On the Transient Response of a β -plane Ocean*

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1. Introduction

In addition to inertio-gravity waves two mechanisms exist in the ocean for the propagation of large scale disturbances. The first is that of horizontal divergence which occurs together with a change in the free surface height. (This mechanism is necessary also for the generation of inertio-gravity waves). The second mechanism is due to the variation of the coriolis parameter—the so-called β -effect. The transient nature of the latter mechanism can be seen from the following example.

Suppose that a rotationally symmetric (cyclonic) wind begins to blow over the water surface. Locally the ocean will respond in such a way as to come to equilibrium with the stress acting on the surface. Initially, therefore, the sea surface will assume a rotationally symmetric slope with the low point in the center. With non-uniform rotation the water in the northern part of the region will travel slower than the water in the southern section. Water will converge in the eastern part and diverge in the region west of the center. Hence, an additional re-distribution of mass is necessary to bring the system to equilibrium. The system is unsteady because the non-uniformity of rotation requires a finite time to manifest itself.

It is the purpose of this paper to investigate the effects of the above mechanisms for the case of an ocean which is rotating non-uniformly and which is subjected to a localized wind stress. The analysis is carried out on the β -plane (as introduced by ROSSBY (1939), i.e., the coriolis force is considered to be a linear function of latitude when differentiated but it is taken to be constant when undifferentiated. Inertio-gravity waves

are neglected and attention is directed toward the quasi-geostrophic response only. The last restriction is made for both mathematical and physical simplicity. VERONIS and STOMMEL (1956)* have shown that inertio-gravity waves are unimportant in the energetics of the system when the time scale is longer than a half-pendulum day.

In the present paper as in previous ones the ocean is assumed to be infinite in the horizontal, i.e., no lateral boundaries are included. Such a restriction limits the validity of the analysis to a time scale comparable to that which is required for a barotropic ROSSBY wave to reflect from a boundary and interact with other waves. The difficulties which are encountered when boundaries are included are realized very quickly when one attempts to solve the problem of a bounded, frictionless, β -plane ocean, subjected to a wind stress which is zonally uniform and varies sinusoidally in latitude and in time. If the period of the wind stress variation is large, the assumed width of the ocean becomes critical. The reason for this critical dependence on the exact width of the ocean is that barotropic ROSSBY waves have a period of a few days for systems of oceanic scale and resonance occurs for any systems which allows an integral number of ROSSBY wave periods to occur for each period of the wind stress. The difficulty is clearly a mathematical one and is partly due to the fact that friction is neglected. However, since friction is probably not important in the dispersion of large scale disturbances from a localized wind stress, it will be neglected here along with boundary effects.

Finally, a note must be added about the value of the present investigation. Although general analytical solutions to arbitrary forcing functions are often desirable, the useful-

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* Hereafter referred to as AVS.

ness of such solutions are determined by the facility with which they can be applied to actual situations. Unfortunately, the mathematical complexity of the present solution renders it practically useless. One must impose severe restrictions on the system to derive even a qualitative result. Perhaps the principal justification for presenting this study is that several other investigators have expressed an interest in pursuing a similar course. The horrendous computational problem which faces them is given in the final solution.

2. Mathematical model

Consider an infinite two-layer ocean of finite depth and initially at rest with respect to a rotating coordinate system. At time $t=0$ a wind stress begins to act on the surface. The linear equations governing the surface height, thermocline and vertically averaged velocities in each layer can be derived from the equations of motion and continuity for a fluid in a rotating system. It suffices here to consider a one-layer system. The extension to a two-layer fluid is given in AVS and will not be reproduced, since the final numerical results here are restricted to a barotropic system. Thus

$$\frac{\partial u}{\partial t} - fv = -g \frac{\partial \eta}{\partial x} + \frac{\tau_x}{D} \quad (1)$$

$$\frac{\partial v}{\partial t} + fu = -g \frac{\partial \eta}{\partial y} + \frac{\tau_y}{D} \quad (2)$$

$$\frac{\partial \eta}{\partial t} + D \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = 0 \quad (3)$$

where $\tau = (\tau_x, \tau_y)$ is the wind stress vector, D is the depth, η is the free surface deviation, f is the coriolis term and the remaining variables have their usual meanings.

Cross-differentiating (1) and (2) to eliminate η yields the vorticity equation

$$\frac{\partial \zeta}{\partial t} + f \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \beta v = \frac{1}{D} \text{curl}_z \tau \quad (4)$$

where $\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ is the vorticity, and $\beta = \partial f / \partial y = \text{constant}$.

To study the geostrophic response of the ocean one may now make the geostrophic

approximation, viz.,

$$-fv = -g \frac{\partial \eta}{\partial x} \quad (5)$$

$$fu = -g \frac{\partial \eta}{\partial y} \quad (6)$$

then from (3), (4), (5), (6)

$$\begin{aligned} \frac{\partial}{\partial t} \nabla^2 \eta - \frac{1}{\lambda^2} \frac{\partial \eta}{\partial t} - \frac{\beta}{f} \frac{\partial^2 \eta}{\partial t \partial y} + \beta \frac{\partial \eta}{\partial x} \\ = \frac{1}{f \lambda^2} \text{curl}_z \tau \end{aligned} \quad (7)$$

where $\lambda^2 = gD/f^2$ and f is now assumed constant.

If the x and y scales are comparable, then the term $\frac{B}{f} \frac{\partial^2 \eta}{\partial t \partial y}$ may be neglected since for geostrophic response the time scale is assumed to be much longer than a half-pendulum day. Thus, $\frac{\partial \eta}{\partial x} \gg \frac{1}{f} \frac{\partial^2 \eta}{\partial t \partial y}$, and

$$\frac{\partial}{\partial t} \nabla^2 \eta - \frac{1}{\lambda^2} \frac{\partial \eta}{\partial t} + \beta \frac{\partial \eta}{\partial x} = \frac{1}{f \lambda^2} \text{curl}_z \tau \quad (8)$$

The boundary conditions are

$$\eta = 0 \quad x = y = \infty, \quad \text{all } t \quad (9)$$

$$\eta = 0 \quad \text{everywhere at } t = 0$$

3. Solution

The response of the ocean will be determined for a wind stress distribution given by a tweak, i.e., the curl of the wind stress is assumed to act at a point and at one instant of time. Mathematically,

$$\text{Curl}_z \tau = W \delta(\xi, \zeta, \tau) \quad (10)$$

where W is the amplitude of $\text{curl}_z \tau$ and $\delta(\xi, \zeta, \tau)$ is the dirac delta given by

$$\begin{aligned} \delta(\xi, \zeta, \tau) &= \infty & \xi = 0, \zeta = 0, \tau = 0 \\ &= 0 & \text{elsewhere} \end{aligned} \quad (11)$$

The dirac delta can be used to generate more general solutions. Thus from the following property of δ

$$\int_{-\infty}^{\infty} \delta(\xi) \phi(x - \xi) d\xi = \phi(x) \quad (12)$$

it is clear that once the solution for η is known for δ , a general solution for η may be derived by writing the general wind curl distribution as

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(\xi, \zeta, \tau) \operatorname{curl}_{\mathbf{z}} \boldsymbol{\tau}(x-\xi, y-\zeta, t-\tau) d\xi d\zeta d\tau \quad (13)$$

and the final solution as

$$\eta(x, y, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \eta_8(x', y', t') \operatorname{curl}_{\mathbf{z}} \boldsymbol{\tau}(x-x', y-y', t-t') dx' dy' dt' \quad (14)$$

where $\eta_8(x', y', t')$ is the solution for $\operatorname{curl}_{\mathbf{z}} \boldsymbol{\tau} = \delta(\xi, \zeta, \tau)$

One can derive the solution η_8 by taking a LAPLACE transform with respect to time and solving the resultant equation by either a GREEN'S function method or by FOURIER transforms. The mathematical problem then reduces to inverting the expression

$$\bar{\eta} = -\frac{W e^{-\alpha \beta / 2 p}}{2 p f \lambda^2} K_0 \left[r \sqrt{\left(\frac{\beta}{2 p} \right)^2 + \frac{1}{\lambda^2}} \right] \quad (15)$$

where $r = \sqrt{x^2 + y^2}$, is the LAPLACE transform variable and K_0 is the BESSEL function of the second kind. Thus it is necessary to find

$$\eta = \frac{1}{2 \pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \bar{\eta} e^{pt} dp \quad (16)$$

BASSETT'S integral representation of the K_0 function

$$K_0(r\alpha) = \int_0^{\infty} \frac{\cos ru}{\sqrt{u^2 + \alpha^2}} du, \quad r > 0, \\ |arg \alpha| < \pi/2$$

Hence,

$$K_0 \left[r \sqrt{\left(\frac{\beta}{2 p} \right)^2 + \frac{1}{\lambda^2}} \right] \\ = \int_0^{\infty} \frac{\cos ru}{\sqrt{u^2 + 1/\lambda^2} \sqrt{p^2 + \frac{b^2}{4(u^2 + 1/\lambda^2)}}} du \quad (17)$$

If (17) be substituted into (16) and the integral signs are interchanged (the latter operation can be proved to be admissible in the present case) and if the following LAPLACE inversions are employed:

$$\text{for } a > 0, \quad 1 - e^{a/p} \sim \left(\frac{a}{t} \right)^{1/2} J_1(2\sqrt{at})$$

$$\text{for } a < 0, \quad 1 - e^{a/p} \sim - \left(\frac{a}{t} \right)^{1/2} I_1(2\sqrt{at})$$

$$\frac{1}{(p^2 + \gamma^2)^{1/2}} \sim J_0(\gamma t)$$

one finds that

$$\eta = \frac{W}{2 \pi f \lambda^2} \int_0^{\infty} \frac{\cos ru}{\sqrt{u^2 + 1/\lambda^2}} \int_0^t \left(\frac{x\beta}{2t} \right)^{1/2} J_1 \left(2\sqrt{\frac{x\beta\tau}{2}} \right) J_0 \left[\frac{\beta(t-\tau)}{\sqrt{u^2 + 1/\lambda^2}} \right] d\tau du \quad (18) \\ - \frac{W}{2 \pi f \lambda^2} \int_0^{\infty} \frac{\cos ru J_0 \left(\frac{\beta t}{2\sqrt{u^2 + 1/\lambda^2}} \right)}{\sqrt{u^2 + 1/\lambda^2}} du \quad \text{for } x > 0$$

For $x < 0$, $-J_1$ must replace J_1 .

Expression (18) represents η_8 in equation (14). Thus a general solution for the geostrophic response of the sea surface to an arbitrary wind stress distribution is obtained.

For $\beta = 0$, (18) reduces to

$$\eta = -\frac{W}{2 \pi f \lambda^2} K_0 \left(\frac{r}{\lambda} \right), \quad t > 0 \quad (19)$$

which can also be derived by neglecting β originally.

The other variables for both upper and lower layers may be found from (5) and (6) and from the expressions given in AVS for the baroclinic mode.

4. Quasi-non-divergent flow

Expression (18) presents one with a rather severe computational task. Hence, an approximate solution for quasi-non-divergent flow (very large λ^2) is given below. Numerical values can be computed relatively simple for the latter solution.

If the scale length of the disturbances is L , say, and if $\frac{\lambda}{L} \gg 1$, then the divergence is small and one can derive an approximate solution by neglecting the term $\frac{1}{\lambda^2}$ in the radical in (18) or by neglecting the divergence (i.e., the term $\frac{1}{\lambda^2} \frac{\partial \eta}{\partial t}$) in (8) and solving the resultant problem. Then,

$$\eta = -\frac{W}{2 \pi f \lambda^2} \int_0^t \frac{2 K_0(2\sqrt{r\beta\tau}) \cos \sqrt{2(x+r)\beta(t-\tau)}}{\sqrt{\tau(t-\tau)}} d\tau$$

A convenient method for evaluating from (20) is given by THOMPSON (1952). Let $\tau = t \sin^2 \phi$, $\alpha = \beta r t$, $\gamma = \cos \frac{\theta}{2}$, $x = r \cos \theta$.

Then

$$\eta = -\frac{W}{2\pi f\lambda^2} \int_0^\infty \frac{J_0(2\alpha\sqrt{\gamma^2+z^2})}{\sqrt{1+z^2}} dz \quad (21)$$

The integral for η may be shown to satisfy

$$\frac{\partial^2 \eta}{\partial \alpha \partial \gamma} + 4\alpha\gamma\eta = 0 \quad (22)$$

The partial differential equation (22) can be evaluated numerically provided values of η are known along an $\alpha=\text{constant}$ and a $\gamma=\text{constant}$ line in the α, γ plane. The sevalues can be computed directly from (21).

5. Results

Before the numerical results are presented, a few facts about the approximate solution should be noted.

The first of the two mechanisms listed in the introduction, i.e., horizontal divergence, is absent in the approximate solution. The essential effect of horizontal divergence is to bring about a re-distribution of mass in this model in response to the imposed distribution of wind stress. Therefore, the re-distribution takes place instantaneously. Thus the mechanism for the removal* of water from the origin is not included.

* The total stress curl is assumed positive here. In general, the integrated value of the curl will vanish and the water will simply be re-distributed rather than removed from the system.

Also, the influence function $W\delta(\xi, \zeta, \tau)$ is used here as the total wind stress instead of as the auxiliary function for the more general wind stress distribution. Hence, for the particular storm considered, the entire influence of the storm is concentrated at one point in space and at one instant of time. Clearly, the numerical results have only a qualitative significance and even the latter must be restricted to regions removed from the point of application of the wind stress and to times well after the wind stress is applied.

The amplitude W in (10) thus contains the entire effect of the storm on the ocean. A wind system of solid rotation acting for a period of one day and over a region of 500 km. radius is assumed. The velocity at the outer edge of the system is chosen to be 30 m/sec. The transverse wind stress component, τ_θ , (there is no radial component here) can be computed from

$$\tau_\theta = \kappa \rho v_\theta^2$$

where ρ =density of air, $\kappa=2.3 \times 10^{-3}$ and v_θ is the transverse wind velocity. From the above values

$$W = 2\pi \times 10^{12} \text{ m}^3/\text{sec}$$

With

$$f = 10^{-4} \text{ sec}^{-1}, \quad \lambda^2 = 4 \times 10^{12} \text{ m}^2$$

one can compute the magnitude of η .

Fig. 1. Contour heights (in cm.) of surface deviation from its equilibrium value. Distances along any radial line are measured in units of $\sqrt{\beta\tau t}$. Graph is symmetric about the X axis.

Figure 1 shows both the time and space distribution of η . Proceeding along a radial line indicates the change of the surface heights at either (a) a point in space as time increases or (b) an instant of time with increasing distance from the origin. The

effect of β is indicated by the asymmetry in the eastwest direction. For constant f the contour lines would be symmetrical about the origin. The large negative values of η near the origin reflect of an imposed positive vorticity.

Fig. 2. η at a distance of 2000 km from tweak as a function of t .

In figure 2 the surface height is plotted as a function of time for points 2000 kms. from the tweak and at points to the east, north and west of the disturbance. The origin reflects the unrealistic nature of the disturbance. To the east and north there are wave crests (after the initial depression) and to the west there is a single trough. Then the qualitative picture described in the introduction is borne out here. There is an additional effect of the east-west asymmetry which does not enter the simple picture given previously but which must enter into the re-distribution of mass.

The waves of shortest period (approximately five days) occur due east of the tweak and the periods become longer as the direction is rotated about the origin from east to west. The waves are not waves in the simple Fourier sense since there is some dispersion. It is interesting to note that the relative amplitude of each wave decreases slowly with time. To the east of the tweak the height of the second peak is still one third that of

the first and it occurs six days later. For points closer to the origin the wave "periods" would be shorter.

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