A Rising-Plume Problem*

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The problem: To estimate the amount of nutrients brought up from the nutrient-rich, stably stratified deep-water into the surface layer by the convective currents which are generated from a maintained source of heat placed on the bottom of the sea or in the deep water. For the sake of simplicity, the density gradient in the deep water is taken to be constant and no allowance is made for the motion of the ambient water.

The convective currents which rise from heat sources have been studied much in connection with atmospheric phenomena. Among mathematical models proposed, one by MORTON, TAYLOR and TURNER (1956) seems to be adequate to use in the present problem. The physics of the rising plume is as follows.

When heat is produced from a continuous source, the water close above the source acquires buoyancy and starts to move upwards in a form of plume. The rising fluid will, sooner or later, be in turbulent motion, the turbulence being generated by the motion in the plume itself. The plume widens as it rises owing to the turbulent mixing by the plume; the ambient water at rest must be entrained into the column of rising water. The rate of entrainment at the edge of the plume may be assumed as proportional to some characteristic velocity, say the mean vertical velocity on the axis of the plume at that depth.

In a stably stratified water the density of the plume increases steadily owing to the entrainment, while that of the ambient water decreases steadily upwards. At some depth above the source, therefore, the density difference vanishes, i.e. the buoyancy vanishes, though the plume continues to rise because of its inertia. Above this depth the downward force of buoyancy acts on the plume fluid so as to reduce its momentum, and ultimately the upward motion of the plume ceases at a certain height from the source ("ceiling height"). The continuous supply of heat makes the topmost part of the rising water spread horizontally, thus a mushroom-like plume is developed. The profiles of mean vertical velocity, density and nutrient concentration in horizontal sections of the plume are assumed to be of similar form at all depths.

If the equations representing conservation of fluxes of volume, vertical momentum and potential density are integrated over the horizontal section of an axisymmetrical plume, they take the form (M–T–T, 1956):

\[ \frac{d}{dx}(\pi b^2 u) = 2\pi bu_e = 2\pi ba u, \quad (1) \]
\[ \frac{d}{dx}(\pi b^2 u^2) = \pi b^2 g \frac{\rho_e - \rho}{\rho_s}, \quad (2) \]
\[ \frac{d}{dx}(\pi b^2 \rho u) = 2\pi ba u \rho_e, \quad (3) \]

where

- \( b \): radius of plume,
- \( u \): vertical velocity,
- \( \rho \): potential density inside the plume,
- \( \rho_e \): potential density outside the plume,
- \( \rho_s \): a reference density of water,
- \( g \): acceleration of gravity,
- \( \alpha \): the proportionality constant relating the entrainment velocity at the edge of the plume, \( u_e \), to the vertical velocity in the plume.

The \( x \)-axis is taken vertically upwards. Equation (3) may be replaced by the equation for the vertical flux of buoyancy, with the aid of (1),

\[ \frac{d}{dx} \left( \pi b^2 u g \frac{\rho_e - \rho}{\rho_s} \right) = \pi b^2 u \frac{g}{\rho_s} \frac{d\rho_e}{dx}. \quad (4) \]
The boundary conditions at the source of buoyancy only are that at \( x=0 \)
\[ \pi b^2 u = \pi b^2 u^2 = 0, \quad \pi b^2 u 0 \frac{d\sigma - \sigma}{\rho} = Q \text{ (constant)}. \] (5)

In order words, we regard the size of the source as infinitesimally small compared with dimension of plume, and the source releases heat alone with a constant rate.

The character of solutions to equations (1), (2), (4) and (5) must depend on the two physical quantities specified in the problem, namely, the rate of release of heat (buoyancy) at the source, \( Q \), and the stability of the ambient water \(- \frac{g}{\rho_1} \frac{d\sigma}{dx} = E \text{ (constant)}. \) Thus the ceiling height \( x_c \) should be expressed as
\[ x_c = \beta Q^{1/4} E^{-3/8} \]
where the non-dimensional constant \( \beta \) may depend, in general, on \( \alpha \). The volume flux can also be expressed as
\[ \pi b^2 u = \gamma Q^{3/4} E^{-5/8} f \left( \frac{x}{Q^{1/4} E^{-3/8}} \right), \]
where \( \gamma \) is a non-dimensional constant dependent on \( \alpha \). The determination of the coefficients \( \beta \) and \( \gamma \) and of the form \( f \left( \frac{x}{Q^{1/4} E^{-3/8}} \right) \) requires to solve the set of equations.

To this end, we reduce the equations into non-dimensional form by the use of the scaling parameters \( Q \) and \( E \). The resultant non-dimensional equations become equivalent to those used by M-T-T in their numerical computations. Thus we have
\[ x_c \equiv 1.37 \alpha^{-1/2} Q^{1/4} E^{-3/8}, \]
\[ \pi b^2 u \equiv 2.05 \alpha^{1/2} Q^{3/4} E^{-5/8} w(x_c), \] (6)
where \( x_c = 2^{5/8} \alpha^{1/2} Q^{1/4} E^{-3/8} \) and the solution for \( w(x_c) \) is tabulated with respect to \( x_c \), by M-T-T.

The dependence of \( x_c \) on the parameters implies that relatively large increase in the strength of source is necessary for the plume to rise slightly higher and that the stability of the environmental water has comparatively weak influence on the ceiling height. The \( Q^{1/4} \) behavior of the ceiling height will discourage us, should a heat source, say nuclear reactor, be used in artificial cultivation of the ocean surface-layer by lifting nutrients from the deeper layer.

The upward flux of excess nutrient at the ceiling height is expressed by
\[ F_N = \int_0^{x_c} n \pi b^2 u dx, \] (7)
where \( n \) is the vertical gradient of nutrient concentration in the deep water. The flux of excess nutrient can then be estimated by substituting (6) into (7).

We shall give an example. Let \( \alpha = 0.1, Q = 1.48 \times 10^7 (\text{cm}^4/\text{sec}^3) \) equivalent to a heat source of 100 Mega Watts, \( E = 5 \times 10^{-5} / \text{cm} \), and \( n = 2 \times 10^{-5} \mu\text{g-at.} / \text{L} / \text{cm} \) (Phosphate-P). The ceiling height is estimated to be approximately 110 meters. The flux of excess phosphate at the depth of the ceiling height is then computed as 354 gm-P/sec. If this flux of nutrient supplied into the surface layer (100 m thick) is balanced by the horizontal turbulent transfer, the excess concentration of phosphate-P in the surface layer right above the plume becomes 0.1 \( \mu\text{g-at.} / \text{L} \), and the value at 1 km distant from the plume is 0.002 \( \mu\text{g-at.} / \text{L} \) provided that the horizontal eddy diffusivity is taken as \( 10^6 \text{cm}^2/\text{sec} \). On the other hand, if the horizontal flow generated by the plume in the surface layer transfers, without horizontal diffusion, the nutrient brought up, we gain the nutrient content uniformly in the surface layer by the amount of 0.1 \( \mu\text{g-at.} / \text{L} \).

Reference