Intensity of Illumination at a Small Plate Lowered in Lake or Marine Water*

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Abstract: Supposing a small plate is horizontally lowered into lake or marine water, the author has calculated, on both surfaces of it, the intensity of illumination due to the light which, coming from the direct solar radiation penetrating into the water, is primarily scattered by the water. Assuming at first the isotropic scattering and then the Rayleigh scattering the calculation has been promoted, and the following results have been revealed.

In both cases, as the depth of the water down to the lowered plate increases the intensity at the surface facing upward increases at first, then it reaches its maximum value, and then gradually decreases, showing a point of inflexion at a certain depth.

And the intensity at the surface facing downward exponentially decreases in both cases, as the depth increases.

I Introduction

Lake or marine water generally contains a great number of suspended particles. They are various in size and shape, and most of them are supposed to be fairly large compared with the colloidal particles.

Sometimes the surface of a lake or the part of the sea is calm, but generally it is more or less in motion, and further, there take place various kinds of circulation of water throughout the year. From these phenomena, the author thinks that the suspended particles in the epilimnion of many lakes and some parts of the sea are brought to uniform distribution with irregular orientations; though there are several actions to classify themselves into different layers.

When the ray of light penetrates into the water it partially polarises, but its degree is very small if the angle of refraction of light is not conspicuous; and the phenomenon of scattering of light caused by a comparatively larger particle is supposed to be closely connected with the reflection and diffraction; and with the fact of irregular orientations of the particles, the author proceeds to think that if we take a small volume of water containing fairly large number of suspended particles, the total volume of light scattered by the matter in it may sometimes be supposed to be in nearly uniform distribution in all directions.

On the conception of isotropic scattering at first, with the assumption that there are no self-luminous particles or substances which exhibit the phenomena of fluorescence, and also neglecting the Raman effect, he promoted the calculations of the intensities at both surfaces of the lowered small plate.

The scattering of light, however, cannot always be assumed to be uniform in all directions in all lakes and in all parts of the sea; he therefore calculated the intensities, as one of the cases of non-uniformity, on the assumption that Rayleigh scattering occurs either in lake water or marine water.

In both cases, the primary scattering alone has been considered; about the intensity due to the secondary scattering, and that due to the sky light, he wishes to report in successive papers.

II Intensity at the surface of the plate facing upward in the case of isotropic scattering

Take a vertical line passing through a certain point on the lowered plate as ζ-axis downward direction as positive, and the point...
a of intersection of this line with the surface of the undisturbed water as the origin of the coordinate axes. Instead of the rectangular coordinates, take the cylindrical coordinates and denote the coordinates by \( \theta, \phi, \zeta \).

Consider a volume element \( r \, d\theta \, dr \, d\zeta \) at a point whose coordinates are \((r, \theta, \zeta)\). Denote this volume element by \( A \) (Fig. 1). Denote it on the horizontal surface at \( A \) will be given by

\[
I = I_0 e^{-\alpha \sec \phi},
\]

where \( I_0 \) denotes the intensity due to the direct solar radiation on the horizontal surface just below the surface of the lake or sea, \( \alpha \) the extinction coefficient, and \( e \) the base of natural logarithm.

From the above formula we have

\[
dI = -\alpha \sec \phi I_0 e^{-\sec \phi \zeta} \, d\zeta,
\]

from which we have

\[
dI_r = -\alpha \sec \phi I_0 e^{-\sec \phi \zeta} \, r \, d\theta \, dr \, d\zeta,
\]

which would show the decrement of the light in transit through the volume element \( A \).

Part of it is truly absorbed by the matter in \( A \), and the rest of it is scattered in all directions. The amount of light scattered by \( A \) is given by

\[
a_\sigma \sec \phi I_0 e^{-\sec \phi \zeta} \, r \, d\theta \, dr \, d\zeta,
\]

where \( a_\sigma \) denotes the coefficient of scattering.

Part of the light scattered by \( A \), again having its intensity decreased by absorption and secondary scattering, reaches the lowered plate. As the scattering is assumed to be uniform in all directions, the amount of light which reaches the plate from \( A \) is given by

\[
a_\sigma \sec \phi I_0 e^{-\sec \phi \zeta} \, \frac{\delta \omega}{4\pi} e^{-\alpha \sqrt{\varphi^2 + (z-\zeta)^2}}
\]

Therefore, if the primarily scattered light alone is considered, the amount of the light reaching the surface of the plate facing upward would be given by the following integral

\[
\int_{\theta_0}^{\theta_1} \int_{\phi_0}^{\phi_1} \int_{\zeta_0}^{\zeta_1} a_\sigma \sec \phi I_0 e^{-\sec \phi \zeta} \, \frac{\delta \omega}{4\pi} e^{-\alpha \sqrt{\varphi^2 + (z-\zeta)^2}} \, r \, d\theta \, dr \, d\zeta,
\]

in which \( \delta \omega \) is substituted by (1). Here the primarily scattered light, which is reflected back into the water at the surface of the lake or sea, is not considered.

The primarily scattered light which contributes to the illumination of the surface of the plate facing upward, is the light scattered by all the water which fills the

\[
(2)
\]
upper part of the volume of the lake limited by the plane which contains the plane of the plate. It is, however, to be remarked here that the light scattered by the water which is fairly distant from the plate contributes very little to the illumination. Therefore, if the place at which the plate is lowered is distant in some degree from the shore of a fairly deep lake or sea, the limits of integration of (2) may be taken for $\theta$, from 0 to $2\pi$, for $r$, from 0 to infinity, and for $\zeta$ from 0 to $z$.

The intensity of illumination $I_2$ at the plate is the quotient of (2) divided by $\delta s$. Now integrating it with respect to $\theta$, and arranging the result we obtain

$$I_2 = \frac{1}{2} \alpha_s \sec \varphi I_0 \int_0^\pi (z-\zeta) e^{-\alpha \sec \varphi} \int_0^\infty e^{-\frac{\alpha^2 (z-\zeta)^2}{r^2 + (z-\zeta)^2}} r dr d\zeta. \quad (3)$$

Put

$$\alpha \sqrt{r^2 + (z-\zeta)^2} = u,$$

then

$$\int_0^\infty e^{-\alpha \sqrt{r^2 + (z-\zeta)^2}} r dr = \alpha \int_0^\infty \frac{1}{2} u^2 e^{-\frac{u}{2}} du = \frac{1}{2} \alpha \gamma + \alpha \int_0^\infty e^{-u} \log u du \quad (4)$$

where $\gamma$ is Euler's constant with the value of 0.577215.................

Replace $\alpha (z-\zeta)$ by $v$, and integrate the last term successively repeating the integration by part, then we obtain an infinites series

$$e^{-v} \sum_{n=1}^\infty \frac{1}{n!} u^n (\log v - h_n),$$

where

$$h_n = \sum_{k=1}^n \frac{1}{k}.$$

The above infinites series is convergent for all finite values of $v$.

Now substitute the result in (4), and substitute the (4) in (3), then we obtain

$$I_2 = \frac{1}{2} \alpha_s \sec \varphi I_0 \left\{ \int_0^\pi e^{(\sec \varphi - 1)\varphi} \right\} \int_0^\infty e^{(\sec \varphi - 1)\varphi} \left( \sum_{n=1}^\infty \frac{1}{n!} v^n (\log v - h_n) \right) dv \quad (3)$$

And

$$\int_0^\infty e^{(\sec \varphi - 1)\varphi} dv = \frac{1}{\sec \varphi - 1} \left\{ e^{(\sec \varphi - 1)\varphi} - 1 \right\} \quad (6)$$

$$\int_0^\infty e^{\sec \varphi \varphi} v^n dv = \frac{\alpha \sec \varphi}{\sec \varphi - 1} - \frac{1}{\sec^2 \varphi} e^{\sec \varphi \varphi}$$

$$+ \frac{1}{\sec^2 \varphi}. \quad (7)$$

And successively repeating the integration by part, we obtain

$$\int_0^\infty e^{(\sec \varphi - 1)\varphi} \log v dv = e^{(\sec \varphi - 1)\varphi} \sum_{n=2}^\infty \frac{1}{n!} (-1)^{n-2} \frac{(\sec \varphi - 1)^{n-2}}{n!} (\log \alpha z - h_n + 1) \quad (8)$$

Now, as $\varphi$ is always less than $\frac{\pi}{2}$, $\sec \varphi$ is always finite. Therefore, the above infinites series is convergent for all finite values of $\alpha z$.

The general term of the last integral of (5) is

$$\int_0^\infty e^{(\sec \varphi - 1)\varphi} \frac{1}{n!} v^{n+1} (\log v - h_n) dv,$$

which can be integrated with a successive repetition of integration by part, and the result of it is an infinites series

$$e^{(\sec \varphi - 1)\varphi} \sum_{k=1}^\infty \frac{n+1}{k+1} \frac{(-1)^{k-2}(\sec \varphi - 1)^{k-2}}{k+1} \left( \log \alpha z - h_n + k \right) \left( \frac{1}{n+1} \right).$$

Therefore

$$\int_0^\infty e^{(\sec \varphi - 1)\varphi} \sum_{n=1}^\infty \frac{1}{n!} v^n (\log v - h_n) dv = e^{(\sec \varphi - 1)\varphi} \sum_{n=1}^\infty \sum_{k=1}^\infty \frac{n+1}{(n+k)!} (-1)^{k-2} \left( \frac{1}{n+1} \right).$$

(3)
This infinite series is convergent for all finite values of $\alpha z$.

In order to reduce the formula a simpler form, add the results of (8) and (9), and arrange the series in ascending power of $\alpha z$, then the result of it is

$$e^{(\sec \varphi-1)az} \sum_{n=2}^{\infty} \frac{1}{n!} (\alpha z)^n \left( \Phi_{n-2} + K_{n-2} (\log \alpha z - h_n) \right),$$

(10)

where $\Phi_{n-2}$ and $K_{n-2}$ are the abbreviations of the following series respectively

$$\Phi_{n-2} = \sum_{j=0}^{n-2} (-1)^j (\sec \varphi - 1)^j$$

$$K_{n-2} = \sum_{j=0}^{n-2} \Phi_j.$$ 

Substitute the results of (6), (7) and (10) in (5), then we finally have

$$I_2 = \frac{1}{2} \frac{\alpha}{\sec \varphi} \frac{1}{(\sec \varphi + 1)_{\infty}} \left( e^{(\sec \varphi-1)z} - 1 \right)$$

$$\times e^{-\sec \varphi \alpha z} + \frac{1}{\sec \varphi} \sec ^2 \varphi + \frac{1}{\sec ^2 \varphi} \times e^{-\sec \varphi \alpha z} + e^{-\alpha \sum_{n=1}^{\infty} \frac{1}{n!} (\alpha z)^n (\Phi_{n-2} + K_{n-2})}$$

$$\times (\log \alpha z - h_n) \right).$$

(11)

III Intensity at the surface of the plate facing downward in the case of isotropic scattering

The surface of the plate facing downward is illuminated by the scattered light alone. And the amount of light, which is scattered by the matter in the volume element $A$ and reaches the plate having its intensity decreased by absorption and scattering, is given by

$$\alpha \frac{\sec \varphi}{\sec \varphi} e^{-\sec \varphi \alpha} \frac{(\zeta - z) \delta s}{4 \pi (r^2 + (\zeta - z)^2)^{3/2}}$$

$$\times e^{-\alpha \sqrt{r^2 + (\zeta - z)^2}} r d\theta d r d \zeta.$$ 

Therefore, the intensity of illumination $I_3$ at the plate, in so as the primary scattering alone is considered, is given by the following integral

$$I_3 = \frac{1}{2} \frac{\alpha}{\sec \varphi} \frac{1}{(\sec \varphi + 1)_{\infty}} \left( e^{(\sec \varphi-1)z} - 1 \right)$$

$$\times e^{-\sec \varphi \alpha z} + \frac{1}{\sec \varphi} \sec ^2 \varphi + \frac{1}{\sec ^2 \varphi} \times e^{-\sec \varphi \alpha z} + e^{-\alpha \sum_{n=1}^{\infty} \frac{1}{n!} (\alpha z)^n (\Phi_{n-2} + K_{n-2})}$$

$$\times (\log \alpha z - h_n) \right).$$

(11)

Put

$$\alpha \sqrt{r^2 + (\zeta - z)^2} = u,$$

then

$$\int \frac{1}{(r^2 + (\zeta - z)^2)^{3/2}} e^{-\alpha \sqrt{r^2 + (\zeta - z)^2}} r d r$$

$$= \alpha \int e^{-u} e^{-a(\zeta - z)}$$

$$+ \alpha e^{-a(\zeta - z)} \log \alpha (\zeta - z) + \alpha \gamma$$

$$+ \alpha \int e^{-a(\zeta - z)} e^{-a} \log u d u.$$ 

Substitute the result in (13), and again put

$$\alpha (\zeta - z) = x$$

and arrange the formula, then
which shows that \( I_3 \) exponentially decreases as the depth \( z \) increases.

Now substitute these results in (14), and arrange the result, then we finally have

\[
I_3 = \frac{1}{2} \frac{\alpha_s}{\alpha} I_0 \left\{ \frac{1}{\sec \varphi + 1} e^{-\sec \varphi \cdot az} + e^{-\sec \varphi \cdot az} \int_0^\infty xe^{-\sec \varphi \cdot x} \log x dx \right. \\
+ \left. ye^{-\sec \varphi \cdot az} \int_0^\infty xe^{-\sec \varphi \cdot x} dx + e^{-\sec \varphi \cdot az} \int_0^\infty xe^{-\sec \varphi \cdot x} \left( \int_0^x e^{-u \log u du} \right) dx \right\}.
\]

(14)

which shows that \( I_3 \) exponentially decreases as the depth \( z \) increases.

Now

\[
\int_0^\infty xe^{-\sec \varphi \cdot x} dx = \frac{1}{\sec^2 \varphi}.
\]

\[
\int_0^\infty xe^{-\sec \varphi \cdot x} \left( \int_0^x e^{-u \log u du} \right) dx \\
+ \int_0^\infty xe^{-\sec \varphi \cdot x} \log x dx \\
= \left[ \left( -\frac{x}{\sec \varphi} e^{-\sec \varphi \cdot x} - \frac{1}{\sec^2 \varphi} e^{-\sec \varphi \cdot x} \right) \right]_0^\infty \\
\times \left( \int_0^\infty e^{-u \log u du} \right)^n + \int_0^\infty \left( \frac{x}{\sec \varphi} e^{-\sec \varphi \cdot x} \\
+ \frac{1}{\sec^2 \varphi} e^{-\sec \varphi \cdot x} \right) e^{-x \log x} dx \\
+ \int_0^\infty xe^{-\sec \varphi \cdot x} \log x dx = -\frac{1}{\sec^2 \varphi}
\]

\[
\times \left\{ \gamma + \log (\sec \varphi + 1) \right\} + \frac{1}{\sec \varphi (\sec \varphi + 1)}.
\]

Substitute these results in (14), and arrange the result, then we finally have

\[
I_3 = \frac{1}{2} \frac{\alpha_s}{\alpha} I_0 \left\{ 1 - \frac{\log (\sec \varphi + 1)}{\sec \varphi} \right\} e^{-\sec \varphi \cdot az}.
\]

(15)

Put \( z = 0 \) in (15) then

\[
I_3 = \frac{1}{2} \frac{\alpha_s}{\alpha} I_0 \left\{ 1 - \frac{\log (\sec \varphi + 1)}{\sec \varphi} \right\} e^{-\sec \varphi \cdot az}.
\]

which is the intensity at the surface of the plate facing downward, when the plate is considered to be situated just below the surface of the lake or marine water. The results is coincident with the result obtained by L.V. Whitney (Hutchinson, A Treatise on Limnology, Volume 1, p. 403).

Put \( \varphi = 0 \) in (15), then

\[
I_3 = \frac{1}{2} \frac{\alpha_s}{\alpha} I_0 \left( 1 - \log 2 \right) e^{-\alpha z}
\]

which shows the case when the sun illuminates the lake or marine water vertically downward.

### IV Intensity at the surface of the plate facing upward in the case of Rayleigh scattering

When the Rayleigh's law holds in scattering of light, the amount of light which is scattered by the matter in the volume element \( A \), and reaches the lowered plate, having its intensity decreased by absorption and secondary scattering is given by

\[
\sec \varphi I_0 e^{-\sec \varphi \cdot az} \left( \gamma + \log (\sec \varphi + 1) \right) \times \sin \Theta_{\varphi} \times d\varphi d\theta d\xi d\eta.
\]

(16)

where \( k \) is considered to be constant, because the states of the suspended particles in all volume elements are considered to be similar in average; \( \Theta \) denotes the angle between the ray of solar light and the line joining \( A \) to the plate (Fig. 1.). The total amount of light scattered upon the Rayleigh's law from a volume element \( d\varphi d\theta d\xi d\eta \) must be equal to \( \alpha_s d\varphi d\theta d\xi d\eta \), therefore

\[
k = \frac{3}{16\pi} \alpha_s.
\]

Take now the rectangular coordinates, \( \xi \)-axis and the origin as before, and \( \eta \)-axis perpendicular to the plane of incidence of the solar light, the \( \xi \)-axis accordingly parallel to this plane. Then the direction cosines of the refracted ray are

\[
-\sin \varphi, 0, \cos \varphi
\]

respectively. And those of the ray, which starts from \( A \), whose coordinates are \( (\xi, \eta, \zeta) \), and reaches the lowered plate, whose coordinates are \( (0, 0, z) \), are

\[
\frac{-\xi}{\sqrt{\xi^2 + \eta^2 + (z - \zeta)^2}} , \frac{-\eta}{\sqrt{\xi^2 + \eta^2 + (z - \zeta)^2}} , \frac{z - \zeta}{\sqrt{\xi^2 + \eta^2 + (z - \zeta)^2}}.
\]

Therefore
Substitute 
\[ \xi = r \cos \theta \]
\[ \eta = r \sin \theta \]
in (17) and again substitute the result in (16), and divide the formula by \( \delta s \); then the intensity \( I_s \) at the surface of the plate facing upward is represented by the following integral:

\[
I_s = \int_0^\infty \int_0^\infty I_0 \sec \varphi \cdot e^{-\sec \varphi \cdot \alpha \xi} \frac{3\alpha \xi}{16\pi} \\
\times \left[ \frac{z - \xi}{r^2 + (z - \xi)^2} \left\{ 1 + \frac{r^2 \cos^2 \theta}{r^2 + (z - \xi)^2} \sin^2 \varphi \right\} \\
+ \frac{2(z - \xi) \cos \theta}{r^2 + (z - \xi)^2} \sin \varphi \cos \varphi \right. \\
\left. + \frac{(z - \xi)^2 \cos^2 \varphi}{r^2 + (z - \xi)^2} e^{-\alpha \sqrt{r^2 + (z - \xi)^2}} \right] d\theta d\xi ,
\]

where the intensity due to the direct solar light, the sky light, the light engendered by secondary and higher scatterings, and the light reflected back into the water at the surface of the lake or sea are excluded as in the case of isotropic scattering.

At first integrate (18) with respect to \( \theta \), then replace \( \alpha \sqrt{r^2 + (z - \xi)^2} \) by \( u \) and arrange the result, then

\[
I_s = \frac{3\alpha \xi}{16\pi} \sec \varphi \cdot I_0 \left\{ (2 + \sin^2 \varphi) \int_0^\infty \alpha (z - \xi) \\
\times e^{-\sec \varphi \cdot \alpha \xi} \left( \int_{\alpha(z - \xi)u}^{\infty} e^{-u} du \right) \right\} + \left( 2 \cos^2 \varphi \\
- \sin^2 \varphi \right) \int_0^\infty \alpha \left( z - \xi \right)^3 e^{-\sec \varphi \cdot \alpha \xi} \left( \int_{\alpha(z - \xi)u}^{\infty} \frac{1}{u^4} du \right) \right\},
\]

The integral of the first term in the parenthesis of (19) is of the same form as that when the factor \( \alpha \sqrt{r^2 + (z - \xi)^2} \) in the integral of (3) is substituted by \( u \); therefore referring to (11) the result of integration of it will be written as follows

\[
\frac{1}{\alpha \sec \varphi} \left( e^{(\sec \varphi - 1) u} - 1 \right) e^{-\sec \varphi \cdot u} \\
+ \frac{\gamma}{\alpha \sec \varphi} \left( \sec \varphi - 1 \right) e^{-\sec \varphi \cdot u} \\
+ \frac{1}{\alpha} \sum_{n=1}^{\infty} \frac{1}{n!} (\alpha \xi)^n \left( \Phi_{n-2} + K_{n-2} \right) \\
\times (\log \alpha z - h_n) ,
\]

where

\[
\Phi_{n-2} = \sum_{i=0}^{n-2} (-1)^i (\sec \varphi - 1)^i \\
K_{n-2} = \sum_{j=0}^{n-2} \Phi_j .
\]

In order to integrate the second integral in the parenthesis of (19), transform

\[
\int_\alpha(z - \xi) u^4 e^{-u} du
\]

by repeating the integration by part. The result of integration is as follows

\[
\left\{ \frac{1}{3 \alpha^3 (z - \xi)^3} - \frac{1}{6 \alpha^2 (z - \xi)^2} + \frac{1}{6 \alpha (z - \xi)} \right\} \\
\times e^{-\alpha(z - \xi)} + \frac{1}{6} e^{-\alpha(z - \xi)} \log \alpha (z - \xi) + \frac{1}{6} \gamma \\
+ \frac{1}{6} \int_0^{\alpha(z - \xi)} e^{-u} \log u du .
\]

Now substitute the result in the second integral in the parenthesis of (19), and put

\[ \alpha (z - \xi) = v \]

then

\[
\int_0^{\alpha(z - \xi)} e^{-\sec \varphi \cdot \alpha \xi} \left( \int_{\alpha(z - \xi)u}^{\infty} e^{-u} du \right) d\xi
\]

\[
= \frac{1}{\alpha \sec \varphi} \int_0^{\alpha(z - \xi)} e^{-\sec \varphi \cdot \alpha \xi} \left( \frac{1}{3} - \frac{1}{6} v + \frac{1}{6} v^2 \right) \\
\times e^{-\sec \varphi \cdot (1 - 1) v} dv + \frac{1}{6 \alpha \sec \varphi \cdot \alpha z} \int_0^{\alpha(z - \xi)} v e^{-\sec \varphi \cdot (1 - 1) v} \\
\times \log vdv + \frac{1}{6 \alpha \sec \varphi \cdot \alpha z} \int_0^{\alpha(z - \xi)} v e^{-\sec \varphi \cdot \alpha z} dv
\]

\[
+ \frac{1}{6 \alpha \sec \varphi \cdot \alpha z} \int_0^{\alpha(z - \xi)} v^3 e^{-\sec \varphi \cdot \alpha z} (\int_{\alpha(z - \xi)u}^{\infty} e^{-u} \log u du) dv .
\]

And

(6)
In order to reduce the last formula to a simpler form, rearrange it in ascending power of $\alpha z$, then it becomes

$$\int_0^{\alpha z} \left( \frac{1}{3} - \frac{1}{6} v + \frac{1}{6} v^2 \right) e^{(\sec \varphi - 1)v} \, dv$$

$$= \left\{ \begin{array}{l}
\frac{1}{3(\sec \varphi - 1)} + \frac{1}{6(\sec \varphi - 1)^2} \\
+ \frac{1}{3(\sec \varphi - 1)^3} e^{(\sec \varphi - 1)\alpha z} \\
+ \frac{(\alpha z)^2}{6(\sec \varphi - 1)} - \frac{1}{6(\sec \varphi - 1)} \\
+ \frac{1}{3(\sec \varphi - 1)^2} \alpha z e^{(\sec \varphi - 1)\alpha z}
\end{array} \right\}$$

(22)

$$\int_0^{\alpha z} v e^{(\sec \varphi - 1)v} \, dv = \frac{(\alpha z)^3}{6(\sec \varphi - 1)} - \frac{3(\alpha z)^2}{6(\sec \varphi - 1)^2}$$

$$+ \frac{6\alpha z}{\sec ^3 \varphi} e^{\sec \varphi - \alpha z} - \frac{6}{\sec ^4 \varphi} \left( e^{\sec \varphi - \alpha z} - 1 \right)$$

(23)

Then, we finally have

$$V_{\text{Intensity at the surface of the plate facing downward in the case of Rayleigh scattering}}$$

$$\text{If the coefficients of pure absorption and scattering be constant throughout the water contained in the lake, the intensity } I_5 \text{ at the surface of the lowered plate facing downward will be represented by the following integral}$$

$$I_5 = \int_0^{\alpha z} \int_0^{\sec \varphi - 1} \frac{3}{16 \pi} e^{\sec \varphi - \alpha z}$$

$$\times \left( \frac{1 + \cos^2 \Theta}{r^2 + (\zeta - z)^2} \right) e^{-av^2/2} d\theta d\phi dz \zeta - z$$

(7)
(18), integrate it with respect to $\theta$, put
$$\alpha \sqrt{r^2 + (\zeta - z)^2} = u,$$
and arrange the result, then
$$I_5 = \frac{3\alpha_z}{16} \sec \varphi \left( 2 + \sin^2 \varphi \right) I_0 \int_z^\infty \alpha (\zeta - z)$$
$$\times e^{-\sec \varphi \cdot \alpha \zeta} \left( \int_z^\infty \frac{1}{u^2} e^{-u} du \right) d\zeta$$
$$+ \frac{3\alpha_z}{16} \sec \varphi \left( 2 \cos \varphi \cdot \sin \varphi \right)$$
$$\times \int_z^\infty \alpha^3 (\zeta - z)^3 e^{-\sec \alpha \zeta}$$
$$\times \left( \int_z^\infty \frac{1}{u^2} e^{-u} du \right) d\zeta. \quad (26)$$

The integral of the first term is of the same form as that in (13) when the factor
$$\alpha \sqrt{r^2 + (\zeta - z)^2}$$
is replaced by $u$. Therefore, the result of integration of it will, referring to (15), be written as follows
$$\frac{1}{\alpha} \left\{ \frac{1}{\sec \varphi} - \frac{1}{\sec^2 \varphi} \log (\sec \varphi + 1) \right\} e^{-\sec \varphi \cdot \alpha z} \quad (27)$$

Now transform
$$\int_z^\infty \frac{1}{u^2} e^{-u} du$$
repeating the integration by part, and put
$$\alpha (\zeta - z) = x$$
in the integral of the second term of (26), then
$$\int_z^\infty \alpha^3 (\zeta - z)^3 e^{-\sec \varphi \cdot \alpha \zeta} \left( \int_z^\infty \frac{1}{u^2} e^{-u} du \right) d\zeta$$
$$= \frac{1}{\alpha} e^{-\sec \varphi \cdot \alpha \zeta} \left\{ \frac{1}{3} \int_0^\infty e^{-(\sec \varphi + 1)z} dx \right\}$$
$$- \frac{1}{6} \int_0^\infty xe^{-\sec \varphi + 1)} dx$$
$$+ \frac{1}{6} \int_0^\infty x^2 e^{-(\sec \varphi + 1)z} \log x. dx$$
$$+ \frac{1}{6} \Gamma \int_0^\infty x^3 e^{-\sec \varphi \cdot \alpha \zeta} dx$$
$$+ \frac{1}{6} \int_0^\infty x^3 e^{-\sec \varphi \cdot \alpha \zeta} \left( \int_0^x e^{-u} \log u du \right) dx$$

These integrals are easily integrable. Substitute the results and (27) in (26), then we finally have
$$I_5 = \frac{3}{16} \frac{\alpha_z}{\alpha} I_0 \left( 2 + \sin^2 \varphi \right) \left\{ \frac{1}{\sec \varphi} \right\}$$
$$\times \left\{ \sec \varphi + 1 \right\} + (2 \cos^2 \varphi \cdot \sin^2 \varphi)$$
$$\times \left\{ \frac{1}{3 (\sec \varphi + 1)^2} \right\}$$
$$+ \frac{1}{6 (\sec \varphi + 1)^2} + \frac{1}{2 \sec \varphi (\sec \varphi + 1)^2}$$
$$\times \log (\sec \varphi + 1) \right\} e^{-\sec \varphi \cdot \alpha z}. \quad (28)$$

The formula shows that the intensity $I_5$ exponentially decreases as the depth of the water down to the lowered plate $z$ increases.

Put $\varphi = 0$ in (28), then we have
$$I_5 = \frac{\alpha_z}{\alpha} I_0 \left( \frac{11}{16} - \frac{3}{4} \log 2 \right) e^{-ax},$$
which is the case when the sun illuminates the lake or marine water vertically downward.

VI Results of numerical calculations and discussion

The formulae which represent $I_2$, $I_3$, $I_4$, and $I_5$ are of the shape
$$I_0 \frac{\alpha_z}{\alpha} f(az, \varphi).$$

As this $\alpha_z$ is particular to each lake, the author thinks that it may be said, that the measured value of it represented one of the characteristics of each individual lake or the part of the sea.

The dimensions of $\alpha$ and $a$ are $L^{-1}$ and $L$ respectively, therefore $\alpha z$ is dimensionless and also $f(\alpha z, \varphi)$ is a mere number; and therefore the formula are available not only for all lakes and all parts of the sea, but also for all wave lengths of light, in so far as the same law of scattering holds. Accordingly, the values of the formulae $f(\alpha z, \varphi)$ once numerically calculated corresponding to every value of $\varphi$ and $\alpha z$, are, to the author's
The author has numerically calculated the formulae which appear in (11) and (25), the former the case of isotropic scattering and the latter the case of Rayleigh scattering, for several values of \( \alpha \zeta \), taking
\[ \varphi = 10^\circ. \]

When \( \varphi = 10^\circ \), the zenith distance of the sun corresponding to it is \( 13^\circ \ 23' \), which is the case frequently realized in many lakes in the world.

The results of numerical calculations are shown in the second and third column of the Table, and they are graphically represented in the Figure 2. Both graphs much resemble each other, and there are no conspicuous differences between the two series of the calculated values. Both graphs show that, at first, near the surface, the intensity increases comparatively rapidly as the depth of the water down to the lowered plate increases, then it reaches its maximum value and then gradually decreases showing a point of inflexion at a certain depth.

From the table and graphs, it is conjectured that the values of \( \alpha \zeta \) corresponding to the maximum intensities in both cases of isotropic and Rayleigh scattering, are nearly equal and amounts about 0.75.

The fourth and fifth columns of the Table
show the ratio $I_2/I_1$ multiplied by $\alpha/\alpha_s$, and the ratio $I_4/I_1$ multiplied by $\alpha/\alpha_s$ respectively, *viz.* the ratio of $f(\alpha z, \phi)$ to $e^{-\alpha_s \phi \alpha z}$ are shown corresponding to several values of $\alpha z$, $\phi$ being kept to $10^\circ$. From them it is evident that the ratio fairly increases as $\alpha z$ increases in both cases.

The intensity at the surface facing downward, in the case of isotropic scattering as well as in the case of Rayleigh scattering, exponentially decreases as the depth of the water down to the lowered plate increases. And if the ratio of the coefficient of scattering, to that of extinction *viz.* $\alpha_s/\alpha$ be equal in both cases, the ratio $I_2/I_5$ *viz.* the ratio of the intensity in the case of isotropic scattering to that in the case of Rayleigh scattering becomes 0.9190 when $\phi=10^\circ$.

Concluding this essay, the author wishes to express his sincere thanks to Assist. Prof. Hiroshi AMAKI for his kind advices on the calculations made by the author.

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