26.52	9.12	4.56	M_2	13.84	234.93	14.71	240.15
δ	116.13	140.68	116.82	S_2	9.12	234.96	9.53	233.79
$\tan^{-1} Q/P$	94.27	84.28	161.67	K_2	2.48	234.96	2.60	233.79
$\kappa=\text{sum}$	210.40	234.96	278.46	N_2	4.56	278.49	3.13	237.26
n	0.331	0.272	0.194	L_2	0.81	65.51	0.61	177.47
nH	8.78	2.48	0.88	ν_2	0.88	278.49	0.61	237.26
				μ_2	3.49	153.90	4.48	152.44
				M_4	1.91	297.12	1.49	334.51
				MS_4	1.33	357.53	1.26	333.54