A Note on Ocean Currents Produced by Sources and Sinks*

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Abstract: Ocean currents produced by sources and sinks on a β -plane are treated. In presence of an eastern coast an isolated sink (or source) yields zonal flows only to the west of the sink (or source). The total zonal transport due to the sink (or source) is eastwards (or westwards). The pressure and velocity distributions induced by a point source are determined for the flow with frictional forces.

1. Introduction

GOLDSBROUGH (1933) discussed ocean currents produced by evaporation and precipitation, which are equivalent to distributed sources and sinks. Recently STOMMEL and OTHERS (1958) studied a stationary planetary flow induced by combination of point and distributed sources and sinks, theoretically and experimentally. In this paper is discussed a mathematical model of currents on a β -plane produced by an isolated source or sink.

The model considered here gives a simple example illustrating characteristic features of the β -plane. The result may be applied to a stationary circulation in the deep layer where intense vertical currents serve as a source or sink, by supplying or subtracting water in a small area.

2. Mathematical Model

Consider the ocean on a β -plane. The equations of motion become

$$-fv = -\partial p / \partial x + F_x, \quad fu = -\partial p / \partial y + F_y \quad (1)$$

in which f is the Coriolis' parameter, F_x and F_y are components of friction forces and x and y are coordinates pointing to the east and north, respectively. In presence of sources or sinks, the equation of continuity becomes

$$\partial u / \partial x + \partial v / \partial y = -q \quad (2)$$

The vorticity equation derived from (1) is

$$\beta v - fq = \partial F_y / \partial x - \partial F_x / \partial y \quad (3)$$

For a flow without friction forces this equation becomes

$$v = f^{-1}(\partial p / \partial x) = f\beta^{-1}q \quad (4)$$

Assume an eastern coast at $x = x_1$ where $u = 0$ or $\partial p / \partial y = 0$. The integration of (4) under this boundary condition gives

$$p = f^2\beta^{-1} \int_{x_1}^x q dx \quad (5)$$

in which $x < x_1$. The second equation of (1) leads to

$$u = 2 \int_x^{x_1} q dx + f\beta^{-1} \int_x^{x_1} (\partial q / \partial y) dx \quad (6)$$

In order to understand the meaning of these relations a schematical model shown in figure 1 is considered. An isolated sink (positive q), symmetrical about the x -axis, is assumed to have a north-south dimension of about 1000 km. The equation (4) indicates that v vanishes outside the sink area. According to (6), the zonal current begins at the eastern edge of the sink and its velocity becomes zonally constant to the west of the sink.

The first term in the right hand side of (6) gives an eastward flow to the west of the sink as shown by the curve (A) of figure 1. The second term—a solid curve of (B)—represents the westward flow in the northern half and eastward flow in the southern half. Owing to northward increase of f the profile indicates a north-south asymmetry, contrary to the symmetrical distribution corresponding to the constant f (a broken line of (B) in figure 1). For a sink or source of the dimension of figure 1 the second term of (6) is predominant over the first term.

The total zonal transport due to a sink or source is obtained by integration of (6) with y from $-\infty$ to ∞ . The integral of the

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second term of (6) is given by

$$\int_x^{x_1} \left\{ f\beta^{-1}q \right\}_{-\infty}^{\infty} - \int_{-\infty}^{\infty} q dy \Big\} dx = - \int_x^{x_1} \int_{-\infty}^{\infty} q dy dx$$

since q is zero at $y = \pm\infty$. Therefore, we have

$$\int_{-\infty}^{\infty} u dy = \int_{-\infty}^{\infty} \int_x^{x_1} q dx dy. \quad (7)$$

This equation indicates that the total water supply (or subtraction) becomes the zonal transport from the source (or to the sink).

Fig. 1. Arrangements of an isolated sink and an eastern coast at $x=x_1$. The curves (A) and (B) represent meridional distributions of zonal currents given by the first and second term of the r.h.s. of the equation (6), respectively. (A broken curve of (B) is for $f=f_0$, the value of f at $y=0$.)

STOMMEL and OTHERS (1958) derived a conclusion that a point source (or sink) produces a simple westward (or eastward) flow to the west of the source (or sink), in contrast with a meridional structure of u in this model for a source with a finite area. However, this model cannot be applied to a point source, because the second term of the right hand side of (6) becomes infinite as the dimension of the source diminishes to zero. Actually dissipative forces which are neglected in this model will be important as the area of the source decreases.

In presence of a western coast, the formal solution of (4) is the same as (5). However, the equations (1) are no longer valid near the western boundary owing to the importance of inertia terms neglected in (1) (MORGAN, 1956; ICHIYE, 1959). In this paper, the effect of a western coast is not discussed.

3. The flow with friction forces

In order to obtain a circulation induced by a point source, friction forces are introduced.

Only for understanding the effect of friction, these forces may simply be taken as

$$F_x = -ku, \quad F_y = -kv \quad (8)$$

in which k is assumed to be constant.

The equation of p becomes

$$kf\nabla^2 p + \beta(f\partial p/\partial x - k\partial p/\partial y) = f(k^2 + f^2)q. \quad (9)$$

The value of k was estimated by STOMMEL (1948) as 10^{-6} (sec^{-1}) from comparison of observed and computed transports of the western current. This value indicates that $k \ll f$. Thus the equation (9) can be simplified as

$$\nabla^2 p + \beta k^{-1} \partial p / \partial x = f^2 k^{-1} q. \quad (10)$$

The solution of p for a point source can be obtained by the Fourier integral method, in which q is taken as

$$q = q_0 \text{ for } |x| \leq c \text{ and } |y| \leq c'; \\ q = 0 \text{ elsewhere,} \quad (11)$$

and c is put to zero. The Fourier transform of p about y becomes

$$\hat{p} = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{ia y} P(x, a) da. \quad (12)$$

The equation for P is given by

$$\partial^2 P / \partial x^2 + 2b \partial P / \partial x - a^2 P = f^2 k^{-1} Q \quad (13)$$

in which

$$Q(x, a) = \int_{-\infty}^{\infty} e^{-ia y} q(x, y) dy \quad (14)$$

and $2b = \beta k^{-1}$.

Under the boundary condition at an eastern coast that

$$P = 0 \quad \text{at } x = x_1, \quad (15)$$

the general solution of (13) is expressed by

$$P = A \{ \exp \delta_1 (x - x_1) - \exp \delta_2 (x - x_1) \} \\ + \frac{1}{2} f^2 (\lambda k)^{-1} \int_{x_1}^x Q \{ e^{\delta_1 (x-t)} - e^{\delta_2 (x-t)} \} dt \quad (16)$$

in which

$$\delta_1 = -b + \lambda, \quad \delta_2 = -b - \lambda, \quad \lambda^2 = a^2 + b^2. \quad (17)$$

The other boundary condition at a western coast may determine the constant A of (16). In this model, instead, the condition that

$$P = \text{finite} \quad \text{at } x = -\infty \quad (18)$$

is considered. The function Q corresponding to the distribution of q given by (11) is

$$Q = Q_0 = 2q_0 a^{-1} \sin ac', \quad \text{for } |x| \leq c; \\ Q = 0, \quad \text{for } |x| \geq c \quad (19)$$

The boundary condition (18) and the relation (19) determines A as

$$A = f^2 Q_0 (\delta_2 \lambda k)^{-1} \sinh \delta_2 c \cdot \exp \delta_2 x_1 \\ = A_0 \exp \delta_2 x_1. \quad (20)$$

Thus the solution (15) with A given by (20) is written as

$$P = A_0 \{ \exp (\delta_1 x - 2\lambda x_1) - \exp \delta_2 x \}, \text{ for } x \geq c; \\ P = \{ A_0 \exp (-2\lambda x_1) - B_0 \} \exp \delta_1 x, \\ \text{for } x \leq -c \quad (21)$$

in which

$$B_0 = f^2 Q_0 (\delta_1 \lambda k)^{-1} \sinh \delta_1 c. \quad (22)$$

For numerical values of $k=10^{-6}$ (sec $^{-1}$) and $\beta=2 \times 10^{-13}$ (cm $^{-1}$ sec $^{-1}$), b is equal to 10^{-7} (cm $^{-1}$). Thus when the source is apart from the eastern coast by more than 100 km, we can put $\exp(-2\lambda x_1) \ll 1$. Then (21) can be written as

$$P = -A_0 \exp \delta_2 x, \text{ for } x \geq c; \\ P = -B_0 \exp \delta_1 x, \text{ for } x \leq -c. \quad (23)$$

Further, if c is small enough to put $\delta_2 c \ll 1$, (23) is expressed by

$$P = -f^2 Q_0 c (\lambda k)^{-1} \exp (-bx - \lambda |x|), \\ \text{for } |x| \geq c \quad (24)$$

(In a model with a western coast at $x=x_1$ (<0), the formal solution of (13) is again given by (16). The condition that P is finite at $x=\infty$ determines the value of A as

$$A = -B_0 \exp \delta_1 x_1. \quad (20')$$

Corresponding to (21), we have

$$P = (B_0 \exp 2\lambda x_1 - A_0) \exp \delta_2 x, \text{ for } x > c; \\ P = B_0 \{ \exp (\delta_2 x + 2\lambda x_1) - \exp \delta_1 x \}, \text{ for } x < -c \quad (21')$$

Therefore, when $\lambda |x_1| \gg 1$, the same approximation of P for $x > c$ as in (23) is valid. However, for $x < -c$ the approximation similar to (23) is valid except near the western coast, where the term $\exp (\delta_2 x + 2\lambda x_1)$ becomes important. This corresponds to the effect of intensification of currents due to the western coast [ICHIYE, 1958].

Under the condition that c' is small enough to insure $\sin ac' \approx ac'$, the Fourier integral (12) becomes

$$p = -2f^2 q_0 cc' (\pi k)^{-1} \int_0^\infty \cos aye^{\delta_1 z} \lambda^{-1} da. \quad (25)$$

This integral is expressed by a modified

Bessel function $K_0(x)$ as (ERDELYI *et al*, 1954)

$$p = -2f^2 q_0 cc' (\pi k)^{-1} e^{-bx} K_0(br) \quad (26)$$

in which $r^2 = x^2 + y^2$. This equation represents the Green function for the equation (10) (VERONIS, 1958). However, it should be noted that the dimensions of the source c and c' must be less than 100 km for $k=10^{-6}$ (sec $^{-1}$) (or less than 1000 km for $k=10^{-5}$) so that the approximation leading from (23) to (24) and (25) might be valid.

The geostrophic component of a zonal velocity is expressed by

$$u_g = -f^{-1} (\partial p / \partial y) = 4bS e^{-bx} \{ 2K_0(br) \\ - fy(2kr)^{-1} K_1(br) \} \quad (27)$$

in which $S = q_0 cc' / \pi$. The profiles of u_g at $x = -500$ km and -1000 km are plotted with a unit of $4bS$ in figures 2(A) and 2(B) for $k = 10^{-6}$ and 10^{-5} (sec $^{-1}$), respectively. These profiles indicate that u_g is westward in the northern half and eastward in the southern half for positive q_0 (sink), as is the case of

Fig. 2. Distributions of u_g (in a unit of $4bS$) against y (in 100 km) at $x = -500$ km (full lines) and at $x = -1000$ km (broken lines). Figure (A) is for $k = 10^{-6}$ and (B) for $k = 10^{-5}$ (sec $^{-1}$).

figure 1. The north-south gradient and maximal absolute value of u_g decrease as the distance from the source increases and as the value of k increases. As usually is the case, friction forces tend to decrease the gradient and absolute values of velocity. The pressure field given by (26) is plotted for $k=10^{-6}$ in figure 3(A), which shows rapid decrease of p to the east of the source.

The total zonal transport of a geostrophic current is given by

$$\int_{-\infty}^{\infty} u_g dy = -f^{-1}p \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \beta f^{-2} p dy. \quad (28)$$

Since p vanishes at $y=\pm\infty$, (28) becomes

$$\int_{-\infty}^{\infty} u_g dy = 8bS e^{-bx} \int_0^{\infty} K_0(br) dy. \quad (29)$$

The formula of a modified Bessel function (ERDÉLYI *et al.*, 1953) gives

$$\int_0^{\infty} K_0[b(x^2+y^2)^{1/2}] dy = (\pi/2b) \exp(-b|x|). \quad (30)$$

Therefore, for $x < 0$, we have

$$\int_{-\infty}^{\infty} u_g dy = 4cc'q_0. \quad (31)$$

This relation indicates that the total water supply from the source is carried away by a geostrophic component of zonal currents.

When the source or sink is due to the vertical current w at the top of the deep layer of a thickness h , q_0 is equal to w/h . Using $c=c'=10^7$ (cm), $w=\pi \times 10^{-4}$ (cm/sec) and $h=2 \times 10^5$ (cm), we have $4bS=2 \times 10^{-8}k^{-1}$. This gives the maximal value of u_g of 0.16 (cm/sec) at the distance of 500 km west of the sink, for $k=10^{-6}$ (sec $^{-1}$). The total water supply $4cc'w$ from the source is equal to 0.126×10^6 (m 3 /sec).

The water supply at the source in the North Atlantic postulated by STOMMEL (1958) in his scheme of deep circulation 20×10^6 (m 3 /sec) leads to the vertical velocity of 5×10^{-2} (cm/sec) for the same values of c and c' as above. Then the maximal u_g at 500 km west of the source is about 25 (cm/sec) for the same magnitudes of k and h .

4. A point source at an eastern coast

The pressure distribution for a model with

a point source at an eastern coast can be obtained by putting $x_1=c$ in the second equation of (21) and letting $c \rightarrow 0$. When the third and higher powers of c in the expansions of A_0 and B_0 in (21) are neglected, we have

$$P = -f^2 k^{-1} c' c^2 q_0 \exp \delta_1 x, \text{ for } x < -c \quad (32)$$

The Fourier integral for p becomes

$$p = -4f^2(k\pi)^{-1} c' c^2 q_0 e^{-bx} \int_0^{\infty} \cos ay e^{-\lambda|z|} da. \quad (33)$$

This is again expressed by a modified Bessel function (ERDÉLYI *et al.*, 1954) like

$$p = -4f^2(k\pi)^{-1} c' c^2 q_0 b|x| r^{-1} e^{-bx} K_1(br). \quad (34)$$

The distribution of p computed by (34) is plotted in figure 3(B) with a unit of $0.4f_0^2(k\pi)^{-1} \cdot c' bc^2 q_0$, in which f_0 is the value of f at $y=0$. If $bc=\frac{1}{2}$ (or $c=50$ km for $k=10^{-6}$), this unit is equal to the one used in figure 3(A). The pressure distribution of (34) is similar to figure 3(A) except near the source, though the equations are different. The source in figure 3(B) forms a singular point to which all isobars converge, but it is a singular point corresponding to $p=\infty$ in figure 3(A).

The total zonal transport of a geostrophic component is given by

$$\int_{-\infty}^{\infty} u_g dy = 8(\pi)^{-1} b^2 c' c^2 q_0 |x| e^{-bx} \int_{-\infty}^{\infty} r^{-1} K_1(br) dy. \quad (35)$$

Again, from the relation of a modified Bessel function (ERDÉLYI *et al.*, 1953), we have

$$\int_0^{\infty} (t^2+z^2)^{-1/2} K_1[a(t^2+z^2)^{1/2}] dt = \pi(2az)^{-1} e^{-a|z|}, \quad (36)$$

(35) is written as

$$\int_{-\infty}^{\infty} u_g dy = 4bc'c^2q_0 \quad (37)$$

which is different from the total supply $4cc'q_0$ from the source. In order to obtain a zonal geostrophic transport equal to this supply, the higher power of c must be taken into account in the expansion of (21).

The water from the source is transported neither by a non-geostrophic zonal component nor by meridional currents through the sides A and C of figure 4, as the y -coordinates y_0 and $-y_0$ of A and C approach to $\pm\infty$, respectively.

From (1) and (8), the velocity components

are given by

$$u = -f(\partial p/\partial y + k\partial p/\partial y)(f^2 + k^2)^{-1}$$

$$v = (-k\partial p/\partial y + f\partial p/\partial y)(f^2 + k^2)^{-1}. \quad (38)$$

First, it will be proved that

$$\int_A v dx, \quad \int_C v dy \rightarrow 0 \quad \text{as } y_1 \rightarrow \infty. \quad (39)$$

Fig. 3. Pressure patterns: (A) computed from (26) (in a unit of $-0.2f_0^2(k\pi)^{-2}cc'q_0$) and (B) computed from (34) (in a unit of $-0.4f_0^2(k\pi)^{-1}bc^2c'q_0$). The distances in x - and y -direction are shown in 100 km.

Fig. 4. The paths of integration around a point source for determining a net transport.

relation (39) is proved to be valid for a finite value of x_0 , the x -coordinate of the side B (Fig. 4).

Since $f^2 \gg k^2$, a non-geostrophic component of zonal currents of (38) is given by $u_n = -kf^{-2}(\partial p/\partial x)$. The integration of u_n along the side B gives

$$\int_{-\infty}^{\infty} u_n dy = 4\pi^{-1}c'c^2bq_0 \frac{\partial}{\partial x}$$

$$\times \left[|x| e^{-bx} \int_{-\infty}^{\infty} r^{-1} K_1(br) dy \right], \quad (40)$$

From the relation (36), the expression under partial differentiation about x is equal to π . Thus the right hand side of (40) vanishes. It is proved that the non-geostrophic component of zonal currents does not transport a net amount of water. The same argument can be applied in the previous section to

prove that only a geostrophic zonal component to the west of the source transports the net water supplied by the source.

5. Conclusion

The analysis for a point source with friction as well as for an isolated but finite-dimension source gives different distributions of a zonal velocity from the experiment in a rotating basin by STOMMEL and OTHERS (1958). Such discrepancy is reasonable, because the condition at the slots which supply water into the basin of the experiment may be quite different from the theoretical model treated here.

Neither the circulation pattern induced by a source at the eastern coast has similarity to the spread of the Mediterranean water into the North Atlantic indicated by salinity distribution (Wüst, 1936). However, the vortices with a length scale of 1000 km found at the intermediate depth (Montgomery, 1938) may be attributed to a permanent sink or source in a small, limited area.

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