The Polar Forms of $\text{Sinh } i^{\frac{1}{2}}x$, etc. and Their Application to the Ekman Spiral*

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In some theoretical studies, the final results are expressed as complicated functions involving both hyperbolic and trigonometric functions. Too often, the functions are so involved that it becomes impossible to interpret them without recourse to laborious computations involving several parameters and expressing the results as a series of curves. Even so, physical interpretation under such circumstances can be extremely difficult and many features of the solution will be lost.

As an example, consider the case of the Ekman Spiral for a finite depth and constant eddy viscosity. Physically, we assume water of constant depth and a constant uniform wind of velocity W blowing over the surface. Taking the Coriolis acceleration into consideration, the velocity components u and v (in the x and y directions) are (Hidaka, 1955)

$$u = A \text{Sinh } a(h-z) \cos a(h-z) - B \text{Cosh } a(h-z) \sin a(h-z) \quad (1)$$

$$v = A \text{Cosh } a(h-z) \sin a(h-z) + B \text{Sinh } a(h-z) \cos a(h-z) \quad (2)$$

where

$$A = \frac{\tau_a D}{\pi \mu} \frac{\text{Cosh } ah \cos ah + \text{Sinh } ah \sin ah}{\text{Cosh } 2ah + \cos 2ah} \quad (3)$$

$$B = \frac{\tau_a D}{\pi \mu} \frac{\text{Cosh } ah \cos ah - \text{Sinh } ah \sin ah}{\text{Cosh } 2ah + \cos 2ah} \quad (4)$$

$$a = \pi/D$$

$$\tau_a = \text{wind stress}$$

$$h = \text{depth of water}$$

$$z = \text{depth coordinate, positive downwards}$$

$$D = \pi \sqrt{\mu/\rho \omega} \sin \phi$$

$$\mu = \text{eddy viscosity}$$

$$\rho = \text{density}$$

$$\omega = \text{angular velocity of the earth}$$

$$\phi = \text{latitude (assumed constant)}$$

Obviously, one cannot deduce anything about the behavior of the wind current merely by looking at equations (1), (2), (3) and (4). The best that can be done is to compute the currents at a particular depth under a given set of conditions.

The writer (Olson, 1959) has derived the equations for the Ekman Spiral for the case similar to this one except that the eddy viscosity was assumed to vary linearly with the depth. Although the solution involved the more sophisticated Modified Bessel Functions, the form of solution was such that the properties were readily apparent. If then, the solution for a simpler case is awkward and inelegant, one may conjecture that it is expressed in an unpropitious form and that there must exist a better way of expressing the solution. This conjecture was investigated by studying the properties of functions of the type $\text{Cosh } x \cdot \cos x$, $\text{Sinh } x \cdot \sin x$, etc. It was found that these functions are obvious and interesting generalizations of the exponential and trigonometric functions, and that they are conveniently studied when expressed as $\text{Sinh } i^{\frac{1}{2}}x$, etc. They are closely related to the "hyperbolic functions of order n " (where $n=4$) for which a brief review and pertinent literature is given by Erdelyi, *et al* (1955).

The polar forms of $\text{Sinh } i^{\frac{1}{2}}x$, $\text{Cosh } i^{\frac{1}{2}}x$ and $\text{Tanh } i^{\frac{1}{2}}x$.

Let

$$\sin x = s$$

$$\cos x = c$$

$$\text{Sinh } x = S$$

$$\text{Cosh } x = C$$

Using identities of the form $\text{Sinh } x = \frac{1}{2}(e^x - e^{-x})$, it can be shown that

$$\text{Sinh}(i^{\frac{1}{2}}x\sqrt{2}) = M_s(x) e^{iP_s(x)} \quad (5)$$

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$$\text{Cosh}(\frac{1}{2}x\sqrt{2}) = M_c(x) e^{iP_c(x)} \quad (6)$$

$$\text{Tanh}(i\frac{1}{2}x\sqrt{2}) = M_T(x) e^{iP_T(x)} \quad (7)$$

where

$$M_s^2(x) = S^2c^2 + C^2s^2 \quad (8)$$

$$M_c^2(x) = C^2c^2 + S^2s^2 \quad (9)$$

$$M_T(x) = M_s(x)/M_c(x) \quad (10)$$

$$P_s(x) = \tan^{-1}(Cs/Sc) \quad (11)$$

$$P_c(x) = \tan^{-1}(Ss/Cc) \quad (12)$$

$$P_T(x) = \tan^{-1}(cs/CS) \quad (13)$$

The arguments of the right hand sides of (5), (6) and (7) are x instead of $\sqrt{2}x$.

Table 1 gives values of M_s , M_c , M_T , P_s , P_c and P_T for $0 \leq x \leq 6.2832$ corresponding to $0^\circ \leq x \leq 360^\circ$ in 15° increments. Further, the Table gives values of Ss , Cc , Sc and Cs , as well as these four functions normalized by dividing them by $\sqrt{C^2c^2 + S^2s^2}$.

The normalized Cc , Cs , Ss and Sc functions.

These are shown in Figure 1. For x greater than π they closely approximate the cosine and sine functions. This is particularly important in trying to interpret more complicated expressions involving them. For $0 \leq x \leq \pi$, the functions are well behaved and only the normalized Sc function departs strongly

Table 1b.

Deg.	(Ss)*	(Sc)*	(Cs)*	(Cc)*
0	0	0	0	1.000
15	0.075	0.281	0.267	0.998
30	0.268	0.464	0.556	0.964
45	0.548	0.548	0.836	0.836
60	0.804	0.464	0.030	0.594
75	0.955	0.256	1.105	0.296
90	1.000	0	1.169	0
105	0.962	-0.258	1.013	-0.272
120	0.859	-0.496	0.886	-0.512
135	0.700	-0.700	0.714	-0.714
150	0.496	-0.859	0.501	-0.868
165	0.258	-0.961	0.259	-0.966
180	0	-0.996	0	-1.000
195	-0.258	-0.963	-0.259	-0.966
210	-0.500	-0.865	-0.500	-0.866
225	-0.710	-0.710	-0.710	-0.710
240	-0.866	-0.500	-0.866	-0.505
255	-0.966	-0.259	-0.966	-0.259
270	-1.000	0	-1.000	0
285	-0.966	0.259	-0.966	+0.259
300	-0.866	0.500	-0.866	0.500
315	-0.707	0.707	-0.707	0.707
330	-0.500	0.866	-0.500	0.866
345	-0.258	0.963	-0.258	0.963
360	0	1.000	0	1.000

Table 1c.

Deg.	Radians	Ss	SC	Cs	Cc
0	0.0000	0	0	0	1.000
15	0.2618	0.075	0.281	0.267	0.998
30	0.5236	0.274	0.475	0.570	0.987
45	0.7854	0.614	0.614	0.936	0.936
60	1.0472	1.08	0.625	1.39	0.800
75	1.3090	1.66	0.444	1.92	0.514
90	1.5708	2.15	0	2.51	0
105	1.8326	2.94	-0.789	3.10	-0.830
120	2.0944	3.46	-2.00	3.57	-2.06
135	2.3562	3.70	-3.70	3.76	-3.76
150	2.6180	3.41	-5.91	3.44	-5.97
165	2.8798	2.30	-8.58	2.31	-8.63
180	3.1416	0	-11.6	0	-11.6
195	3.4034	-3.89	-14.5	-3.90	-14.6
210	3.6652	-9.76	-16.9	-9.77	-16.9
225	3.9270	-17.9	-17.9	-17.9	-17.9
240	4.1888	-28.5	-16.5	-28.6	-16.5
255	4.4506	-41.4	-11.1	-41.4	-11.1
270	4.7124	-55.6	0	-55.6	0
285	4.9742	-69.8	18.7	-69.8	18.7
300	5.2360	-81.4	47.0	-81.4	47.0
315	5.4978	-86.4	86.4	-86.4	86.4
330	5.7596	-79.3	137	-79.3	137
345	6.0214	-53.4	199	-53.4	199
360	6.2832	0	268	0	268

Deg.						
0	1.000	0	0	45.0	0	45.0
15	1.001	0.388	0.388	46.3	4.3	42.0
30	1.024	0.742	0.724	50.2	15.5	34.7
45	1.119	1.119	1.000	56.7	33.3	23.4
60	1.346	1.521	1.130	65.7	53.5	12.2
75	1.736	1.969	1.134	77.0	72.7	4.3
90	2.146	2.508	1.169	90.0	90.0	0
105	3.056	3.195	1.046	104.3	105.7	-1.4
120	4.029	4.091	1.015	119.3	120.8	-1.5
135	5.276	5.275	1.000	134.5	135.3	-0.8
150	6.872	6.837	0.995	149.8	150.3	-0.5
165	8.927	8.884	0.995	164.9	165.1	-0.2
180	11.59	11.55	0.996	180	180.0	0
195	15.06	15.02	0.997	195	194.9	+0.1
210	19.53	19.52	0.999	210	210	0
225	25.26	25.37	1.004	225	225	0
240	32.96	32.97	1.000	240	240	0
255	42.85	42.86	1.000	255	255	0
270	55.65	55.65	1.000	270	270	0
285	72.21	72.21	1.000	285	285	0
300	93.99	93.99	1.000	300	300	0
315	122.2	122.2	1.000	315	315	0
330	158.6	158.6	1.000	330	330	0
345	206.0	206.0	1.000	345	345	0
360	267.8	267.8	1.000	360	360	0

from the trigonometric function.

Fig. 1. "Normalized" Cc, Cs, Sc and Ss functions.

The Ekman Spiral as an example.

Instead of following the usual treatment which leads to (1), (2), (3) and (4), we shall employ complex variables to their fullest extent and will show that by so doing the final results can be expressed in an elegant and physically understandable manner.

The equations of motion as given by Sverdrup *et al* (1942) are:

$$\frac{\rho}{\mu c}v + \frac{d^2u}{dz^2} = 0 \quad (14)$$

$$-\frac{\rho}{\mu c}u + \frac{d^2v}{dz^2} = 0 \quad (15)$$

to which we add the conditions that

$$u=v=0 \text{ at } z=h \quad (16)$$

$$\frac{dv}{dz} = -\frac{\tau_a}{\mu} \text{ at } z=0 \quad (17)$$

$$\frac{dv}{dz} = 0 \text{ at } z=0 \quad (18)$$

Multiplying (15) by i , adding to (14) and setting

$$V = u + iv \quad (19)$$

we get

$$\frac{d^2V}{dz^2} - \frac{\rho}{\mu c}iV = 0 \quad (20)$$

Also, from (16),

$$V=0 \text{ at } z=h \quad (21)$$

and

$$\frac{dV}{dz} = -\frac{i\tau_a}{\mu} \text{ at } z=0. \quad (22)$$

The substitution

$$z = \sqrt{\frac{\mu c}{\rho}} \zeta \quad (23)$$

takes (20), (21) and (22) into

$$\frac{d^2V}{d\zeta^2} - i = 0 \quad (24)$$

$$V=0 \text{ at } \zeta=\zeta_0 = h\sqrt{\frac{\mu c}{\rho}} \quad (25)$$

$$\frac{dV}{d\zeta} = -i\tau_a\sqrt{\frac{c}{\mu\rho}} = -ik \text{ at } \zeta=0 \quad (26)$$

The solution of (24) is

$$V = A_1 e^{i\frac{1}{2}\zeta} + A_2 e^{-i\frac{1}{2}\zeta} \quad (27)$$

Evaluating the constants A_1 and A_2 by imposing the boundary conditions, we find that

$$V = \tau_a \sqrt{\frac{c}{\mu\rho}} \frac{M_s \left[\frac{\sqrt{2}}{2} (\zeta_0 - \zeta) \right]}{M_c \left[\frac{\sqrt{2}}{2} \zeta_0 \right]} e^{i \left\{ P_s \left[\frac{\sqrt{2}}{2} (\zeta_0 - \zeta) \right] - P_T \left[\frac{\sqrt{2}}{2} \zeta_0 \right] + \frac{\pi}{4} \right\}} \quad (28)$$

This is the desired result, equivalent to (1), (2), (3) and (4). Instead of expressing the velocity in terms of its components u and v , the magnitude is simply $\tau_a(c/\mu\rho)^{\frac{1}{2}}(M_s/M_c)$ and the direction is given by $P_s - P_T + \frac{\pi}{4}$.

Further, the variation of magnitude with depth is given by M_s ; the variation of direction with depth is given by P_s . These functions are tabulated in Table 1 and allow us to interpret (28) with little difficulty.

References

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