The Natural Coordinates on the Equiscalar Surface Undulating with Time*

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1. Introduction

HOLMBOE and his collaborators (1945) used the natural coordinates extensively in "Dynamical Meteorology", but their treatise was restricted within the horizontal flow in level surface, casting somewhat important role to geometrical intuition in the process of expressing equations by the natural coordinates. ARAKAWA (1951) tried to derive more consistently the equations of motion in the natural coordinates. However, they could not touch on the second derivative because their definition of the natural coordinates is mathematically obscure.

In a previous paper, the present author introduced analytically the natural coordinates basing on the extended directional differentiation and referring to the isobaric surfaces, and the geometry and kinematics of this coordinates were treated applying some part of them to the Kuroshio System (KAWAI, 1957). Though the vertical motion was taken into consideration and the second derivative was referred to, in order to simplify the mathematical expression there were adopted three assumptions upon the isobaric surfaces: the steady state of pressure field, the coincidence of isobaric surfaces with the surfaces of equal depth below the sea surface and the neglect of the curvature of isobaric surfaces. In short, this system was a natural coordinates on the isobaric surface, whose curvature is negligibly small. Thus, it is needful to combine more strictly the natural coordinates with isobaric surface or further with any equiscalar surface undulating with time.

The quasi-Lagrangian coordinates proposed by STARR (1945) were investigated in detail by ELIASSEN (1949), but the condition of the hydrostatic equilibrium was assumed and the vertical \( p \)-velocity divided by gravity force is substituted for the vertical velocity itself, neglecting the time change of the level of isobaric surfaces. The natural coordinates on an equiscalar surface in the present paper seems to be a combination of the natural coordinates with the quasi-Lagrangian coordinates, but it has not the character of the quasi-Lagrangian coordinates because the coordinate replaced for the vertical coordinate in the present system is not a scalar quantity but is a metric coordinate along the normal of the equiscalar surface.

In the course of deriving the present system it is not necessary to assume such conditions as assumed in HOLMBOE's natural coordinates and in ELIASSEN's quasi-Lagrangian coordinates, and the present system has the merit that we can derive, in its use, dynamical laws in complete form analytically without any aid of geometrical figures where finite difference is replaced for differential. As an application, the potential vorticity in the ocean is analytically derived (§7).

Since the natural coordinates on an equiscalar surface has the character of the so-called metric coordinates being essentially different from the curvilinear coordinates, the chief disadvantage in its use is that the respective coordinate is not independent of time and each other, which makes integration of the equation difficult, while the merit of the metric coordinates is that we can readily make practical calculations in its use. As another application using this merit, the orders of magnitude of terms appearing in the equations of motion expressed by the natural coordinates on the isobaric surface are estimated, and the equations are reduced to simpler forms dropping smaller terms.

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2. Orientation of the coordinate axes

Let the x and y axes take on a level surface, directed positive to the east and to the north respectively, and let the z axis take vertically, positive upward. When a current vector \( V \) is given at a point \( O \), we can determine at \( O \) three unit vectors along the axes of the natural coordinates on an equiscalar surface as

\[
\begin{align*}
q &= \frac{\pm \text{grad} f}{|\text{grad} f|} \quad (\text{grad } f \neq 0) \\
\mathbf{n} &= \frac{\mathbf{q} \times \mathbf{V}}{|\mathbf{q} \times \mathbf{V}|} \quad (|\mathbf{q} \times \mathbf{V}| = |\mathbf{u}| \neq 0) \\
\mathbf{s} &= \mathbf{n} \times \mathbf{q}
\end{align*}
\]

where \( f \) indicates the scalar quantity whose equiscalar surface is designated as the coordinate surface. In the following, the equiscalar surface of \( f \) designated as the coordinate surface is called the \textit{reference equiscalar surface} distinguishing it from the equiscalar surface of any other scalar quantity \( F \), and the natural coordinates on an equiscalar surface of \( f \) are called the \textit{natural f-coordinates} for simplicity. The double sign in front of grad \( f \) in the first equation of (1) has to be selected so that the unit vector \( q \) is directed upward.

The definition (1) is represented in words as follows; \( s \) is the unit vector of \( u \) which is the projected component of \( V \) normally upon the reference equiscalar surface of \( f \) passing through \( O \), and \( n \) is the unit vector lying on the same equiscalar surface as \( s \) and is perpendicular, positive counter-clockwise, to \( s \), while \( q \) is the unit normal of the reference equiscalar surface, positive upward. The system of \( s, n \) and \( q \) forms a right-handed orthogonal system locally at each point. If a vector field of current and a scalar field of \( f \) under consideration are given at an instant three-dimensionally, \( s, n \) and \( q \) are determined at each point so far as the current direction does not coincide with the normal of the equiscalar surface of \( f \) and grad \( f \) does not vanish.

Let \( s' \) and \( n' \) denote the vertically projected vectors of \( s \) and \( n \) upon a level surface respectively, then \( s' \) is not at right angles to \( n' \). Let \( \theta_1 \) and \( \theta_2 \) denote the angles of \( s' \) and \( n' \) made with the x and y axes, positive counter-clockwise, respectively. Let \( \varepsilon_1 \) and \( \varepsilon_2 \) denote the upward inclinations of \( s \) and \( n \) to a level surface respectively. Using these angles, from Fig. 1 the unit vectors along the axes of the natural f-coordinates \((s, n, q)\) are expressed by the unit vectors along the axes of the Cartesian coordinates \((i, j, k)\) as follows:

\[
\begin{align*}
\mathbf{s} &= T_{11}i + T_{12}j + T_{13}k \\
\mathbf{n} &= T_{21}i + T_{22}j + T_{23}k \\
\mathbf{q} &= \mathbf{s} \times \mathbf{n} = T_{31}i + T_{32}j + T_{33}k
\end{align*}
\]

where \( T_{ij} \) has the meaning:

\[
\begin{align*}
T_{11} &= \cos \varepsilon_1 \cos \theta_1, & T_{21} &= -\cos \varepsilon_2 \sin \theta_2 \\
T_{31} &= \cos \varepsilon_1 \sin \varepsilon_2 \sin \theta_1 - \sin \varepsilon_1 \cos \varepsilon_2 \cos \theta_2 \\
T_{12} &= \cos \varepsilon_1 \sin \theta_1, & T_{22} &= \cos \varepsilon_2 \cos \theta_2 \\
T_{32} &= -\left(\cos \varepsilon_1 \sin \varepsilon_2 \cos \theta_1 + \sin \varepsilon_1 \cos \varepsilon_2 \sin \theta_2\right) \\
T_{13} &= \sin \varepsilon_1, & T_{23} &= \sin \varepsilon_2 \\
T_{33} &= \cos \varepsilon_1 \cos \varepsilon_2 \cos (\theta_2 - \theta_1)
\end{align*}
\]

Since any direction can be designated completely by three angles, there must be a relation among the four angles \( \varepsilon_1, \varepsilon_2, \theta_1 \) and \( \theta_2 \), and it is expressed from the perpendicularity of \( n \) to \( s \) by the formula:

\[

\sin (\theta_2 - \theta_1) = \tan \varepsilon_1 \tan \varepsilon_2
\]

or

\[

T_{33} = \cos \varepsilon_1 \cos \varepsilon_2 \cos (\theta_2 - \theta_1) = \sqrt{1 - \sin^2 \varepsilon_1 - \sin^2 \varepsilon_2}
\]

Differentiating (4), we have

\[

T_{33} d\theta_1 + (\sin \varepsilon_2 / \cos \varepsilon_1) d\varepsilon_1 = T_{33} d\theta_2 - (\sin \varepsilon_1 / \cos \varepsilon_2) d\varepsilon_2.
\]

We can further verify from (4)
where \( \delta \) indicates the Kronecker's symbol.

Accordingly, we get from (2) and (5)

\[
\begin{align*}
 i &= T_{31} s + T_{32} n + T_{33} q \\
 j &= T_{13} s + T_{23} n + T_{33} q \\
 k &= T_{23} s + T_{32} n + T_{33} q.
\end{align*}
\]

Using (2), (3), (4), (6) and (4'), the differentials of the unit vectors \( s, n \) and \( q \) are expressed as

\[
\begin{align*}
 ds &= nd\theta_s - qd\theta_n - qd\theta_q, \\
 dq &= sd\theta_n - nd\theta_q,
\end{align*}
\]

where

\[
\begin{align*}
 d\theta_s &= T_{33}/\cos\epsilon_2 d\epsilon_2 + \sin\epsilon_1 d\theta_2, \\
 d\theta_n &= -T_{33}/\cos\epsilon_1 d\epsilon_1 + \sin\epsilon_2 d\theta_1, \\
 d\theta_q &= T_{33}d\theta_1 + \sin\epsilon_2/\cos\epsilon_1 d\epsilon_1 - T_{33}d\theta_2 - \sin\epsilon_1/\cos\epsilon_2 d\epsilon_2.
\end{align*}
\]

Now, we shall consider the geometrical meaning of \( d\theta_s, d\theta_n \) and \( d\theta_q \). The infinitesimal rotations \( d\theta_1 \) and \( d\theta_2 \) have the same unit axial vector \( k \), while \( d\theta_1 \) and \( d\theta_2 \) have the unit axial vectors \( n'' \) and \( s'' \) respectively.

Here, \( n'' \) and \( s'' \) are normal to the \( s'-s \) plane and to the \( n'-n \) plane respectively (Fig. 1), and they are expressed by

\[
\begin{align*}
n'' &= i \sin \theta_1 - j \cos \theta_1 \\
&= q \sin \epsilon_1/\cos \epsilon_1 - n T_{33}/\cos \epsilon_1 \\
s'' &= i \cos \theta_1 + j \sin \theta_1 \\
&= -q \sin \epsilon_1/\cos \epsilon_2 + s T_{33}/\cos \epsilon_2.
\end{align*}
\]

On the other hand, from the above equations, the last equation of (6) and (8), the infinitesimal rotations around \( s, n \) and \( q \) axes are expressed by

\[
\begin{align*}
s \cdot (kd\theta_1 + n''d\epsilon_2) &= d\theta_s, \\
n \cdot (kd\theta_1 + n''d\epsilon_2) &= d\theta_n, \\
q \cdot (kd\theta_1 + n''d\epsilon_2) &= d\theta_q.
\end{align*}
\]

In words, \( d\theta_s, d\theta_n, d\theta_q \) signify the infinitesimal rotations of this coordinate system around the respective coordinate axes.

### 3. The directional differentiation along the axes of the natural \( f \)-coordinates

In this paragraph and the next, geometry and kinematics in the natural \( f \)-coordinates are introduced with the aid of the directional differentiation.

When a scalar function \( F(x, y, z, t) \) is differentiable, the directional derivatives in the direction of \( s, n \) and \( q \) are defined as follows:

\[
\begin{align*}
 \frac{\partial F}{\partial s} &= T_{11} \frac{\partial F}{\partial x} + T_{12} \frac{\partial F}{\partial y} + T_{13} \frac{\partial F}{\partial z}, \\
 \frac{\partial F}{\partial n} &= T_{21} \frac{\partial F}{\partial x} + T_{22} \frac{\partial F}{\partial y} + T_{23} \frac{\partial F}{\partial z}, \\
 \frac{\partial F}{\partial q} &= T_{31} \frac{\partial F}{\partial x} + T_{32} \frac{\partial F}{\partial y} + T_{33} \frac{\partial F}{\partial z}.
\end{align*}
\]

Putting \( F = \varphi \) (geopotential) in (9) and putting \( F = f \) (reference scalar quantity) in the first equation of (10), we get from the conditions: \( \frac{\partial \varphi}{\partial x} = \frac{\partial \varphi}{\partial y} = 0 \) and \( \frac{\partial f}{\partial s} = \frac{\partial f}{\partial m} = 0 \) the formulas:

\[
\begin{align*}
 \frac{\partial \varphi}{\partial s} &= g \sin \epsilon_1, \\
 \frac{\partial \varphi}{\partial m} &= g \sin \epsilon_2, \\
 \frac{\partial \varphi}{\partial q} &= g T_{33}, \\
 \frac{\partial f}{\partial z} &= T_{33} \frac{\partial f}{\partial q}.
\end{align*}
\]

From the definition (9) it is verified that the directional differentiations of sum, product and quotient of two or more functions and a function of another function etc. can be calculated under the same rule as in the usual differentiation, even if \( \epsilon_1, \epsilon_2, \theta_1 \) and \( \theta_2 \) are variable with respect to \( x, y, z \) and \( t \).

However, when considering the derivatives of the second order, the difference between the usual differentiation and the variable directional differentiation revealed as shown below.

Differentiating (9) with respect to time \( t \), we get from (10), (3), (8), (4) and (4')

\[
\begin{align*}
 \frac{\partial}{\partial t} \left( \frac{\partial F}{\partial s} \right) &= \frac{\partial^2 F}{\partial s \partial t}, \\
 \frac{\partial}{\partial t} \left( \frac{\partial F}{\partial n} \right) &= \frac{\partial^2 F}{\partial n \partial t}, \\
 \frac{\partial}{\partial t} \left( \frac{\partial F}{\partial q} \right) &= \frac{\partial^2 F}{\partial q \partial t}.
\end{align*}
\]

Putting \( F = f \) in (12), we get

\[
\begin{align*}
 \frac{\partial}{\partial s} \left( \frac{\partial f}{\partial t} \right) &= \frac{\partial^2 f}{\partial s \partial t}, \\
 \frac{\partial}{\partial m} \left( \frac{\partial f}{\partial t} \right) &= \frac{\partial^2 f}{\partial m \partial t}, \\
 \frac{\partial}{\partial q} \left( \frac{\partial f}{\partial t} \right) &= \frac{\partial^2 f}{\partial q \partial t}.
\end{align*}
\]
Substituting \( x, y, \) or \( z \) for \( t \) in (12) and (13), these equations are established, whereas \( s, n \) or \( q \) cannot be substituted for \( t \) in (12) and (13), and the corresponding equations have to be obtained newly from (9), (10), (3), (8), (4) and ('4) as below:

\[
\frac{\partial}{\partial t} \left( \frac{\partial F}{\partial q} \right) - \frac{\partial}{\partial q} \left( \frac{\partial F}{\partial t} \right) = \frac{\partial F}{\partial q}, \quad \text{\( \partial \partial \)}\]

Putting \( F = f \) in (14), we get

\[
\frac{\partial f}{\partial s} + \frac{\partial f}{\partial n} = 0
\]

Substituting \( x, y \) and \( z \) respectively for \( t \) in (12), we get from (9) and (10)

\[
\Delta F = \frac{\partial^2 F}{\partial s^2} + \frac{\partial^2 F}{\partial n^2} + \frac{\partial^2 F}{\partial q^2} + \left( \frac{\partial \theta_a}{\partial s} - \frac{\partial \theta_n}{\partial n} \right) \frac{\partial F}{\partial s} + \left( \frac{\partial \theta_n}{\partial n} - \frac{\partial \theta_q}{\partial q} \right) \frac{\partial F}{\partial n} + \left( \frac{\partial \theta_q}{\partial q} - \frac{\partial \theta_s}{\partial s} \right) \frac{\partial F}{\partial q}. \tag{16}
\]

4. Vector operations

When a scalar function \( F \) is differentiable, we get from (6), (10) and (5)

\[
\text{grad} F \equiv i \frac{\partial F}{\partial x} + j \frac{\partial F}{\partial y} + k \frac{\partial F}{\partial z} = s \frac{\partial F}{\partial s} + n \frac{\partial F}{\partial n} + q \frac{\partial F}{\partial q}, \tag{17}
\]

so that the gradient on the equiscalar surface of \( f \) is defined as

\[
\text{grad}_F \equiv s \frac{\partial F}{\partial s} + n \frac{\partial F}{\partial n}. \tag{17'}
\]

When a vector \( A \) is given, it is dissolved into three components in the two coordinates systems as

\[
A = iA_s + jA_y + kA_z = sA_s + nA_n + qA_q. \tag{18}
\]

Considering (6) and (2), we have from (18)

\[
\begin{align*}
A_s &= T_{11}A_s + T_{12}A_y + T_{13}A_z \\
A_n &= T_{21}A_s + T_{22}A_y + T_{23}A_z \\
A_q &= T_{31}A_s + T_{32}A_y + T_{33}A_z
\end{align*}
\]

and

\[
\begin{align*}
A_s &= T_{11}A_s + T_{12}A_y + T_{13}A_z \\
A_n &= T_{21}A_s + T_{22}A_y + T_{23}A_z \\
A_q &= T_{31}A_s + T_{32}A_y + T_{33}A_z
\end{align*}
\]

Proceeding now to the expression of \( \text{div} A \) by the natural \( f \)-coordinates, we have from (18)

\[
\text{div} A = A_s \text{div} s + A_n \text{div} n + A_q \text{div} q + \frac{\partial A_s}{\partial s} + \frac{\partial A_n}{\partial n} + \frac{\partial A_q}{\partial q}. \tag{21}
\]

Further, we get from (2), (10), (4) and ('4)

\[
\begin{align*}
\text{div} s &= \frac{\partial \theta_s}{\partial s} - \frac{\partial \theta_n}{\partial n} \\
\text{div} n &= \frac{\partial \theta_n}{\partial q} \\
\text{div} q &= \frac{\partial \theta_q}{\partial q} - \frac{\partial \theta_s}{\partial s}
\end{align*}
\]

The divergence on the equiscalar surface of \( f \) is represented from (21) and (22) similarly as in the case neglecting the curvature of the equiscalar surface of \( f \) (Kawai, 1957) by the formula:

\[
\text{div} f A = \frac{\partial A_s}{\partial s} + \frac{\partial A_n}{\partial n} + A_s \frac{\partial \theta_s}{\partial s} - A_n \frac{\partial \theta_n}{\partial s}. \tag{21'}
\]

On the other hand, \( \text{rot} A \) is expressed from (18) and (17) by the formula:

\[
\text{rot} A = A_s \text{rot} s + A_n \text{rot} n + A_q \text{rot} q + s \left( \frac{\partial A_n}{\partial n} - \frac{\partial A_q}{\partial q} \right) + n \left( \frac{\partial A_q}{\partial q} - \frac{\partial A_s}{\partial s} \right) + q \left( \frac{\partial A_s}{\partial s} - \frac{\partial A_n}{\partial n} \right). \tag{23}
\]

Further, we have from (7) and (22)

\[
\begin{align*}
\text{rot} s &= -s \left( \frac{\partial \theta_n}{\partial n} + \frac{\partial \theta_q}{\partial q} \right) + n \frac{\partial \theta_n}{\partial s} + q \frac{\partial \theta_q}{\partial s} \\
\text{rot} n &= s \frac{\partial \theta_n}{\partial q} + n \left( \frac{\partial \theta_s}{\partial s} + \frac{\partial \theta_q}{\partial q} \right) + q \frac{\partial \theta_n}{\partial q} \\
\text{rot} q &= s \frac{\partial \theta_q}{\partial s} + n \frac{\partial \theta_n}{\partial q}
\end{align*}
\]

The rotation on the equiscalar surface of \( f \) is nothing but the \( q \)-component of \( \text{rot} A \) : \( \text{rot}_A \)

\[
\begin{align*}
\frac{\partial A_n}{\partial n} + A_s \frac{\partial \theta_s}{\partial s} + A_n \frac{\partial \theta_n}{\partial n}. \tag{23'}
\end{align*}
\]

Since the current vector \( V \) has not \( n \)-component as shown in

\[
V = su + qv, \tag{25}
\]

the equations (21), (22), (21'), (23), (24) and
5. **Equation of motion and vorticity equation**

In this paragraph the equation of motion and the vorticity equation are expressed in the frame of the natural $f$-coordinates. The equation of frictionless motion in atmosphere and ocean has the form:

$$\frac{\partial V}{\partial t} + (W \times V) + \alpha \text{ grad } p + \text{grad} (\frac{? + V^2}{2}) = 0, \quad (28)$$

where

\[ W = \text{rot } V + 2Q. \quad (29) \]

The symbols used here and in the following have the meaning:

- $W$: absolute vorticity vector
- $O$: angular velocity of the earth's rotation
- $\alpha$: specific volume
- $\rho$: density
- $\omega$: geographical latitude
- $p$: pressure
- $\lambda, \lambda'$: Coriolis parameters

Considering (25) and (7), the term of acceleration in (28) is expressed by the natural $f$-coordinates as

$$\frac{\partial V}{\partial t} = s\left(\frac{\partial u}{\partial t} + w \frac{\partial \theta_s}{\partial t}\right) + n\left(u \frac{\partial \theta_q}{\partial t} - w \frac{\partial \theta_s}{\partial t}\right) + q\left(\frac{\partial \theta_s}{\partial t} - w \frac{\partial \theta_q}{\partial t}\right). \quad (30)$$

Expressing the angular velocity vector of the earth's rotation as

$$2O = \lambda \alpha + \lambda' \alpha = 2sO_x + n2O_n + q2O_q,$$

the absolute vorticity vector is dissolved by (29) and (27) into three components:

$$W = sX + nY + qZ,$$

$$X = \xi + 2O_n,$$

$$Y = \eta + 2O_n,$$

$$Z = \zeta + 2O_n.$$

Accordingly, the second term of (28) is expressed as

$$W \times V = swY + n(uZ - wX) + qwY. \quad (32)$$

Using (17), (30) and (32), (28) is dissolved into three components:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial s} \left(\frac{V^2}{2} + \alpha\right) + \alpha \frac{\partial p}{\partial s} = -w \left(Y + \frac{\partial \theta_n}{\partial t}\right)$$

$$\frac{\partial}{\partial n} \left(\frac{V^2}{2} + \alpha\right) + \alpha \frac{\partial p}{\partial n} = -u \left(Z + \frac{\partial \theta_q}{\partial t}\right) + w \left(X + \frac{\partial \theta_s}{\partial t}\right)$$

$$\frac{\partial w}{\partial t} + \frac{\partial}{\partial q} \left(\frac{V^2}{2} + \alpha\right) + \alpha \frac{\partial p}{\partial q} = u \left(Y + \frac{\partial \theta_n}{\partial t}\right). \quad (33)$$

Here, it must not be overlooked that the respective component of the angular velocity due to the rotation of this coordinate system around the coordinate axis is added to the respective component of the absolute vorticity, though $s$- and $n$-components of them are numerically minute as shown in §6.

Considering (27), (33) becomes

$$\frac{Du}{Dt} + \frac{\partial}{\partial s} \left(\frac{V^2}{2} + \alpha\right) + \alpha \frac{\partial p}{\partial s} = -w \left(D\frac{\partial \theta_n}{\partial t} + 2O_n\right)$$

$$\frac{\partial}{\partial n} \left(\frac{V^2}{2} + \alpha\right) + \alpha \frac{\partial p}{\partial n} = -u \left(D\frac{\partial \theta_s}{\partial t} + 2O_s\right)$$

$$\frac{Dw}{Dt} + \frac{\partial}{\partial q} \left(\frac{V^2}{2} + \alpha\right) + \alpha \frac{\partial p}{\partial q} = u \left(D\frac{\partial \theta_n}{\partial t} + 2O_n\right). \quad (34)$$

Operating rot to (28), we get

$$\frac{\partial W}{\partial t} + (V \cdot \text{grad } W) - (W \cdot \text{grad } V) = -\text{grad } p \times \alpha + W \text{ div } V = 0. \quad (35)$$

This equation is dissolved by (17), (25), (31), (7) and (15) into three components, and the $q$-component is expressed by

$$\frac{DZ}{Dt} - \left(\frac{\partial w}{\partial s} + \frac{\partial \theta_n}{\partial s} + \frac{w \partial \theta_s}{\partial s}\right)X$$

$$- \left(\frac{\partial w}{\partial n} + \frac{\partial \theta_q}{\partial n} + \frac{w \partial \theta_n}{\partial n}\right)Y$$

$$- \left(\frac{\partial w}{\partial q} - u \frac{\partial \theta_s}{\partial q} - \text{div } V\right)Z - \tilde{N}_t = 0, \quad (36)$$

where

$$\tilde{N}_t = \frac{\partial p}{\partial s} \frac{\partial \alpha}{\partial n} - \frac{\partial p}{\partial n} \frac{\partial \alpha}{\partial s}.$$

Extending the vertical $p$-velocity, the vertical $f$-velocity is defined as
and its directional derivative is shown from (13) and (15) by
\begin{equation}
(37)
\end{equation}
Considering (37), the q-component of the vorticity equation (35) becomes
\begin{equation}
(38)
\end{equation}
This is a new expression of the vorticity equation.

6. Equations of motion in the natural coordinates on the isobaric surface

When \( f \) is equal to \( \varrho \) and \( w \) vanishes in the natural \( f \)-coordinates, Holmboe's natural coordinates is derived. Putting \( f=\sigma_t \) and \( f=p \) in the natural \( f \)-coordinates, we get the natural \( \sigma_t \)-coordinates and the natural pressure coordinates respectively. Though the natural coordinates on the equiscalar surfaces of other physical quantities may be considered, they are not significant in the dynamical oceanography. In this paragraph, the orders of magnitude of terms in the equations of motion expressed by the natural pressure coordinates are estimated, and these equations are reduced to simpler forms, one of which comes to the equations derived in a previous paper.

Considering (11), (34) is reduced to
\begin{equation}
DwDt + g \sin \varepsilon_2 = -w(D\theta_nDt + 2O_n) \quad (34', 1)
\end{equation}
\begin{equation}
g \sin \varepsilon_2 = -u(D\theta SDt + 2O_s) + w(D\theta SDt + 2O_s) \quad (34', 2)
\end{equation}
\begin{equation}
DwDt + gT_{33} = u(D\theta SDt + 2O_s) - \alpha \frac{\partial p}{\partial q}, \quad (34', 3)
\end{equation}
because \( \frac{\partial p}{\partial s} \) and \( \frac{\partial p}{\partial n} \) both vanish in the present coordinates.

We are now in a position to evaluate the orders of magnitude of all quantities appearing in the equations of motion expressed by the natural pressure coordinates in the ocean, after such method that Charney (1948) tested for the atmospheric motion, adopting the characteristic magnitudes as shown below:

- Radius of curvature of the isobaric stream line \( R \sim 10^6 \) cm
- Length along the isobaric stream line \( S \sim 10^6 \) cm
- Length across the isobaric stream lines \( N \sim 10^6 \) cm
- Vertical thickness \( H \sim 10^4 \) cm
- Current velocity \( \sim 10^2 \) cm sec\(^{-1}\)
- Wave velocity \( c \sim 10^2 \) cm sec\(^{-1}\)
- Gravity acceleration \( g \sim 10^3 \) cm sec\(^{-2}\)
- Coriolis parameters \( \lambda \sim 10^{-4} \) sec\(^{-1}\)

Here, \( R \) corresponds to the radius of the eddy of intermediate size (Spilhaus, 1940), \( S \) the mean length along the isobaric stream line between points at which velocity takes extreme values, \( N \) the width of the horizontal shear-zone, and \( H \) the thickness of the vertical shear-layer respectively. Roughly speaking, \( S \) is the mean distance between trough and wedge in the stream line pattern. Of course, the characteristic scales, \( S, N \) and \( H \), defined for current velocity are different from the scales defined for other quantity, but the former is nearly of the same order with the latter.

The time dimension is characterized by \( V \) and \( c \), which corresponds to the mean speed of propagation of the stream line pattern, neglecting such motions on a small scale as gravity waves (\( c \sim 10^4 \) cm sec\(^{-1}\)) and tidal waves (\( c \sim 10^5 \) cm sec\(^{-1}\)).

The orders of magnitude of the spatial derivatives can be estimated by replacing differentials by finite increment and expressing the incremental ratios in term of the characteristic magnitudes. The orders of magnitude of the time derivatives can be estimated by expressing as
\begin{equation}
\frac{\partial}{\partial t} \sim \frac{D}{Dt} \sim c \frac{\partial}{\partial s} \sim V \frac{\partial}{\partial s}.
\end{equation}
Here, the symbol "\( \sim \)" denotes equality in orders of magnitude, while the symbol "\( \approx \)" appearing later on denotes the sign of equality in the approximate equation.

Since the inclination of the isobaric surface is numerically minute, the equations (3), (8) and (9) can be respectively approximated as
\[ T_{11} \approx \cos \theta_1, \quad T_{21} \approx -\sin \theta_2 \]
\[ T_{31} \approx \varepsilon_2 \sin \theta_1 - \varepsilon_1 \cos \theta_2 \]
\[ T_{32} \approx \sin \theta_1, \quad T_{33} \approx \cos \theta_2 \]
\[ T_{32} \approx -\varepsilon_2 \cos \theta_1 + \varepsilon_1 \sin \theta_2 \]
\[ T_{33} \approx \varepsilon_1 \]
\[ T_{33} \approx 1 - \varepsilon_1^2/2 \varepsilon_2^2/2 \]
\[ \frac{d\theta_1}{dt} = \varepsilon_2 \delta + \varepsilon_1 \theta_1 \]
\[ \frac{d\theta_2}{dt} = -\varepsilon_1 \varepsilon_2 \delta + \varepsilon_2 \theta_1 \]
\[ \frac{d\theta_2}{dt} = -\varepsilon_1 \varepsilon_2 \delta + \varepsilon_2 \theta_1 \]
\[ \frac{\partial F}{\partial s} \approx \cos \theta_1 \frac{\partial F}{\partial x} + \sin \theta_1 \frac{\partial F}{\partial y} + \varepsilon_1 \frac{\partial F}{\partial z} \]
\[ \frac{\partial F}{\partial n} \approx -\sin \theta_2 \frac{\partial F}{\partial x} + \cos \theta_2 \frac{\partial F}{\partial y} + \varepsilon_2 \frac{\partial F}{\partial z} \]
\[ \frac{\partial F}{\partial q} \approx (\varepsilon_2 \sin \theta_1 - \varepsilon_1 \cos \theta_2) \frac{\partial F}{\partial x} + \varepsilon_1 \frac{\partial F}{\partial y} \]
\[ \frac{\partial F}{\partial q} \approx (\varepsilon_2 \sin \theta_1 - \varepsilon_1 \cos \theta_2) \frac{\partial F}{\partial x} + \varepsilon_1 \frac{\partial F}{\partial y} \]

Since \( \varepsilon_1 \) and \( \varepsilon_2 \) are minute, we can assume
\[ |\varepsilon_1 F_s| \ll |\varepsilon_2 F_s|, \quad |\varepsilon_2 F_n| \ll |\varepsilon_1 F_n| \]

so we get from (9A)
\[ \frac{\partial F}{\partial s} \ll F_s, \quad \frac{\partial F}{\partial n} \ll F_n \quad \text{and} \quad \frac{\partial F}{\partial q} \ll \frac{\partial F}{\partial z}, \quad \text{(9A')} \]

where \( F_s \) denotes the order of the greater magnitude of \( \frac{\partial F}{\partial x} \) and \( \frac{\partial F}{\partial y} \).

The assumptions:
\[ |V| > |w|, \quad \frac{D\theta_1}{Dt} > \frac{D\theta_2}{Dt} \quad \text{(40)} \]

are held for the current pattern of small scale. Considering
\[ \frac{D\theta_1}{Dt} \sim \frac{D\theta_2}{Dt} \sim \frac{V}{R} \sim 10^{-4}, \quad \text{(41)} \]
we get from (31), (3A) and (40)
\[ \frac{D\theta_1}{Dt} + 2\varepsilon \frac{D\theta_2}{Dt} = 2\varepsilon \frac{D\theta_1}{Dt} + 2\varepsilon \frac{D\theta_2}{Dt} + 2\varepsilon \sim 10^{-4}. \quad \text{(42)} \]

Further, considering that
\[ \frac{Du}{Dt} \sim \frac{V^2}{S} \sim 10^{-3}, \quad \text{and assuming loosely for the present that} \]
\[ w < 10 \text{ cm sec}^{-1}, \text{we get from (34.1), (34.2)} \]
and (34.3)
\[ \varepsilon_1 \sim 10^{-6}, \varepsilon_2 \sim 10^{-5} \quad \text{and} \quad \frac{\partial p}{\partial q} \sim g \cdot 10^3 \quad \text{(43)} \]

These orders of magnitude of \( \varepsilon_1 \) and \( \varepsilon_2 \) suppress the relative error of the approximate equations of (3A) less than \( 10^{-10} \).

The angular velocity components of the rotation of coordinate axes can be estimated roughly as shown below. Considering
\[ \frac{\partial \varepsilon_1}{\partial t} \sim \frac{V}{S} \sim 10^{-11}, \quad \frac{\partial \varepsilon_2}{\partial t} \sim \frac{V}{S} \sim 10^{-10} \]
and
\[ \frac{\partial \theta_1}{\partial t} + \varepsilon_2 \frac{\partial \varepsilon_1}{\partial t} \sim \frac{\partial \theta_1}{\partial t} - \varepsilon_1 \frac{\partial \varepsilon_2}{\partial t} \]
which is derived from the third equation of (8A), we get from (41)
\[ \frac{\partial \theta_2}{\partial t} \approx \frac{\partial \theta_1}{\partial t} \approx \frac{\partial \theta_2}{\partial t} \sim 10^{-4}. \quad \text{(44. q)} \]

Accordingly, we get from the first and the second equations of (8A)
\[ \frac{\partial \theta_1}{\partial t} \approx \frac{\partial \theta_2}{\partial t} + \varepsilon_1 \frac{\partial \varepsilon_2}{\partial t} \sim 10^{-10} \quad \text{(44. s)} \]
\[ \frac{\partial \theta_2}{\partial t} \approx -\varepsilon_1 \frac{\partial \varepsilon_1}{\partial t} + \varepsilon_2 \frac{\partial \varepsilon_1}{\partial t} \sim 10^{-9}. \quad \text{(44. n)} \]

The angular variation of coordinate axes along \( s \) and \( n \) can be estimated roughly as shown below.

Considering
\[ \frac{\partial \varepsilon_1}{\partial s} \sim \frac{\varepsilon_1}{S} \sim 10^{-13}, \quad \frac{\partial \varepsilon_2}{\partial s} \sim \frac{\varepsilon_2}{S} \sim 10^{-12}, \quad \frac{\partial \theta_1}{\partial s} = \frac{1}{R} \sim 10^{-6} \]
and
\[ \frac{\partial \theta_1}{\partial s} \approx \frac{\partial \theta_1}{\partial s} \approx \frac{\partial \theta_2}{\partial s} \sim 10^{-6}. \quad \text{(45. q)} \]

Accordingly, we get from the first and the second equations of (8A)
\[ \frac{\partial \varepsilon_1}{\partial s} \sim \frac{\varepsilon_2}{s} \sim 10^{-12}, \quad \frac{\partial \varepsilon_1}{\partial s} \sim \frac{\varepsilon_1}{s} \sim 10^{-11} \quad \text{(45. n)} \]

Considering
\[ \frac{\partial \theta_1}{\partial n} \sim \frac{\varepsilon_1}{N} \sim 10^{-12}, \quad \frac{\partial \theta_1}{\partial n} \sim \frac{\varepsilon_2}{N} \sim 10^{-11} \]
and \[ \frac{\partial \theta_1}{\partial n} = -\frac{\partial \theta_2}{\partial n} \sim 10^{-12}, \quad \text{(46. n)} \]
which is obtained from the first equation of (15) and (45. s), we get from the second equation of (8A)
\[ \frac{\partial \theta_1}{\partial n} \approx \frac{1}{\varepsilon_2} \left( \frac{\partial \theta_2}{\partial n} + \frac{\partial \varepsilon_1}{\partial n} \right) \sim 10^{-7}. \]
Thus, we get from the third equation of (8A)
\[
\frac{\partial \theta_s}{\partial n} \approx \frac{\partial \theta_s}{\partial m} \approx \frac{\partial \theta_s}{\partial q} \approx 10^{-7}, \quad (46. q)
\]
and from the first equation of (8A)
\[
\frac{\partial \theta_s}{\partial n} \approx \frac{\partial \theta_s}{\partial m} + \varepsilon_1 \frac{\partial \theta_s}{\partial q} \approx 10^{-11}. \quad (46. s)
\]

The angular variation of coordinate axes along \( q \) cannot be estimated so easily as along \( s \) and \( n \). Differentiating (34.3) with respect to \( n \), and considering that the density difference between two points on both sides across the Kuroshio Front has the orders of magnitude \( \Delta \rho \approx 10^{-3} \), we get
\[
\frac{\partial}{\partial m} \left( \frac{\partial \phi}{\partial q} \right) \sim g \frac{\partial \rho}{\partial m} \sim g \frac{\Delta \rho}{N} \sim 10^{-6}.
\]

Considering the third equation of (43) and the above equation, we get from the second equation of (15)
\[
\frac{\partial \theta_s}{\partial q} = -\frac{\partial}{\partial m} \left( \frac{\partial \phi}{\partial q} \right) \sim 10^{-9}. \quad (47. s)
\]

Seeing that
\[
\frac{\partial \varepsilon_1}{\partial q} \approx \frac{\varepsilon_1}{H} \sim 10^{-10} \quad \text{and} \quad \frac{\partial \varepsilon_2}{\partial q} \approx \frac{\varepsilon_2}{H} \sim 10^{-9},
\]
we get from the first equation of (8A) and (47.s)
\[
\frac{\partial \theta_s}{\partial q} \approx \frac{1}{\varepsilon_1} \left( \frac{\partial \theta_s}{\partial q} - \frac{\partial \varepsilon_2}{\partial q} \right) \sim 10^{-3},
\]
so that
\[
\frac{\partial \theta_s}{\partial q} \approx \frac{\partial \theta_1}{\partial q} + \varepsilon_2 \frac{\partial \varepsilon_1}{\partial q} \approx \frac{\partial \theta_1}{\partial q} - \varepsilon_1 \frac{\partial \varepsilon_2}{\partial q}
\]
we get
\[
\frac{\partial \theta_s}{\partial q} \approx \frac{\partial \theta_1}{\partial q} \approx \frac{\partial \theta_2}{\partial q} \sim 10^{-3}. \quad (47. q)
\]

We get, also, from the second equation of (8A) and (47.q)
\[
\frac{\partial \theta_n}{\partial q} \approx -\frac{\partial \varepsilon_1}{\partial q} + \varepsilon_2 \frac{\partial \varepsilon_1}{\partial q} \sim 10^{-9}. \quad (47. n)
\]

The order of magnitude of \( w \) can be estimated from the condition that \( \text{div} \, V \) vanishes in the incompressible fluid. Thus, we get from (26)
\[
\frac{\partial w}{\partial q} \approx w \left( \frac{\partial \theta_n}{\partial q} - \frac{\partial \theta_n}{\partial n} \right)
\]
\[
= -\frac{\partial u}{\partial s} - u \left( \frac{\partial \theta_n}{\partial n} - \frac{\partial \theta_n}{\partial q} \right) \sim 10^{-8}.
\]

Considering that
\[
\frac{\partial w}{\partial q} \sim \frac{w}{H} \sim 10^{-4} w \left( \frac{\partial \theta_n}{\partial s} - \frac{\partial \theta_n}{\partial n} \right) w,
\]
we get
\[
\frac{\partial w}{\partial q} \approx -\frac{\partial u}{\partial q} \approx 10^{-5}.
\]

Accordingly, we obtain that \( w \sim 10^{-2} \).

From (44.q), (45.q), (46.q) and (47.q) the third equation of (8A) is reduced to
\[
\frac{\partial \theta_s}{\partial q} \approx \frac{\partial \theta_1}{\partial q} \approx \frac{\partial \theta_2}{\partial q} \approx \frac{\partial \theta_3}{\partial q},
\]
so we can reject the suffixes 1, 2 and \( q \) hereafter, whereas the suffixes \( s \) and \( n \) of \( \theta \) cannot be omitted.

Considering the orders of magnitude of \( \varepsilon_1 \) and \( \varepsilon_2 \) given in (43), the condition (39) becomes
\[
10^{-5} F_o \ll \frac{\partial F}{\partial z} \ll 10^6 F_o.
\]

Then the equations (9A) become
\[
\begin{bmatrix}
\frac{\partial F}{\partial s} \\
\frac{\partial F}{\partial n} \\
\frac{\partial F}{\partial q}
\end{bmatrix}
= \begin{bmatrix}
\cos \theta \frac{\partial F}{\partial x} + \sin \theta \frac{\partial F}{\partial y} \\
-\sin \theta \frac{\partial F}{\partial x} + \cos \theta \frac{\partial F}{\partial y} \\
\frac{\partial F}{\partial q}
\end{bmatrix}
\quad (9B)
\]

Corresponding to (48), we can assume that
\[
10^{-5} F_o \ll \frac{\partial F}{\partial q} \ll 10^6 F_o,
\]
where \( F_o \) denotes the order of the smaller magnitude of \( \frac{\partial F}{\partial s} \) and \( \frac{\partial F}{\partial n} \). Then the equations (12) become from (40)
\[
\begin{bmatrix}
\frac{\partial}{\partial t} \left( \frac{\partial F}{\partial s} \right) \\
\frac{\partial}{\partial t} \left( \frac{\partial F}{\partial n} \right) \\
\frac{\partial}{\partial t} \left( \frac{\partial F}{\partial q} \right)
\end{bmatrix}
= \begin{bmatrix}
\frac{\partial}{\partial s} \left( \frac{\partial F}{\partial s} \right) \\
\frac{\partial}{\partial n} \left( \frac{\partial F}{\partial n} \right) \\
\frac{\partial}{\partial q} \left( \frac{\partial F}{\partial q} \right)
\end{bmatrix} \sim 10^{-9} \frac{\partial F}{\partial s}.
\quad (12A)
\]

On the basis of (49), the equations (14) become
\[
\begin{bmatrix}
\frac{\partial}{\partial n} \left( \frac{\partial F}{\partial s} \right) - \frac{\partial}{\partial s} \left( \frac{\partial F}{\partial n} \right) \\
\frac{\partial}{\partial n} \left( \frac{\partial F}{\partial q} \right) - \frac{\partial}{\partial q} \left( \frac{\partial F}{\partial n} \right) \\
\frac{\partial}{\partial n} \left( \frac{\partial F}{\partial q} \right) - \frac{\partial}{\partial q} \left( \frac{\partial F}{\partial q} \right)
\end{bmatrix}
= \begin{bmatrix}
\frac{\partial}{\partial s} \left( \frac{\partial F}{\partial s} \right) \\
\frac{\partial}{\partial n} \left( \frac{\partial F}{\partial n} \right) \\
\frac{\partial}{\partial q} \left( \frac{\partial F}{\partial q} \right)
\end{bmatrix} \sim 10^{-9} \frac{\partial F}{\partial s}.
\quad (14A)
\]

In regard to the vector operations we can approximate their expressions by the present natural coordinates as shown below.

Considering (47.n), (26) becomes
\[
\text{div} \, V \approx \frac{\partial u}{\partial s} + \frac{\partial w}{\partial q} + u \frac{\partial \theta}{\partial q}.
\quad (26A)
\]

Also, the equations (27) and (31) become
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\[ \xi \approx \frac{\partial w}{\partial n} - u \frac{\partial \theta}{\partial q} - u \frac{\partial \theta}{\partial q} \sim 10^{-1} \]  
\[ \eta \approx \frac{\partial u}{\partial q} - \frac{\partial w}{\partial s} \approx \frac{\partial u}{\partial q} \sim 10^{-2} \]  
\[ \zeta \approx \frac{\partial w}{\partial n} + u \frac{\partial \theta}{\partial s} \sim 10^{-4} \]  
and

\[ X \approx \xi + \lambda \sin \theta \sim 10^{-1} \]  
\[ Y \approx \eta + \lambda \cos \theta \sim 10^{-2} \]  
\[ Z \approx \zeta + \lambda \sim 10^{-4} \]  
respectively.

Accordingly, (34') is reduced to

\[ \frac{\partial \theta}{\partial s} + \frac{\partial u}{\partial q} \approx 0 \]  
\[ \frac{\partial \theta}{\partial n} + u \left( \frac{\partial \theta}{\partial t} + \lambda \right) \approx 0 \]  
\[ \frac{\partial \theta}{\partial z} + \alpha \frac{\partial p}{\partial z} \approx 0 \]  
(34'A)

Similarly, the equations of motion (33) are reduced to

\[ \frac{\partial u}{\partial t} + \frac{\partial}{\partial n} \left( \frac{V^2}{2} - \Omega \right) \approx -wY \]  
\[ \frac{\partial}{\partial n} \left( \frac{V^2}{2} + \Omega \right) \approx (Z + \frac{\partial \theta}{\partial t}) + uX \]  
\[ \frac{\partial w}{\partial t} + \frac{\partial}{\partial z} \left( \frac{V^2}{2} + \Omega \right) \approx UY - \alpha \frac{\partial p}{\partial z} \]  
(33'A)

Since the equations (9B), (12A), (14A), (26A), (27A), (31A) and (33A) derived above have the same forms with the equations derived under some assumptions upon the isobaric surface in a previous paper (Kawai, 1957), we can conclude that the effects of the motion and the curvature of isobaric surfaces are negligibly small in the motion of the scale mentioned at the beginning of this paragraph.

7. Various forms of the vorticity equation

The vorticity equation (38) is the most original form, for the various expressions of the vorticity equation may be derived from it when giving various scalar quantities to \( f \) in (38). The common assumptions made in deriving the following equations are that the motion is frictionless and that the fluid is incompressible.

(i) Referred to level surface

Putting \( f = \alpha = gz \), the vertical \( f \)-velocity be-
comes \( \omega = \frac{Dg}{Dt} = gw \), and we have \( \partial f/\partial q = g \).

Considering the above equations and the incompressibility of the fluid, the vorticity equation (38) is reduced to the usual form:

\[ \frac{DZ}{Dt} = (W \cdot \text{grad } w) + \tilde{N}_e. \]  
(38.1)

(ii) Referred to isobaric surface

Putting \( f = p \), \( \tilde{N}_e \) vanishes. Accordingly, the vorticity equation (38) is reduced to

\[ \frac{D}{Dt} \left( Z \frac{\partial p}{\partial q} \right) = (W \cdot \text{grad } \omega). \]  
(38.2)

If the motion is isobaric, the vertical \( p \)-velocity \( \frac{DP}{Dt} \) vanishes, so that \( Z \frac{\partial p}{\partial q} \) is conserved.

(iii) Referred to the equiscalar surface of conservative quantity \( f_\sigma \)

In this case the vertical \( f \)-velocity vanishes, so that the vorticity equation (38) is reduced to

\[ \frac{D}{Dt} \left( Z \frac{\partial f_\sigma}{\partial q} \right) = \tilde{N}_{e\sigma} \frac{\partial f_\sigma}{\partial q}. \]  
(38.3)

(iv) Referred to isentropic surface

In the ocean, the \( \sigma \)-surfaces can be considered as being nearly equivalent to the isentropic surfaces in a dry atmosphere. Putting \( f = \sigma \), and considering the incompressibility of the water, the vertical \( \sigma \)-velocity \( \frac{D\sigma}{Dt} \) and the solenoidal term \( \tilde{N}_\sigma \) both vanish.

Accordingly, the vorticity equation (38) is reduced to

\[ \frac{D}{Dt} \left( Z \frac{\partial \sigma}{\partial q} \right) = 0. \]  
(38.4)

This equation corresponds to the conservation equation of the potential vorticity in a dry atmosphere derived by Charney (1948), supposing a cylinder with sides perpendicular to two isentropic surfaces infinitesimally close together and with infinitesimal cross-section. A clear physical insight contributed much to his success, whereas the vorticity equation (38.4) could be derived analytically.

Thus, the potential vorticity in the ocean may be defined, most reasonably, as \( Z \frac{\partial \sigma}{\partial q} \)}
or $Z \, \text{grad} \, \sigma_t$, where $\frac{\partial}{\partial q}$ signifies the directional differentiation along the upward normal of the $\sigma_t$ surface, and $Z$ the vorticity component normal to the $\sigma_t$ surface.

Numerically, $\frac{\partial}{\partial q}$ is nearly equal to $\frac{\partial}{\partial z}$ for the $\sigma_t$ surface, so that $10^{-3} \frac{\partial \sigma_t}{\partial q}$ is equivalent to the vertical stability in the upper layer of the ocean (HESSELBERG and SVERDRUP, 1915). On the other hand $Z$ indicates the horizontal stability in zonal or circular flow. Accordingly, the equation (38.4) signifies that the product of the vertical stability and the horizontal stability is conserved for the isentropic motion in the incompressible water, assuming that the friction is very small. In the water mass stable horizontally and vertically ($\frac{\partial \sigma_t}{\partial q} > 0$, $Z > 0$), if the vertical stability of the water mass increases during motion, its horizontal stability decreases; while if the former decreases, the latter increases, namely—the two stabilities are complementary each other.

8. Concluded summary

The geometry and kinematics in the natural coordinates on an equiscalar surface, which undulates with time, were derived basing on the extended directional differentiation, and the equations of motion and the vorticity equation were represented in the frame of this coordinates. The merit in the use of the present coordinates is that we can analytically derive the dynamical laws in complete form, affording a more natural view in the transformation of equations. For instance, the law of the parallel solenoidal field (KAWAI, 1955) and the law of the conservation of the potential vorticity can be derived in a more fundamental form. As another appplication it was verified that we can numerically neglect the effects of the undulation of isobaric surfaces in the equations of motion referred to isobaric surfaces. Thus, the equations derived in part I of the previous paper (KAWAI, 1957), neglecting the curvature of isobaric surfaces, can be established as they were, in so far as the characteristic scale of the phenomenon under consideration is large to some extent. However, when the phenomenon of small scale is considered, the present coordinate system will be effective.

References

CHARNEY, J. G., 1948:
"On the scale of the atmospheric motions." Geofys. Publ., 17 (2).

ELIASSEN, A., 1949:
"The quasi-static equations of motion with pressure as independent variable." Geofys. Publ., 17 (3).

HESSELBERG, Th. and H. U. SVERDRUP, 1915:

HOLMBOE, J., G. E. FORSYTHE and W. GUSTIN, 1945:
"Dynamical meteorology." New York.

KAWAI, H., 1955:

KAWAI, H., 1957:

SPIHLAUS, A. F., 1940:

STARR, V. P., 1945:

{ 10 }