Formulation of a Three-Dimensional, Steady Ocean Circulation Model Using a Similarity Hypothesis

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Abstract: This model portrays steady, wind-driven flow applicable to the interior region of a beta-plane ocean in which density varies continuously with depth. Under the assumption of steady flow in an incompressible fluid, the vector velocity can be determined from a vector potential. Moreover, the velocity is expressed as the vector product of the density gradient with the gradient of an undetermined potential function, an assumption which automatically satisfies continuity and the energy relation that potential density be conserved along stream lines (in the absence of mixing). A similarity hypothesis regarding the dependence of the horizontal variation in potential density and the potential function are introduced and discussed. Boundary conditions appropriate to a baroclinic mode in a deep ocean are introduced. The resulting mathematical problem is highly nonlinear, and we have had little success in obtaining solutions for the three-dimensional velocity field. The results obtained to date and the problems encountered, however, may provide some insight into physical restrictions imposed by the introduction of such similarity hypotheses into the general circulation problem.

It is shown that the similarity hypothesis is general enough to include several published density models as special cases—in fact, the dynamical results of two such models are derived as special cases of the present problem.

I. Introduction

In the well-known model of Sverdrup (1947), describing steady, wind-driven circulation in the interior of an ocean, the approach of vertically integrating the momentum conservation relations from the surface to some great depth was used. This leads to a relation between distributions of horizontal mass transport and applied wind stress. However, vertical integration negates all possibility of describing the internal velocity field and, consequently, of gaining insight into relations between the density and velocity distributions.

Models comprised of two or more layers with distinct, uniform density can and have been formulated and solved, e.g., Nowlin (1967), with kinematic as well as density stratification for interior oceanic circulation. Such models provide first approximations to relations involving the internal velocity field.

Of considerable interest is the study of steady, wind-driven circulation for an oceanic interior with a general, arbitrary density profile. Highly desirable would be the attainment of solutions for the three velocity components in such a general interior model. The formulation of such models in terms of basic equations is easy; obtaining solutions is not.

In recent years there have appeared in the literature many mathematical models for the thermohaline circulation in an idealized oceanic interior. Essentially these papers constitute extensions and/or refinements of the formulation by Robinson and Stommel (1959) and Stommel and Webster (1962). Despite mathematical difficulties, each of these studies resulted in limited or approximate solutions to a set of equations modeling the thermocline. Most such models utilize some form of similarity hypothesis regarding the distribution of certain variables at any two distinct depths and thereby threat the depth dependence parametrically when considering the horizontal distributions of dependent variables.
In the present paper we formulate the steady, wind-driven, baroclinic, interior circulation for an arbitrary density profile and three-dimensional motion. As with the thermohaline models, we introduce a hypothesis regarding the similarity in depth dependence of the density field and certain aspects of the velocity field. The present model differs significantly from thermohaline models in that no mixing is allowed.

Most of this paper is concerned with the formulation of the model in terms of the similarity hypothesis, the investigation of restrictions imposed by this formulation and the demonstration of the generality and relation of this model to other studies. The present similarity hypothesis and resulting formulation were suggested by Professor R. O. Reid in 1965 as a logical extension of his density model for the equatorial Pacific (1948).

We allow wind stress and consider the model to be turbulent, although lateral turbulent momentum exchange is neglected along with nonlinear field accelerations. The mean velocity field is thus sought. One artificiality of the model, which should be emphasized, is that vertical turbulent momentum exchange is allowed in a density stratified model but no associated mixing of heat and salt is assumed, since the flow is along isopycnal surfaces.

II. The Model

(1) Geometry and assumptions

We consider three-dimensional motion in a model ocean. The vector fluid velocity is represented by \( \mathbf{V} = iu + jv + kw \), where \( i, j \) and \( k \) are \( x, y \) and \( z \)-directed unit vectors.* The cartesian coordinates \( x, y \) and \( z \) increase eastward, northward and upward, respectively. The level surface corresponding to mean sea level is taken as the \( z=0 \) datum plane.

The principal assumptions employed in the development of this model are that:

1. The motion is steady;
2. The fluid is incompressible;
3. The nonlinear field accelerations and horizontal turbulent transfer of momentum are negligible in the interior ocean region considered;
4. The surface wind stress represents the only external force applied;
5. The vertical pressure distribution is related to the density field through the hydrostatic relation;
6. The model ocean is situated on a beta plane;
7. The vertical transfer of momentum by Boussinesq forms for \( x \)- and \( y \)-directed stress over horizontal planes, \( i.e., K \frac{\partial u}{\partial z} \) and \( K \frac{\partial v}{\partial z} \), respectively;
8. The velocity and stress components approach zero at very great depth; and
9. Mixing of heat and salt is neglected and hence the flow is along isopycnal surfaces.

(2) The equations

Under the foregoing assumptions conservation of horizontal momentum is expressed by the relations,

\[
-fv + \frac{1}{\rho} \frac{\partial p}{\partial x} = \frac{\partial}{\partial z} \left( K \frac{\partial u}{\partial z} \right), \quad (\text{II-1})
\]

\[
fu + \frac{1}{\rho} \frac{\partial p}{\partial y} = \frac{\partial}{\partial z} \left( K \frac{\partial v}{\partial z} \right), \quad (\text{II-2})
\]

where \( \rho \) is fluid density, \( p \) is pressure and \( K \) is the kinematic coefficient of vertical eddy viscosity (c.g.s. units cm\(^2\)sec\(^{-1}\)). The locally vertical component of the earth's vorticity, the Coriolis parameter, is denoted by \( f \). The hydrostatic equation can be written

\[
\frac{\partial p}{\partial z} = -\rho g(1+S), \quad (\text{II-3})
\]

where density is expressed in the form

\[\rho = \rho_0(1+S).\]

For this representation \( \rho_0 \) is taken as constant, and \( S \) is a disguised measure of \( \sigma \) for this incompressible fluid.

The continuity equation is

\[\nabla \cdot \mathbf{V} = 0. \quad (\text{II-4})\]

Moreover, for flow along isopycnal surfaces,
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\[ \nabla \cdot P \rho = 0 \]
\[ \nabla \cdot P S = 0, \]  \hspace{1cm} (II-5)

The system of 5 scalar relations, (II-1) thro' (II-5), involves the 5 dependent variables: \( u, v, w, p \) and \( S \). There is, therefore, the possibility of obtaining solutions for these dependent variables, provided that we stipulate the appropriate boundary conditions.

The surface conditions, at \( z = 0 \), can be written

\[ \begin{align*}
\tau_x &= K \frac{\partial u}{\partial z}, \\
\tau_y &= K \frac{\partial v}{\partial z},
\end{align*} \]  \hspace{1cm} (II-6)

where \( \tau = \tau_x \hat{i} + \tau_y \hat{j} \) is the kinematic form of the mean surface wind stress, a specified function of horizontal position \( x \) and \( y \). The deep-water conditions are expressed

\[ u, v, w, \frac{\partial u}{\partial z}, \frac{\partial v}{\partial z} \rightarrow 0 \text{ as } z \rightarrow -\infty \]  \hspace{1cm} (II-7)

Equations (II-1) and (II-2) are simplified by replacing \( \rho \) by \( \rho_0 \) and taking \( K \) as constant. Thus

\[ \begin{align*}
-f_v - \frac{1}{\rho_0} \frac{\partial \rho}{\partial x} &= K \frac{\partial^2 u}{\partial z^2}, \\
f_v + \frac{1}{\rho_0} \frac{\partial \rho}{\partial y} &= K \frac{\partial^2 v}{\partial z^2}.
\end{align*} \]  \hspace{1cm} (II-8)

III. A Reformulation of the Problem

If \( \hat{V} \) is a solenoidal vector field, \( i.e. \), one which satisfies equation (II-4), then \( \hat{V} \) may be represented in the form

\[ \hat{V} = \Gamma_1 \times \Gamma_2, \]

where \( \Gamma_1 \) and \( \Gamma_2 \) are scalar fields (cf. PHILLIPS 1933; p. 103-106). With this representation, a curve defined by the intersection of an iso-\( \Gamma_1 \) surface with an iso-\( \Gamma_2 \) surface is seen to be a stream line (Fig. 1). Equiscalar surfaces for \( \Gamma_1 \) or \( \Gamma_2 \) are stream surfaces. It is to be noted that for a given solenoidal field \( \hat{V} \), the functions \( \Gamma_1 \) and \( \Gamma_2 \) are unique only to within a contact transformation.

In view of equation (II-5), which maintains that isopycnal surfaces (\( S \) uniform) are also stream surfaces, it is natural to select \( S \) as one of the scalar fields \( \Gamma_1 \) and \( \Gamma_2 \); let \( \phi \) denote the other. Thus,

\[ \hat{V} = P S \times P \phi. \]  \hspace{1cm} (III-1)

It is seen that this choice of the form for \( \hat{V} \) satisfies (II-4) and (II-5) automatically. Our problem is then reduced to seeking solutions for \( S, \phi \) and \( P \) from the system comprised of relations (II-1) through (II-3) and the boundary conditions, with the velocity components expressed by (III-1) in terms of \( S \) and \( \phi \).

Now, \( S = \text{constant} \) and \( \phi = \text{constant} \) surfaces are stream surfaces. The intersections of such stream surfaces with the surface \( z = 0 \), on which we require \( w = 0 \), must represent stream lines. Therefore, isolines of \( S \) and \( \phi \) must correspond at \( z = 0 \). Moreover, in view of the condition \( w \rightarrow 0 \) as \( z \rightarrow -\infty \), isolines of \( \phi \) must approach coincidence to isolines of \( S \) at very great depth. In summary, we can assert that \( S = \text{function (} \phi \text{ only)} \) both at \( z = 0 \) and for \( z \rightarrow -\infty \).

(1) Similarity hypothesis

We have seen that our upper and lower kinematic conditions require coincidence of \( S \)- and \( \phi \)-isolines at these boundaries of our model ocean. We now consider the consequences of extending to intermediate depths some degree of similarity between \( S \) and \( \phi \). In order to separate the horizontal and depth dependence, carrying the later in parametric form, we make the similarity hypothesis that \( S \) and \( \phi \) can be represented by the forms:

\[ S(x, y, z) = A(q)G(mz) + B(r)[1 - G(mz)], \]  \hspace{1cm} (III-2)

\[ \]
\[ \phi(x, y, z) = q(x, y)F(mz) + r(x, y)[1 - F(mz)] \]

with \( m^{-1} \) a vertical scale factor which varies generally with the horizontal coordinates \( x \) and \( y \). In order to obtain coincidence of \( S \) and \( \phi \) isolines at upper and lower boundaries, we can require

\[ F = 1, \quad G = 1 \quad (\text{at } z = 0) \quad (\text{III-4}) \]

and

\[ F, G \to 0 \quad (\text{as } z \to -\infty) \quad (\text{III-5}) \]

The question of whether an adequately general representation of the velocity field can be obtained using representations of the forms (III-2) and (III-3) for \( S \) and \( \phi \) is naturally one of some concern. On the other hand, it is reasonable to ask whether forms for \( S \) and \( \phi \) even simpler than (III-2) and (III-3) would not suffice for representing the velocity field in some instances. We defer these questions until the present model has been posed in terms of the proposed functional forms and a degree of familiarity with these forms for \( S \) and \( \phi \) thereby attained.

We immediately note one consequence of our choice of the basic form \( \phi \) for the velocity, however. This form is applicable only to the physical situation of continuous density distributions, if we are to interpret \( S \) as a measure of density, namely the fractional density anomaly \((\rho - \rho_0)/\rho_0\). The case of a homogeneous ocean, i.e., for which \( S \) is constant, is seen to lead to \( F = 0 \) and consequently to \( \nabla \phi = 0 \) everywhere in the system. Likewise, an idealized ocean composed entirely of layers having uniform density is not representable by the present formulation. In general, this representation is applicable only to continuously stratified media, although some layers in which density does not vary with elevation are allowable. If density is assumed a function of depth only, however, then \( F \) is vertical and it follows that the velocity is entirely horizontal.

We now express the model in terms of the functions given by (III-2) and (III-3). This approach involves replacing the three velocity components \( u, v \) and \( w \), each a function of \( x, y \) and \( z \), by three functions of \( x \) and \( y \), namely \( q(x, y) \), \( r(x, y) \) and \( m(x, y) \), and the single variable functions \( A(q) \), \( B(r) \), \( F(mz) \) and \( G(mz) \).

\(_{\text{2}}\) The velocity representations and the boundary conditions

The gradients of \( S \) and \( \phi \) as given by (III-2) and (III-3) are

\[ FS = A'Gq + B'(1 - G)r + (A - B)GF(z), \]

\[ F\phi = Fq + (1 - F)r + (q - r)F'(z), \]

where \( A' \) and \( B' \) denote derivatives with respect to \( q \) and \( r \), respectively, and the primes on \( G \) and \( F \) denote differentiation of these functions with respect to \( mz \).

Using equation (III-1), it can be shown that the cartesian velocity components are

\[ \begin{align*}
  u &= -m\left( F_1 \frac{\partial q}{\partial y} + F_2 \frac{\partial r}{\partial y} \right), \\
  v &= m\left( F_1 \frac{\partial q}{\partial x} + F_2 \frac{\partial r}{\partial x} \right), \\
  w &= F_0J \left( \frac{r}{x, y} \right) + F_1zJ \left( \frac{m, q}{x, y} \right) + F_2zJ \left( \frac{m, r}{x, y} \right)
\end{align*} \]

where

\[ \begin{align*}
  F_0 &= F(1 - G)B' - (1 - F)G'A' \\
  F_1 &= FG'(A - B) - F'G(q - r)A' \\
  F_2 &= (1 - F)G'(A - B) - F'(1 - G)B',
\end{align*} \]

and the \( J \) denotes the Jacobian operator, e.g.,

\[ J \left( \frac{r, q}{x, y} \right) = \frac{\partial r}{\partial x} \frac{\partial q}{\partial y} - \frac{\partial r}{\partial y} \frac{\partial q}{\partial x}. \]

We express the surface and deep-water conditions, (II-6) and (II-7), in terms of \( S \) and \( \phi \). These conditions impose requirements, in addition to (III-4) and (III-5), on the functional representations of \( S \) and \( \phi \).

The surface condition (II-6a) requires that at \( z = 0 \)

\[ F_0J \left( \frac{r, q}{x, y} \right) = 0, \]

which in turn requires \( F_0 = 0 \) at \( z = 0 \). It is seen from (III-7) that condition (III-4) is sufficient
to satisfy this surface requirement. We note that, at \( z=0 \),
\[
\begin{align*}
u &= -m F_1 \frac{\partial \psi}{\partial y} = -m F_1 \frac{\partial \phi}{\partial y}, \\
v &= m F_1 \frac{\partial \psi}{\partial x} = m F_1 \frac{\partial \phi}{\partial x}.
\end{align*}
\]
At this level the function \( \psi \) thus serves as a velocity stream function except for the scale factor \( m F_1 \).

With regard to the deep-water condition (II-7), it is seen that we must require that
\[
\begin{align*}
\epsilon &= \epsilon_0, \\
\Delta &= \Delta_0.
\end{align*}
\]
In addition to (III-5), i.e., \( F, G \rightarrow 0 \) as \( z \rightarrow \infty \), the requirements needed to satisfy (II-7) are just those which will insure that
\[
\begin{align*}
z[(A-B)G'-(q-r)B'] &\rightarrow 0, \\
z[F'G'(A-B)-F'G(q-r)A'] &\rightarrow 0, \\
F'G'(A-B)+F''G(q-r)A' &\rightarrow 0.
\end{align*}
\]
and
\[
\begin{align*}
(1-F)G'G(A-B)-F'G(q-r)B' &\rightarrow 0, \\
(1-G)(q-r)B' + F''G'(q-r)B' &\rightarrow 0.
\end{align*}
\]
as \( z \rightarrow \infty \). The restrictions that \( F, G, G'z, G'' \rightarrow 0 \) and that \( G'' \) and \( F'' \) be bounded as \( z \rightarrow -\infty \) suffice to eliminate all terms of these requirements except
\[
\begin{align*}
G''(A-B) &\rightarrow 0, \\
F''(q-r)B' &\rightarrow 0.
\end{align*}
\]
These are satisfied provided \( G'', F'' \rightarrow 0 \) as \( z \rightarrow -\infty \) or \( G'' \rightarrow 0 \) as \( z \rightarrow -\infty \) and \( B = \text{constant} \).

We choose the latter in view of the fact that we will later find \( B = \text{constant} \) is a requirement needed to insure a reasonable value of the pressure gradient at great depth. In summary, then, the deep-water condition (II-7) is automatically satisfied by our similarity hypothesis provided that
\[
F, G, F'z, G'z, G'' \rightarrow 0 \text{ and } F''
\]
bounded as \( z \rightarrow -\infty \)
\[
\left\{ \begin{array}{l}
B = \text{constant}.
\end{array} \right.
\]

(3) The Pressure field

The hydrostatic approximation (II-3) may be vertically integrated to obtain
\[
\rho = \rho_0 C(x,y) + \rho_0 \int_0^z (1 - S) dz, \quad (III-9)
\]
where \( \rho_0 C(x,y) \) is a function of horizontal position introduced on integration to represent the pressure at \( z=0 \). We can obtain an approximation to \( C(x,y) \) by evaluating (III-9) at \( z=\eta \), the sea surface. Neglecting the magnitude of \( \int_0^z S dz \) in comparison with that of \( \int_0^\eta dz \), we obtain the approximation
\[
\rho_0 C(x,y) \approx \rho_a + \rho_0 \eta,
\]
where \( \rho_a = \rho(x,y,\eta) \) is the sea-level atmospheric pressure.

Defining a dimensionless measure of depth by
\[
\zeta = mz, \quad (III-10)
\]
the functional form for \( S \) given by the similarity hypothesis (III-2) can be written
\[
S = A(q)G(\zeta) + B(r)[1-G(\zeta)].
\]
Let
\[
Q = \int_0^1 G(\zeta) d\zeta. \quad (III-11)
\]
Introducing the foregoing expression for \( S \) into (III-9) yields
\[
\frac{p}{\rho_0} = C - g z + \frac{q}{m} A Q - \frac{q}{m} B (m z + Q). \quad (III-12)
\]
Operating on \( \frac{p}{\rho_0} \) with the horizontal gradient operator,
\[
\begin{align*}
\nabla_h &= \frac{\partial}{\partial x} + j \frac{\partial}{\partial y},
\end{align*}
\]
we obtain the horizontal pressure gradient
\[
\begin{align*}
\frac{1}{\rho_0} \nabla_h p &= \nabla C + q \frac{A}{m} \frac{Q}{m} \nabla q - q \frac{B}{m} (\xi + Q) \frac{\partial}{\partial m} \\
&\quad + \frac{q}{m^2} (B G \zeta + B Q - A G \zeta - A Q) \frac{\partial}{\partial m}.
\end{align*}
\]
As a check, we examine horizontal pressure gradients on the \( z=0 \) surface. \( G=1 \) and \( Q=0 \)
on this surface, the foregoing equation yields \( \nabla H = \rho \nabla C \), as expected.

Considering the equations of motion (II-8) for this model and the deep-water conditions (II-7), that velocities and horizontal stresses tend to zero with increasing depth, we expect that the horizontal pressure gradients should approach zero at great depth. To investigate \( \frac{1}{\rho_0} \gamma_{Hp} \) as \( z \to -\infty \), first note that \( H \to 0 \) and that \( Q \to Q_1 \), a constant. Thus for very great depth

\[
\frac{1}{\rho_0} \gamma_{Hp} = \frac{g}{m} A' Q_1 \gamma_{qp} - q B' \left( z + \frac{Q_1}{m} \right)
\]

\[
\gamma_{r} = \frac{g}{m} Q_1 (B - A) \gamma_{lm} m.
\]

In view of the term \( g B' (z + \frac{Q_1}{m}) \gamma_{r} \), it is seen that if \( \gamma_{Hp} \) is to remain finite (much less approach zero) as \( z \to -\infty \), we require that \( B' \gamma_{r} = 0 \). If \( \gamma_{r} = 0 \), this implies \( r \) is constant which leads to \( B = \) constant and \( B' = 0 \). Thus, the less restrictive choice will be just to require that \( B' = 0 \) which was anticipated and given as one of the deep-water conditions (III-8).

With the added requirement \( B' = 0 \), some of our foregoing equations can be simplified. The principal simplification results in reducing the functions \( F_0 \) and \( F_3 \), as given by (III-7), to the forms

\[
\begin{align*}
F_0 &= - (1 - F) G A' \\
F_1 &= FG' (A - B) - F' G (q - r) A' \\
F_2 &= (A - F) G' (A - B),
\end{align*}
\]  

which somewhat simplify the velocity components given by (III-6).

Returning to our discussion of the pressure field, if we are to require the horizontal pressure gradient to vanish at great depth, then

\[
\lim_{z \to -\infty} \frac{1}{\rho_0} \gamma_{Hp} = \gamma_{C} + \frac{g}{m} A' Q_1 \gamma_{qp} + \frac{q}{m^2} Q_1
\]

\[
(B - A) \gamma_{lm} = 0,
\]

or

\[
\gamma_{C} = - \frac{g Q_1}{m} \left[ A' \gamma_{qp} + \frac{B - A}{m} \gamma_{lm} \right].
\]  

So, the horizontal pressure gradient force for an arbitrary depth may be expressed in the form

\[
\frac{1}{\rho_0} \gamma_{Hp} = \frac{g A'}{m} (Q - Q_1) \gamma_{qp} + \frac{g}{m^2} (B - A)
\]

\[
[G' + (Q - Q_1)] \gamma_{lm}.
\]  

(III-14)

(4) The volume transport stream function

Integrating our continuity equation \( \gamma_{C} \gamma_{l} = 0 \) with respect to depth and approximating by zero the upper limit of integration \( \eta \), we obtain

\[
\frac{\partial}{\partial x} \int_{-\infty}^{0} u dz + \frac{\partial}{\partial y} \int_{-\infty}^{0} v dz = 0.
\]

Thus, there must exist a volume transport stream function \( \theta(x, y) \) such that

\[
\frac{\partial \theta}{\partial x} = - \int_{-\infty}^{0} u dz, \quad \frac{\partial \theta}{\partial y} = \int_{-\infty}^{0} v dz.
\]  

(III-15)

We now express the differential of \( \theta \) in terms of the functions introduced by the similarity hypothesis.

Using (III-6)

\[
\int_{-\infty}^{0} u dz = - m \frac{\partial q}{\partial y} \int_{-\infty}^{0} F_2 dz - m \frac{\partial r}{\partial y} \int_{-\infty}^{0} F_2 dz.
\]

Then, defining

\[
N_0 = \int_{-\infty}^{0} F G'd \zeta.
\]  

(III-16)

and using (III-7'), we have

\[
m \int_{-\infty}^{0} F_2 dz = (A - B) N_0 - (q - r) A'(1 - N_0),
\]

\[
m \int_{-\infty}^{0} F_2 dz = (A - B) (1 - N_0).
\]

So,

\[
- \int_{-\infty}^{0} u dz = [(A - B) N_0 - (q - r) A'(1 - N_0)]
\]

\[
[\frac{\partial q}{\partial y} + [(A - B) (1 - N_0)] \frac{\partial r}{\partial y}.
\]

It can be shown in like manner that \( \int_{-\infty}^{0} v dz \) is identical except that derivatives of \( q \) and \( r \) are with respect to \( x \) in that case.

The differential of \( \theta \) is

\[
d \theta = \left[ \int_{-\infty}^{0} u dz \right] dx + \left[ - \int_{-\infty}^{0} u dz \right] dy.
\]

This can be expressed

\[
d \theta = [(A - B) N_0 - (q - r) A'(1 - N_0)] dq
\]

\[
+ [(A - B) (1 - N_0)] dr.
\]
Since $\Theta$ is a stream function, a line integral of $d\Theta$ between any two points in the $rq$-plane is independent of the path of integration, depending only on the end points. Let $\Theta_0$ denote the value of $\Theta$ at $r=q=0$. Now integrate $d\Theta$ over the path consisting of the line segments $(0,0)$ to $(0,q)$ and $(0,q)$ to $(r,q)$, to obtain

$$
\Theta = \Theta_0 + (1-N_0)(A-B)r - (1-N_0)Aq - N_0Bq + \int_0^q A(q) dq. \quad \text{(III-17)}
$$

(5) The momentum conservation equations

We are now in a position to express the model completely in terms of the new functions introduced by the similarity hypothesis, equations (III-2) and (III-3). The boundary conditions (II-6a) and (II-7) have already been reformulated and are automatically satisfied by the similarity hypothesis under the restrictions (III-4) and (III-8), respectively. Using (III-7') and (III-6), the surface stress conditions (II-6b) can be written:

$$
K m^2 \left\{ n_1(A-B) - n_2(q-r)A' \frac{\partial q}{\partial y} \right\} + n_3(A-B) \frac{\partial r}{\partial y} = -\tau_x \tag{III-18}
$$

$$
K^2 \left\{ n_1(A-B) - n_2(q-r)A' \frac{\partial q}{\partial x} \right\} + n_3(A-B) \frac{\partial r}{\partial x} = \tau_y \tag{III-19}
$$

with

$$
\begin{align*}
&n_1 = F'G' + G'' , \\
n_2 = F'G' + F'' , \\
n_3 = -F'G'.
\end{align*}
$$

There seems to be no advantage to be gained from writing the full unintegrated momentum conservation equations (II-8) in the forms determined by the similarity hypothesis. We formulate rather the vertically integrated momentum conservation equations,

$$
-f \frac{\partial \Theta}{\partial x} + \frac{1}{\rho_0} \frac{\partial}{\partial x} \int_{-\infty}^{0} p dz = \tau_x ,
$$

$$
-f \frac{\partial \Theta}{\partial y} + \frac{1}{\rho_0} \frac{\partial}{\partial y} \int_{-\infty}^{0} p dz = \tau_y .
$$

Consider the pressure gradient term,
It was previously stated that layers in which density does not vary with elevation were permissible although this formulation is generally for the case of continuous depth stratification. Let us consider the case (Fig. 2) of a surface boundary layer, \(-h \leq z \leq h\), in which density is independent of depth and given by \(\rho_0 (1 + A)\). In such a case we can not reasonably consider the velocity within this layer through use of the form \(V = S \times \phi\). However, with proper reformulation of the upper stress conditions, it may be possible to use the similarity representation within that region of the ocean underlying the surface boundary layer.

In place of the surface boundary conditions (II-6b), we consider the stress at \(z = -h\). Denoting the components of such stress by \(\tau_{zh}\) and \(\tau_{yh}\) we obtain at \(z = -h\):

\[
\tau_{zh} = K \frac{\partial u}{\partial x}; \quad \tau_{yh} = K \frac{\partial v}{\partial x}.
\]

This gives, in place of (III-18) and (III-19),

\[
K m^2 \left\{ \left[ n_4 (A - B) - n_6 (q - r) A V_{z} \right] \frac{\partial q}{\partial y} + n_6 (A - B) \frac{\partial r}{\partial y} \right\} = -\tau_{zh}
\]

\[
K m^2 \left\{ \left[ n_4 (A - B) - n_6 (q - r) A V_{x} \right] \frac{\partial q}{\partial x} + n_6 (A - B) \frac{\partial r}{\partial x} \right\} = \tau_{yh}
\]

with

\[
\begin{align*}
\tau_{zh} &= F' G' + G'' F, \\
\tau_{yh} &= F' G' + F'' G, \\
\tau_{zh} &= -F' G' + (1 - F) G''
\end{align*}
\]

at \(z = -h\).

**IV. Some Formal Solutions and Considerations**

1. **The case for uniform surface density**

If we consider an ocean for which surface density \(\rho_1\) is uniform, then \(A\) is uniform. For simplicity, choose \(A\) to be zero. So,

\[
S = B [1 - G(mz)],
\]
and the surface density is given \(\rho_0\).

Although the overall vertical contrast of density \((\rho_1 - \rho_0) = B\) is independent of position, horizontal dependence of the density and velocity fields are not completely lost in this case. The \(x\)- and \(y\)-dependence is carried by the depth scale \(h(x, y) \equiv 1/m(x, y)\).

The vertically integrated equations of motion are

\[
\begin{align*}
&fB \left[ N_0 \frac{\partial q}{\partial x} + (1 - N_0) \frac{\partial r}{\partial x} \right] - qBN_0 \frac{\partial h^2}{\partial x} = \tau_x, \\
&fB \left[ N_0 \frac{\partial q}{\partial y} + (1 - N_0) \frac{\partial r}{\partial y} \right] - qBN_0 \frac{\partial h^2}{\partial y} = \tau_y.
\end{align*}
\]

(IV-1)

From (III-18) and (III-19), the surface conditions are

\[
\begin{align*}
&n_1 \frac{\partial q}{\partial y} + n_6 \frac{\partial r}{\partial y} = f_1 \tau_x, \\
&n_1 \frac{\partial q}{\partial x} + n_6 \frac{\partial r}{\partial x} = f_1 \tau_y,
\end{align*}
\]

at \(z = 0\), with \(f_1 = h^2/KB\) and

\[
\begin{align*}
n_1 &= F' G' + G'' \\
n_6 &= -F' G'.
\end{align*}
\]

(IV-2)

(IV-3)

The surface conditions (IV-2) form a system of two linear, first order differential equations with constant coefficients. Note that

\[
D q = R_1, \quad D r = R_2
\]

where

\[
D = \begin{vmatrix}
n_1 \frac{\partial}{\partial y} & n_6 \frac{\partial}{\partial y} \\
n_1 \frac{\partial}{\partial x} & n_6 \frac{\partial}{\partial x}
\end{vmatrix}
\]

(8)
Formulation of a Three-Dimensional, Steady Ocean Circulation Model Using a Similarity Hypothesis

Moreover, since \( \frac{Dq}{Dr} = 0 \), these surface conditions admit solutions for \( q \) and \( r \) only if \( R_1 = R_2 = 0 \). We next investigate the restrictions imposed on the model by these requirements.

It is easily seen that \( R_1 = 0 \) and \( R_2 = 0 \) if, and only if,

\[
\frac{\partial}{\partial x} (h^2 \tau_x) + \frac{\partial}{\partial y} (h^2 \tau_y) = 0. \tag{IV-4}
\]

If the surface wind stress field is divergence free, as is the case commonly considered, we have the restriction

\[
\frac{\partial h^2}{\partial y} \bigg/ \frac{\partial h^2}{\partial x} = -\tau_x / \tau_y. \tag{IV-5}
\]

This simply states that in a horizontal plane isolines of \( h^2 \) are everywhere tangent to the surface stress vector.

Cross-differentiation of equation (IV-1) after division by \( f \) gives the vertically integrated vorticity equation,

\[
-gBN_2 \frac{\partial h^2}{\partial x} = \tau_x + \frac{f}{\beta} \left( \frac{\partial \tau_y}{\partial x} - \frac{\partial \tau_x}{\partial y} \right). \tag{IV-6}
\]

Since \( q \), \( B \) and \( N_2 \) are constants and since \( f \), \( \tau_x \), \( \tau_y \) and their derivatives are assumed specified, the foregoing equation can be integrated to obtain \( h \) to within a function of integration depending at most on \( y \). Depending upon the form of the surface wind stress, it may be possible to determine the \( y \)-dependence of this function of integration through the use of condition (IV-4) arising from the surface boundary condition. It should be emphasized, however, that there remains still another degree of arbitrariness in such a solution for \( h^2 \); the constant

\[
N_2 = \int_{-\infty}^{0} G m^2 \varepsilon \, dz
\]

is not known unless explicit forms for \( m(x, y) \) and \( G(mz) \) are specified or can be determined.

An alternate form of the vertically integrated vorticity equation is obtained by cross-differentiating (IV-1) directly:

\[
BN_2 \frac{\partial q}{\partial x} + B(1-N_0) \frac{\partial r}{\partial x} = -\frac{1}{\beta} \left( \frac{\partial \tau_y}{\partial x} - \frac{\partial \tau_x}{\partial y} \right). \tag{IV-7}
\]

With the second of the surface conditions \((N-2)\), \( \frac{\partial r}{\partial x} \) can be eliminated to obtain

\[
\left( \frac{n_1-N_0}{n_3-1-N_0} \right) \frac{\partial q}{\partial x} = \frac{1}{\beta B(1-N_0)} \left( \frac{\partial \tau_y}{\partial x} - \frac{\partial \tau_x}{\partial y} \right) - \frac{h^2}{Kn_3 B \tau_y}. \tag{IV-8}
\]

Letting \( I \) be defined as the ratio of the right hand side of this equation to \( \left( \frac{n_1-N_0}{n_3-1-N_0} \right) \), we may write

\[
q = \int_{x_0}^{x} I \, dx + M(x_0, y), \tag{IV-8}
\]

where \( M(x_0, y) \) is a function of \( y \) which may be determined by the lateral boundary conditions. Let \( x = x_0(y) \) define the eastern lateral boundary of the oceanic region under consideration. The velocity stream function should be uniform on this boundary. Since \( q = \phi \) on \( x = 0 \), and at this level \( \phi \) is the velocity stream function except for a factor of \( mF_1 \), it is reasonable to specify \( q = 0 \) along \( x = x_0 \).

(2) Application to a particular density distribution with mixed surface layer of uniform density

We consider now the case of a well-mixed surface layer which is horizontally uniform in density. Applying the present model to the density distribution assumed in a familiar study by Reid (1948), we shall establish a relation between our model and previously published approaches to the interior oceanic region.

Suppose the depth of the mixed layer is given by \( h(x, y) \). Let the density distribution be

\[
\rho = \left\{ \begin{array}{ll}
\rho_1, & z \geq -h(x, y) = -1/m \\
\rho_4 - \Delta \rho \, e^{1+my}, & z \leq -h(x, y)
\end{array} \right.
\]

where \( \Delta \rho = \rho_4 - \rho_1 \); and \( \rho_1 \) and \( \rho_4 \) represent surface and deep-water density values. (See Fig. 2.) This is the distribution used by Reid (1948) in his baroclinic model of an oceanic interior with steady, wind-driven currents. Comparing this density distribution with

\[
\rho = \rho_0 (1+S) = \rho_0 [1+AG+B(1-G)],
\]

we see that
This assumed form of the \( z \)-dependence of the density field enables us to evaluate the constants \( F_0, N_2, n_1, n_3, n_4, n_6 \) and \( Q_1 \) and partially determine the constants \( F_1, F_2 \) and \( N_0 \).

Using (III-7)

\[
F_0 = 0, \quad F_1 = -F_0 \frac{d \rho}{\rho_0} e^{i m z}, \quad |m z| \geq 1, \quad F_2 = -(1 - F) \frac{d \rho}{\rho_0} e^{i m z}, \quad |m z| \geq 1.
\]

Using (III-16) and (III-20),

\[
N_0 = \int_{-\infty}^{-1} F e^{i \xi} d\xi, \quad N_2 = \int_{-\infty}^{-1} e^{i \xi} d\xi + \int_{-1}^{0} \xi d\xi = -\frac{5}{2}.
\]

From (III-11)

\[
Q = \begin{cases} m z, & |m z| \leq 1, \\ 2 - e^{i m z}, & |m z| \geq 1, \end{cases}
\]

so

\[
Q_1 = \lim_{z \to -\infty} Q = 2.
\]

From (III-22) or (IV-3)

\[
n_1 = n_4 = 0, \quad n_3 = n_6 = 0.
\]

We will employ Reid's (1948) assumption of a purely zonal wind stress which varies sinusoidally with latitude

\[
\tau_x = \tau_0 + b \sin \omega (\phi - \phi_0), \quad \tau_y = 0,
\]

where \( \phi \) is latitude, \( \phi_0 \) is the latitude for which \( \tau_x = \tau_0 \), \( b \) is constant, and \( \omega \) (a constant) is the meridional angular wave number of the zonal wind stress. The surface wind stress vorticity is then

\[
\frac{\partial \tau_y}{\partial x} = \frac{\partial \tau_x}{\partial y} = -\frac{\omega b}{R} \cos \omega (\phi - \phi_0),
\]

where \( R \) is Earth's radius, so that \( dy = R d\phi \).

Integration of (N-6) gives the \( h^2 \) field as

\[
h^2 = \frac{2 \rho_0 (x - x_0)}{5 g d \rho} \left[ \tau_0 + b \sin \omega (\phi - \phi_0) - \frac{\omega b f}{R} \cos \omega (\phi - \phi_0) \right] + C_1(y),
\]

where the function of integration must be interpreted as \( [h(x, y)]^2 \), the square of the mixed-layer thickness evaluated at the eastern lateral boundary. This functional form is identical with that obtained by Reid (1948). The same is true for the horizontal gradients of vertically integrated pressure which, by examination of (N-1), are seen to be

\[
\frac{\partial P}{\partial x} = \frac{\partial}{\partial x} \int_{-\infty}^{0} p dz = -g \rho_0 B N_2 \frac{d h^2}{d x} = \frac{5}{2} g d \rho \frac{d h^2}{d x},
\]

\[
\frac{\partial P}{\partial y} = \frac{\partial}{\partial y} \int_{-\infty}^{0} p dz = \frac{5}{2} g d \rho \frac{d h^2}{d y}.
\]

Thus,

\[
P = \frac{5}{2} g d \rho h^2.
\]

(3) Vertically uniform densities in a two-layer system

A two-layer, baroclinic model provides a finite difference approximation to a density stratified prototype. Let \( \rho_d \), a uniform constant, represent the density of the lower, motionless layer. Let the density \( \rho_1 \) in the upper layer vary horizontally only. Let \( z = h(x, y) \) represent the position of the interlayer interface. Thus,

\[
\rho = \begin{cases} \rho_1 = \rho_0 [1 + S_1(x, y)], & z \geq -h(x, y) \\ \rho_d, & z \leq -h(x, y) \end{cases}
\]

In this case, we can write \( \rho = \rho_0 (1 + S) \) with \( S = A G + B (1 - G) \) provided that we choose:

\[
G(m z) = \begin{cases} 1, & z \geq -h \\ 0, & z \leq -h \end{cases},
\]

\[
A(q) = S_1(x, y) \frac{\rho_1 - \rho_0}{\rho_0},
\]

\[
B = \frac{\rho_d - \rho_0}{\rho_0}.
\]

We consider now the consequences of the velocity representation.
under the similarity hypothesis (III-2, 3) between $S$ and $\phi$.

Since within the lower layer $S=\text{constant}$, we have

\[ \dot{V} = 0, \quad z \leq -h. \]

This is just as required for a physically correct two-layer model in which the lower layer is very deep. Using equations (III-6, 7'), we have in the upper layer:

\begin{align*}
    u & = h^{-1} F'(q-r) \frac{\partial S_1}{\partial y} \\
    v & = -h^{-1} F'(q-r) \frac{\partial S_1}{\partial x} \\
    w & = -(1-F) \frac{\partial S_1}{\partial q} J\left(\frac{r, q}{x, y}\right) \\
    & \quad - z F'(q-r) \frac{\partial S_1}{\partial q} J\left(\frac{h^{-1}, q}{x, y}\right),
\end{align*}

since:

\[ F_0 = -(1-F) \frac{\partial S_1}{\partial q}, \quad F_1 = -F'(q-r) \frac{\partial S_1}{\partial q}, \quad F_2 = 0. \]

Since $n_1 = n_3 = 0$ and $n_2 = F''|_{z=0} = F_2 = 0$, the surface conditions expressed by (III-18) become

\begin{align*}
    Kh^{-2} (q-r) F''|_{z=0} \frac{\partial S_1}{\partial y} & = \tau_x \\
    Kh^{-2} (q-r) F''|_{z=0} \frac{\partial S_1}{\partial x} & = -\tau_y.
\end{align*}

As in the special case of uniform surface density discussed previously, these boundary conditions yield a restrictive relation. Namely, if equations (IV-11) are to have solutions for $q$ and $r$, then

\[ \tau_x / \tau_y = \left(\frac{\partial x}{\partial y}\right) S_1 \text{constant}. \]

Thus, as shown in Fig. 3, the surface boundary conditions imply that isolines of surface density must be everywhere tangent to the local wind stress vector.
Horizontal momentum neglected.

Under assumptions (1) and (2), equation (IV-14) yields

$$V \rho_1 = \frac{\rho_d - \rho_1}{h} V h,$$  \hspace{1cm} (IV-15)

the relation between the resulting interface slope and the horizontal density distribution after mixing. Applying (IV-15) to the first of (IV-13) gives within the upper layer $V h \rho = -g z V \rho_1$. Under the added assumptions (3) and (4) our model gives the geostrophic approximation in the upper layer,

$$iu + jv = \frac{1}{\rho_0 f} [k \times V h \rho], \hspace{1cm} (IV-16)$$

Moreover, vertical integration gives the volume transports per unit width as

$$\int_{-h}^{0} u \, dz = -\frac{g h^2}{2} \frac{\partial \rho_1}{\partial y}, \hspace{1cm} (IV-17)$$

These results (IV-15 thru 17) are just those of Freeman—the only exception is that $\rho_0$ appears in place of $\rho_1$ in the denominator of equations (IV-16) and (IV-17) due to our linearization of (II-8).

The density model of Yoshida

Yoshida (1965) considered a vertical distribution of density which is somewhat more realistic than those previously considered in this paper. He took the form

$$\rho(x, y, z) = \rho_d - \Delta \rho (1 - mz) e^{mx}, \hspace{1cm} (IV-18)$$

with:

- $\rho_d$, the deep-water density, a uniform constant,
- $\rho_1$, the surface density, dependent only on $y$,
- $\Delta \rho$, the overall vertical density contrast ($\rho_d - \rho_1$), and
- $m$, the effective reciprocal thermocline depth, dependent on $x$ and $y$.

This form provides a realistic vertical density profile for an incompressible ocean— see Fig. 4. Vertical profile of density considered by Yoshida (1965, Fig. 1).

V. Summary

We have attempted to present a rather general method of analysis for steady three-dimensional circulation in the interior of a stratified ocean under the assumption of non-divergent flow which is everywhere tangent equiscalar surfaces of potential density. Under such conditions the flow can be represented as the vector-product of the gradients of two scalar stream functions, one of which is essentially the potential density.
A similarity hypothesis is made in respect to the form of the stream functions. Using this hypothesis, the various physical boundary conditions are then recast in the form of constraints imposed on the parameters and functions introduced in the similarity hypothesis. Specifically, the model calls for the flow, internal shear stress and horizontal pressure gradient to vanish at great depth. Moreover, the surface flow is constrained to be parallel to the free surface, and the velocity gradient at the free surface is related to the wind stress through the usual Boussinesq representation.

The system of equations and conditions to be satisfied by the unknown functions in the similarity representation are highly nonlinear. We have not been able to solve for explicit forms of the $F$ and $G$ functions which satisfy all of the conditions imposed, therefore we have not achieved a general solution for the density structure and associated currents. However, we have shown that the model is sufficiently general to include several published density models as special cases. In these previously published models specific forms have been assumed for the functions $F$ and $G$. It is unlikely that solutions based on these specific forms will satisfy all of the constraints imposed on the general, wind-driven circulation.

We hope that the results obtained and the problems encountered may provide some insight into the physical restrictions imposed by the introduction of similarity hypotheses into the general circulation problem.

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APPENDIX

List of symbols

$A, B$ Dimensionless functions of $q$ and $r$, respectively, which express the depth-independent part of the horizontal dependence of $S$ as shown in (III-2).

$b \text{cm}^2 \text{sec}^{-2}$ Amplitude of meridional variation in zonal wind stress assumed in section IV-2.

$c \text{cm}^2 \text{sec}^{-2}$ Function of $x$ and $y$ defined by (III-9); approximation to pressure at $z=0$, normalized by density.

$C_1 \text{cm}^2$ Function of integration equal $h(x_0)^2$ in section IV-2.

$D \text{cm}^2$ The operator $n_1 n_3 \left( \frac{\partial^2}{\partial x \partial y} - \frac{\partial^2}{\partial y \partial x} \right)$.

$F$ Function of $\zeta$ which carries dependence of $\phi$ as displayed in equation (III-3).

$F_0, F_1, F_2$ Dimensionless functions defined by (III-7).

$f \text{sec}^{-1}$ Coriolis parameter $2\Omega \sin \phi$.

$f_1 \text{sec}^{-1}$ The quantity $h^2/(KB)$.

$G$ Function of $\zeta$ which carries depth dependence of $S$ as displayed in equation (III-2).

$g \text{cm sec}^{-2}$ Local gravity.

$h \text{cm}$ Vertical scale factor which varies generally with $x$ and $y$.

$I \text{cm}^2 \text{sec}^{-1}$ Function defined equal $\frac{\partial q}{\partial x}$ by the first displayed equation preceeding (IV-8).

$i, j, k$ Unit vectors in $x$, $y$ and $z$ directions, respectively.

$J$ The Jacobian operator as defined by the first displayed equation following (III-7).

$K \text{cm}^2 \text{sec}^{-1}$ Kinematic coefficient of vertical eddy viscosity.

$M \text{cm}^3 \text{sec}^{-1}$ Function of integration depending on $y$ only and defined by (IV-8).

$m \text{cm}^{-1}$ Reciprocal scale factor, $1/h$.

$N_0$ Dimensionless constant defined by (III-16).

$N_2$ Dimensionless constant defined by (III-20).

$n_1, n_2, n_3$ Dimensionless constants defined by (III-19).

$n_4, n_5, n_6$ Dimensionless constants defined by (III-22).
$P \, \text{dynes cm}^{-1}$ Vertically integrated pressure used in section IV-2.

$p \, \text{dynes cm}^{-2}$ Pressure.

$p_a \, \text{dynes cm}^{-2}$ Sea-level atmospheric pressure, assumed uniform.

$Q$ Dimensionless vertical integral of $G(\zeta)$ defined by (III-11).

$Q_1$ $\lim_{z \to -\infty} Q$.

$q, r \, \text{cm}^3 \text{sec}^{-1}$ Functions of $x$ and $y$ which express through (III-3) the depth-independent horizontal variation of $\psi$.

$R \, \text{cm}$ Earth's mean radius.

$R_1, R_2 \, \text{cm sec}^{-1}$ The quantities defined by the first displayed equation proceeding equation (IV-4).

$S$ Fractional departure of $\rho$ from $\rho_0$ given by $(\rho - \rho_0)/\rho_0$.

$S_1$ Dimensionless density contrast defined by (IV-9) as $(\rho_1 - \rho_0)/\rho_0$.

$u, v, w \, \text{cm sec}^{-1}$ Components of fluid velocity in $x$, $y$, and $z$-directions, respectively.

$\dot{V} \, \text{cm sec}^{-1}$ Vector fluid velocity.

$x, y, z \, \text{cm}$ Cartesian space coordinates, increasing eastward, northward and up, respectively.

$x_0 \, \text{cm}$ Position of eastern lateral, basin boundary; function of $y$ only.

$\beta \, \text{cm}^{-1} \text{sec}^{-1}$ The derivative of $f$ with respect to $y$.

$\Gamma_1, \Gamma_2$ Arbitrary scalar functions chosen such that $\dot{V} = \Gamma_1 \times \Gamma_2$; product $\Gamma_1 \Gamma_2$ has units cm$^3$ sec$^{-1}$.

$\Delta \rho \, \text{gm cm}^{-3}$ Density contrast $\rho_4 - \rho_1$.

$\zeta \, \text{cm}$ Dimensionless depth defined as $mz$.

$\eta \, \text{cm}$ Directed vertical displacement of sea surface from mean sea level $z=0$.

$\Theta \, \text{cm}^3 \text{sec}^{-1}$ Volume transport stream function defined by (III-15).

$\Theta_0 \, \text{cm}^3 \text{sec}^{-1}$ Value of $\Theta$ at $r=q=0$.

$\rho \, \text{gm cm}^{-3}$ Fluid density.

$\rho_0 \, \text{gm cm}^{-3}$ A constant, characteristic value of density for the model.

$\rho_1 \, \text{gm cm}^{-3}$ Value of $\rho$ at $z=0$, generally function of $x$ and $y$.

$\rho_4 \, \text{gm cm}^{-3}$ Limiting value of $\rho$ as $z \to -\infty$, a constant.

$\sigma_i \, \text{gm cm}^{-3}$ A measure of density in an incompressible fluid defined by $\sigma_i = [\rho(35\%, 0^\circ C, \rho) - 1] \times 10^3$.

$\tau \, \text{cm}^2 \text{sec}^{-2}$ Vector form mean surface wind stress expressed kinematically.

$\tau_0 \, \text{cm}^2 \text{sec}^{-2}$ Value of $\tau_x$ at latitude $\phi_0$.

$\tau_x, \tau_y \, \text{cm}^2 \text{sec}^{-2}$ The $x$- and $y$-directed components of mean surface wind stress expressed kinematically.

$\tau_{xh}, \tau_{yh} \, \text{cm}^2 \text{sec}^{-2}$ The $x$- and $y$-directed components of kinematically expressed vertical shear stress evaluated at level $z=-h$.

$\phi \, \text{Latitude}$.

$\phi_0 \, \text{A specific value of latitude}$.

$\phi_i \, \text{cm}^3 \text{sec}^{-1}$ Scalar function defined such that $\dot{V} = \Theta \times \Phi \phi_i$.

$\omega$ Constant angular wave number of zonal wind stress used in section IV-2.

References


相似仮説に基づく，定常な3次元の海洋循環モデルの組立て

Worth D. Nowlin, Jr. and Robert O. Reid

このモデルでは，密度が深さと共に連続的に変化する大洋で，β平面近似の下では，大洋内部領域での定常的な風成海流をあらわすためにつくられたものである．非圧縮性流体中の定常流という仮定の下で，速度ベクトルはあるベクトルポテンシャルから導かれる．更に，速度を密度勾配と未知のポテンシャル函数の勾配とのベクトル積で表現する．この表現によると，連続条件及び（混合がなければ）流線にそってポテンシャル密度が保存されるというエネルギー条件が自動的に満足される．ポテンシャル密度及びポテンシャル函数の水平変化の深度依存性について相似仮説を導入し，その結果を吟味する．無限に深い大洋中での傾圧性の運動に適合するような境界条件を入れる．こうして出てくる数学的問題は，強い非線型性を示し，われわれは3次元的な速度分布について解を得ることには成功しなかった．しかしながら，現在までに得られた結果及び問題点を述べることは，海洋大循環理論に上述の相似仮説を導入するときに課される物理的制約についていくつかの知見を提供するであろう．

相似仮説はかなり一般的なもので，今までに発表されている種々の密度分布モデルを特殊例として含んでいた．具体例として，今までに出された二つの密度分布による力学的な結論が，現在の理論の特殊例として導かれている．