On the Influence of Bottom Topography on Ocean Currents.

By

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Abstract: Topographic effect on currents has been discussed by the method of small perturbation. An equation for the vertical motion is derived which includes a wide range of phenomena, and it is shown that the character of the deflection of currents by the bottom deformation is complicated since the different kinds of waves become dominant according to the scale of topography, and, moreover, since the horizontal shear of a current takes part in the phenomena. When the scale of the bottom deformation is very small, the stationary internal gravity waves prevail under a certain condition and when the scale is large, planetary waves develop in a horizontal plane.

1. Introduction

It is a well known fact that the distribution of density is considerably modified when an ocean current flows across a submarine ridge. Theoretical attacks to this problem, however, are very scarce owing to difficulties of mathematical analyses. In his very extensive studies on the character of ocean currents in homogeneous water, V. W. Ekman (1923) first examined the effect of bottom topography on slope currents, the velocity of which is constant with depth except in a layer influenced by the bottom friction. His conclusions are: The currents generally tend to follow the bottom contours in higher latitudes and in the northern hemisphere, they deviate to the right, passing over an elevation and to the left, over depressions of the bottom, and vice versa in the southern hemisphere. He extended his analysis later (1932) on the basis of hydrodynamic equations including inertial terms as well as frictional terms, but the major features were almost as before. Similar studies of the problem have been made by many meteorologists concerning to an air flow over a mountain. P. Queney (1947, 1948) has developed an extensive study of perturbations of air flow and discussed the mountain effect, taking the vertical stability of the atmosphere into account. Recently, the influence of a large scale mountain range on westerlies has been discussed by many authors (J. Charney and A. Eliassen, 1949, D. Colson, 1949, B. Bolin, 1950, etc.) on the basis of the vorticity equation. In these studies, frictional effects are neglected and the general flow

is assumed to be zonal and uniform. According to the vorticity equation in a barotropic atmosphere, the horizontal divergence or convergence due to the effect of topography results in an increase or decrease of vorticity, respectively, and the deflection of stream lines can be expected on the downstream side which is followed by the planetary waves of Rossby type.

On the other hand, the effect of bottom topography on relative current in non-homogeneous water has been examined qualitatively by H. U. Sverdrup (1941). The conclusion of his analysis is that the observed deflections of currents, which seem to be maximum at the summit of the bottom ridge, may not be caused by bottom friction as was assumed by Ekman, but they are caused by modifications of the density distribution due to the effect of topography. In his theory, the assumption that the isopycnic surface must rise when a water-mass approaches a ridge, includes essentially the existence of a vertical velocity as well as a horizontal velocity near the bottom, and this is inconsistent with the "law of the parallel solenoids" which is the fundamental concept in his discussion.

In these circumstances, V. B. Stockman (1947, 1948) reviewed the former studies and discussed the topographic effect, using the concepts of mass transport and lateral frictions. He emphasized the primary importance of the distribution of wind system over the ocean which is the main cause of ocean currents, and concluded that the deflection of currents by the bottom topography may not be related to the Coriolis force, but is connected with the vorticity in the wind-field, such that when a current crosses a sea-bottom elevation, the flow-lines deviate over its summit in the same direction as the vorticity in the wind field and just the opposite above the central part of a depression. Though his conclusions are very suggestive, his deduction of results seems to be very doubtful in respect to the procedure of simplifying the equations of motion, by means of taking mean values of various quantities.

At any rate, it is a usual experience that the currents, flowing close to the coast or flowing slowly, are apt to follow the bottom contours. It is, however, not so exactly ascertained that in which direction they will turn when ocean currents pass over an elevation or depression of the bottom. Therefore, in the present paper, the effect of bottom topography on ocean currents has been re-examined on the basis of the hydrodynamic equations including inertial terms, taking the density distribution into consideration as far as possible.

2. Basic Equations of Motion.

We assume that the field is stationary and may be divided into two parts, of which one is the basic field and the other is the field of small perturbations caused by irregularities of the bottom. In spite of the defects involved in the approximation of small perturbations, the qualitative features of a flow pattern may be estimated. We write the pressure, $p = p_0 + p_1$, the density $\rho = \rho_0 + \rho_1$, the horizontal components of velocity $u = u_0 + u_1$, $v = v_1$ and the vertical component of velocity $w = w_1$, where subscripts 0 and 1 refer to the quantities in the basic state and the disturbed state, respectively.

The basic current is assumed geostrophic and zonal defined by the following equations:

$$\frac{1}{\rho_0} \frac{\partial p_0}{\partial y} = -\lambda u_0, \quad (1)$$

and

$$\frac{1}{\rho_0} \frac{\partial p_0}{\partial z} = -g, \quad (2)$$

where the co-ordinate x , y and z are directed to the east, to the north and vertically upward, respectively, with the origin situated at the bottom, and $\lambda (= 2\omega \sin \varphi)$ is the Coriolis parameter and g is the acceleration due to gravity.

The equations of motion for stationary perturbations are:

$$u_0 \frac{\partial u_1}{\partial x} + v_1 \frac{\partial u_0}{\partial y} + w_1 \frac{\partial u_0}{\partial z} = -\frac{1}{\rho_0} \frac{\partial p_1}{\partial x} + \lambda v_1, \quad (3)$$

$$u_0 \frac{\partial v_1}{\partial x} = -\frac{1}{\rho_0} \frac{\partial p_1}{\partial y} - \frac{\rho_1}{\rho_0} \lambda u_0 - \lambda u_1, \quad (4)$$

and

$$u_0 \frac{\partial w_1}{\partial x} = -\frac{1}{\rho_0} \frac{\partial p_1}{\partial z} - \frac{\rho_1}{\rho_0} g, \quad (5)$$

provided that the frictional and other small terms be neglected and that the basic current u_0 is assumed to be independent of x .

When the fluid is incompressible and individual density is conservative, we have:

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} + \frac{\partial w_1}{\partial z} = 0, \quad (6)$$

and

$$u_0 \frac{\partial \rho'}{\partial x} + v_1 \frac{\partial \rho_0}{\partial y} + w_1 \frac{\partial \rho_0}{\partial z} = 0. \quad (7)$$

Cross differentiating (3) and (4) with respect to x and y , and taking (6) and (7) into account, we obtain:

$$\left. \begin{aligned} & \frac{\partial^2 v_1}{\partial x^2} + \frac{\partial^2 v_1}{\partial y^2} + \left(\frac{1}{\rho_0} \frac{\partial \rho_0}{\partial y} \right) \frac{\partial v_1}{\partial y} + \left\{ \frac{1}{u_0} \frac{\partial}{\partial y} \left(\lambda - \frac{\partial u_0}{\partial y} \right) - \frac{1}{u_0} \frac{\partial u_0}{\partial y} \cdot \frac{1}{\rho_0} \frac{\partial \rho_0}{\partial y} \right\} v_1 \\ & = - \frac{\partial^2 w_1}{\partial y \partial z} + \left(\frac{1}{u_0} \frac{\partial u_0}{\partial z} \right) \frac{\partial w_1}{\partial y} + \left\{ \frac{1}{u_0} \left(\lambda - \frac{\partial u_0}{\partial y} \right) - \frac{1}{\rho_0} \frac{\partial \rho_0}{\partial y} \right\} \frac{\partial w_1}{\partial z} \\ & \quad + \left(\frac{1}{u_0} \cdot \frac{\partial^2 u_0}{\partial y \partial z} + \frac{1}{u_0} \frac{\partial u_0}{\partial z} \cdot \frac{1}{\rho_0} \frac{\partial \rho_0}{\partial y} + \frac{\lambda}{u_0} \cdot \frac{1}{\rho_0} \frac{\partial \rho_0}{\partial z} \right) w_1, \end{aligned} \right\} \quad (8)$$

and from (3) and (5), we have:

$$\left. \begin{aligned} & \frac{\partial^2 w_1}{\partial x^2} + \frac{\partial^2 w_1}{\partial z^2} + \left(\frac{1}{\rho_0} \frac{\partial \rho_0}{\partial z} \right) \frac{\partial w_1}{\partial z} - \left\{ \left(\frac{1}{u_0} \frac{\partial u_0}{\partial z} + \frac{g}{u_0^2} \right) \frac{1}{\rho_0} \frac{\partial \rho_0}{\partial z} + \frac{1}{u_0} \frac{\partial^2 u_0}{\partial z^2} \right\} w_1 \\ & = - \frac{\partial^2 v_1}{\partial y \partial z} - \left(\frac{1}{u_0} \frac{\partial u_0}{\partial z} + \frac{1}{\rho_0} \frac{\partial \rho_0}{\partial z} \right) \frac{\partial v_1}{\partial y} - \frac{1}{u_0} \left(\lambda - \frac{\partial u_0}{\partial y} \right) \frac{\partial v_1}{\partial z} \\ & \quad - \left\{ \frac{1}{u_0} \left(\lambda - \frac{\partial u_0}{\partial y} \right) \frac{1}{\rho_0} \frac{\partial \rho_0}{\partial z} - \frac{1}{u_0} \cdot \frac{\partial^2 u_0}{\partial y \partial z} - \frac{g}{u_0^2} \cdot \frac{1}{\rho_0} \frac{\partial \rho_0}{\partial y} \right\} v_1. \end{aligned} \right\} \quad (9)$$

Eqs. (8) and (9) may be simplified when adequate assumptions are introduced.

When the horizontal gradient of density is very small and the basic current may be considered to be almost uniform*, we have, from (8), neglecting the variation of velocities in the y -direction,

$$\frac{\partial^2 v_1}{\partial x^2} + \frac{1}{U} \frac{\partial \lambda}{\partial y} v_1 = \frac{\lambda}{U} \left(\frac{\partial w_1}{\partial z} + \frac{w_1}{\rho_0} \frac{\partial \rho_0}{\partial z} \right), \quad (10)$$

and from (9),

$$\frac{\partial^2 w_1}{\partial x^2} + \frac{\partial^2 w_1}{\partial z^2} + \left(\frac{1}{\rho_0} \frac{\partial \rho_0}{\partial z} \right) \frac{\partial w_1}{\partial z} - \frac{g}{U^2} \left(\frac{1}{\rho_0} \frac{\partial \rho_0}{\partial z} \right) w_1 = - \frac{\lambda}{U} \left(\frac{\partial v_1}{\partial z} + \frac{1}{\rho_0} \frac{\partial \rho_0}{\partial z} \cdot v_1 \right), \quad (11)$$

where U is the uniform basic velocity.

Here we shall transform v_1 and w_1 into v' and w' by the following equations:

$$v_1 = \int_{-\infty}^{\infty} v' e^{ikx} dk, \quad \text{and} \quad w_1 = \int_{-\infty}^{\infty} w' e^{ikx} dk. \quad (12)$$

* From equations (1) and (2), we have,

$$u_0 = U + \frac{g}{\lambda \rho_0} \int_0^z \frac{\partial \rho_0}{\partial y} dz, \quad \text{and} \quad U = - \frac{1}{\lambda \rho_0} \frac{\partial P_{0B}}{\partial y},$$

where P_{0B} is the bottom pressure. The assumption of almost uniform basic current is satisfied when the condition, $\left| \frac{\partial P_{0B}}{\partial y} \right| \gg \left| g \int_0^z \frac{\partial \rho_0}{\partial y} dz \right|$, is fulfilled, in other words, when

the baroclinic effect is neglected. Since the relative currents prevail in the ocean, this assumption can not be accepted. Our discussion, however, will be valid if the effect of the vertical shear of velocity does not play a dominant role. Thus, U may be given, regardless of (1) and (2), in our discussion.

Writting $\frac{1}{\rho_0} \frac{\partial \rho_0}{\partial z} = -s$ and $\frac{\partial \lambda}{\partial y} = \beta$, both of which are assumed to be constant, we have from (10) and (11), under the condition $\left| \frac{1}{w_1} \frac{\partial w_1}{\partial z} \right| \gg \left| \frac{1}{\rho_0} \frac{\partial \rho_0}{\partial z} \right|$:

$$v' = \frac{k_\lambda}{k_\beta^2 - k^2} \cdot \frac{\partial w'}{\partial z}, \quad (13)$$

and

$$\frac{\partial^2 w'}{\partial z^2} + \frac{(k_s^2 - k^2)(k_\beta^2 - k^2)}{(k_\lambda^2 - k^2)} \cdot w' = 0, \quad (14)$$

where $k_\beta^2 = \frac{\beta}{U}$, $k_\lambda^2 = \frac{\lambda^2}{U^2}$, $k_s^2 = \frac{g_s}{U^2}$, and the term containing $\frac{\partial w'}{\partial z}$ is neglected, since s , the coefficient of $\frac{\partial w'}{\partial z}$, is sufficiently small. The orders of magnitude of various parameters are tabulated in Table 1. Here, we assume k_β and k_s to be positive, since we are chiefly concerned with steady and stable perturbations.

Table 1. Order of magnitude of various parameters.

U (cm/sec)	k_s^2 (cm ⁻²)		k_λ^2 (cm ⁻²)	k_β^2 (cm ⁻²)
	$s=10^{-8}$	$s=10^{-6}$	$\lambda=10^{-4}$	$\beta=10^{-13}$
1	10 ⁻⁵	10 ⁻³	10 ⁻⁸	10 ⁻¹³
10	10 ⁻⁷	10 ⁻⁵	10 ⁻¹⁰	10 ⁻¹⁴
100	10 ⁻⁹	10 ⁻⁷	10 ⁻¹²	10 ⁻¹⁵

In (14), we can see that when $k \gg k_\beta$, k_β^2 may be neglected and the equation will hold for any direction of a basic current.

3. Solution of the equations.

A formal solution of (14) is,

$$w' = A e^{ipz} + B e^{-ipz}, \quad (15)$$

where $p^2 = (k_s^2 - k^2)(k_\beta^2 - k^2)/(k_\lambda^2 - k^2)$ and A and B are constants of integration. Since p^2 is positive when $k_s > k > k_\lambda$ or $k < k_\beta$ and negative when $k > k_s$ or $k_\lambda > k > k_\beta$, p is real or pure imaginary according to the value of k . Substituting (15) into (12), we have,

$$w_1 = \int_{-\infty}^{\infty} (A e^{ipz} + B e^{-ipz}) e^{ikx} dk \quad (16)$$

Here, boundary conditions for the vertical velocity w_1 are given by

$$w_1 = U \frac{\partial h}{\partial x}, \quad \text{at the bottom,} \quad (17)$$

and

$$w_1=0, \quad \text{at the surface,}^* \quad (18)$$

where h is the height of the bottom deformation which is assumed to be independent of y corresponding to the assumption that the velocity variation in y -direction is negligible.

In the case when the bottom deformation is expressed by $h=h_0 \cos kx$ and the mean depth is H , we have from (16), (17) and (18), as a particular solution:

$$w_1 = -\frac{Ukh_0}{\sin pH} \sin p(H-z) \sin kx; \quad (19)$$

from (10) and (19),

$$v_1 = \frac{Ukh_0}{\sin pH} \cdot \frac{k_\lambda}{k_\beta^2 - k^2} \cdot p \cos p(H-z) \sin kx; \quad (20)$$

from (6), (19) and (20),

$$u_1 = \frac{Uh_0 p}{\sin pH} \cos p(H-z) \cos kx, \quad (21)$$

and from (7), (19), (20) and (21),

$$\rho' = \left(s \cdot \sin p(H-z) - \frac{tk_\lambda p}{k_\beta^2 - k^2} \cos p(H-z) \right) h_0 \cdot \frac{\cos kx}{\sin pH}, \quad (22)$$

where $t = \frac{1}{\rho_0} \frac{\partial \rho_0}{\partial y}$. In equations (19) to (22), p should be understood as:

$$p = \begin{cases} i\sqrt{k^2 - k_s^2}, & \text{for } k > k_s, \end{cases} \quad (23)$$

$$\sqrt{\frac{k_s^2 - k^2}{k^2 - k_\lambda^2}} \cdot k, \quad \text{for } k_s > k > k_\lambda, \quad (24)$$

$$p = \begin{cases} ik_s \sqrt{\frac{k^2 - k_\beta^2}{k_\lambda^2 - k^2}}, & \text{for } k_\lambda > k > k_\beta, \end{cases} \quad (25)$$

$$\frac{k_s}{k_\lambda} \sqrt{k_\beta^2 - k^2}, \quad \text{for } k < k_\beta. \quad (26)$$

Thus we can easily verify that the wave length in the horizontal direction of the disturbances are the same as that of the bottom configuration, and when $k > k_s$, that is, when the bottom deformation is of very small scale, the amplitude of v_1 is very small compared with those of u_1 and w_1 , and the disturbances may be considered to be confined to the x - z plane. When $k_s > k > k_\lambda$, the amplitude of v_1 is still small compared with that of u_1 , and the variation of velocities in the vertical direction is periodic if the condition, $k_s > \frac{\pi}{H_0}$, is satisfied. When $k_\lambda > k > k_\beta$, that is, when bottom deformation is large, the amplitude of v_1 is large compared with that of w_1 and the disturbances may be considered to be almost

* Strictly speaking, small deformation of the free surface is expected, as in the case of usual internal waves. This effect, however, will be very small and may be neglected in this discussion.

horizontal, and when $k < k_\beta$, which is the case of a very large scale deformation, $w_1 \ll v_1$ and the disturbances are almost horizontal so far as the depth of water, H , is not so great.

These solutions coincide with those obtained by Queney (1947) if the depth H tends to infinity, and k_β tends to zero.

In general, when the condition $|pH| \ll 1$ is fulfilled, (that is, $k < 10^{-7} \text{ cm}^{-1}$ for $H \sim 10^5 \text{ cm}$, $k_s \sim 10^{-4} \text{ cm}^{-1}$, $k_\lambda \sim 10^{-5} \text{ cm}^{-1}$ and $k_\beta \sim 10^{-7} \text{ cm}^{-1}$) the disturbances may be simply written as:

$$u_1 = \frac{U h_0}{H} \cos kx, \quad (27)$$

$$v_1 = \frac{k k_\lambda}{k_\beta^2 - k^2} \cdot \frac{U h_0}{H} \cdot k \sin kx, \quad (28)$$

$$w_1 = -\frac{H-z}{H} \cdot U k h_0 \sin kx, \quad (29)$$

and

$$\rho' = \frac{H-z}{H} \cdot s h_0 \cos kx. \quad (30)$$

These solutions indicate that u_1 and v_1 may be considered as uniform in the vertical direction if the deformation of large scale is concerned. The above solutions can also be derived from (10) assuming that the water is homogeneous, u_1 and v_1 are independent of z , and w_1 is a linear function of z , satisfying the boundary conditions (17) and (18). In (30), the condition $\frac{s}{t} \gg \frac{k_\lambda (H-z)}{k_\beta^2 - k^2} \sim 10^{-4}$ is assumed, and the density distribution is just the same as suggested by Sverdrup. The effect of horizontal variation of density, however, is ineffective to the general pattern of the current, because we have assumed that the basic current is almost independent of the baroclinity of water. In this respect, we cannot discuss the influence of the density modification on the current pattern, which was the main part of Sverdrup's theory. From (20), it is found that the stream lines deviate to the left near the summit of the bottom deformation in the northern hemisphere if $k > k_\beta$ and just the opposite if $k < k_\beta$. This is illustrated in Fig. 1.

Fig. 1. Deflection of stream line by a sinusoidal bottom deformation in the case of $U = \text{const.}$

4. Perturbations caused by the parabolic deformation of the bottom

Let us now investigate the case when bottom deformation is such that

$$h=0, \quad \text{for } |x| > a,$$

and

$$h=h_0(a^2-x^2), \quad \text{for } |x| < a.$$

We may write h by the Fourier integral as follows:

$$h = \frac{2h_0}{\pi} \int_{-\infty}^{\infty} \frac{1}{k^3} (\sin ka - ka \cos ka) \cos kx \cdot dk. \quad (31)$$

Then, from (16), (17) and (18), we have:

$$w_1 = -\frac{2h_0U}{\pi} \int_{-\infty}^{\infty} \frac{\sin p(H-z)}{\sin pH} \cdot \frac{(\sin ka - ka \cos ka)}{k^2} \cdot \sin kx \cdot dk. \quad (32)$$

Again, from (10) and (12), we have:

$$v_1 = \frac{2h_0U}{\pi} \int_{-\infty}^{\infty} \frac{k_\lambda}{k_\beta^2 - k^2} \cdot \frac{p \cos p(H-z)}{\sin pH} \cdot \frac{(\sin ka - ka \cos ka)}{k^2} \cdot \sin kx \cdot dk. \quad (33)$$

Both of these integrals can be evaluated by the method of contour integration in the complex plane.

Here we make the following transformations:

$$\frac{\sin p(H-z)}{\sin pH} = \frac{2\pi \sum_{n=1}^{\infty} \frac{n \sin \frac{\pi}{H} nz}{\left(n \frac{\pi}{H}\right)^2 - p^2}}, \quad (34)$$

and

$$\frac{p \cos p(H-z)}{\sin pH} = -\frac{2\pi^2 \sum_{n=1}^{\infty} \frac{n^2 \cos \frac{\pi}{H} nz}{\left(n \frac{\pi}{H}\right)^2 - p^2}}, \quad (35)$$

where n is a positive integer. We may write approximately under the conditions $k_s^2 \gg k_\lambda^2 \gg k_\beta^2$:

$$\left(n \frac{\pi}{H}\right)^2 = p^2 = \left\{ k^2 \begin{pmatrix} -\alpha_n^2 \\ +\beta_n^2 \end{pmatrix} \right\} \left\{ k^2 \begin{pmatrix} +\gamma_n^2 \\ -\delta_n^2 \end{pmatrix} \right\} / k^2 - k_\lambda^2, \quad (36)$$

where

$$\alpha_n = \sqrt{k_s^2 - \left(n \frac{\pi}{H}\right)^2}, \quad \text{and} \quad \gamma_n = \sqrt{\frac{\left(k_\lambda \frac{n\pi}{H}\right)^2 - k_s^2 k_\beta^2}{k_s^2 - \left(n \frac{\pi}{H}\right)^2}}, \quad \text{for } n < \frac{k_s H}{\pi},$$

and

$$\beta_n = \sqrt{\left(n \frac{\pi}{H}\right)^2 - k_s^2}, \quad \text{and} \quad \delta_n = \sqrt{\frac{\left(k_\lambda \frac{n\pi}{H}\right)^2 - k_s^2 k_\beta^2}{\left(n \frac{\pi}{H}\right)^2 - k_s^2}}, \quad \text{for } n > \frac{k_s H}{\pi}.$$

In general, the conditions $\alpha_n \gg \gamma_n$ and $\beta_n \gg \delta_n$ are satisfied, and all these quantities are real.

Furthermore, we rewrite the terms of sine and cosine in exponential forms such as

$$\left. \begin{aligned} & (\sin ka - ka \cos ka) \sin kx \\ &= -\frac{1}{4} [\{e^{ik(a+x)} - e^{ik(a-x)} - aki(e^{ik(a+x)} - e^{ik(a-x)})\} \\ & \quad + \{e^{-ik(a+x)} - e^{-ik(a-x)} + aki(e^{-ik(a+x)} - e^{-ik(a-x)})\}], \quad \text{for } |x| < a, \end{aligned} \right\} \quad (37)$$

and

$$\left. \begin{aligned} &= -\frac{1}{4} [\{e^{ik(x+a)} - e^{ik(x-a)} - aki(e^{ik(x+a)} + e^{ik(x-a)})\} \\ & \quad + \{e^{-ik(x+a)} - e^{-ik(x-a)} + aki(e^{-ik(x+a)} + e^{-ik(x-a)})\}], \quad \text{for } |x| > a. \end{aligned} \right\} \quad (38)$$

Thus, we must evaluate these integrals separately in each region, taking a suitable path of integration. The path may be taken from $-R$ to R , excluding poles, on the real axis and upper or lower semi-circle of radius R , so as to make the integral on the semi-circle to vanish when $R \rightarrow \infty$. If we take the principal values in Cauchy's sense, and substitute the conditions that $v' = 0$ and $w' = 0$ at $x = -\infty$, definite integral values can be obtained. (See Lamb, H.: Hydrodynamics. 6th Ed. P. 408)

The results are:

For $x < -a$,

$$\left. \begin{aligned} v_1 = & -2h_0 U \left\{ \sum_{n=1}^{n=n_0} P_{\gamma n} e^{\gamma_n x} (\sinh \gamma_n a - a \gamma_n \cosh \gamma_n a) \right. \\ & \left. + \sum_{n=n_0+1}^{n=\infty} P_{\beta n} e^{\beta_n x} (\sinh \beta_n a - a \beta_n \cosh \beta_n a) \right\}, \end{aligned} \right\} \quad (39)$$

and

$$\left. \begin{aligned} w_1 = & -2h_0 U \left\{ \sum_{n=1}^{n=n_0} C_n e^{\gamma_n x} (\sinh \gamma_n a - a \gamma_n \cosh \gamma_n a) \right. \\ & \left. + \sum_{n=n_0+1}^{n=\infty} B_n e^{\beta_n x} (\sinh \beta_n a - a \beta_n \cosh \beta_n a) \right\}. \end{aligned} \right\} \quad (40)$$

For $-a < x < a$,

$$\left. \begin{aligned} v_1 = & 2h_0 U [P_0 x - P_1 \{\sin k_\beta (a+x) - a k_\beta \cos k_\beta (a+x)\} \\ & + \sum_{n=1}^{n=n_0} \{P_{\gamma n} (1 + a \gamma_n) e^{-\gamma_n a} \sinh \gamma_n x \\ & \quad - P_{\alpha n} (\sinh \alpha_n (a+x) - a \alpha_n \cos \alpha_n (a+x))\} \\ & + \sum_{n=n_0+1}^{n=\infty} \{P_{\beta n} (1 + a \beta_n) e^{-\beta_n a} \sinh \beta_n x \\ & \quad - P_{\delta n} (\sinh \delta_n (a+x) - a \delta_n \cos \delta_n (a+x))\}], \end{aligned} \right\} \quad (41)$$

and

$$\begin{aligned}
 w_1 = 2h_0 U [& P_2 x + \sum_{n=1}^{n=n_0} \{ C_n (1 + a\gamma_n) e^{-\gamma_n a} \sinh \gamma_n x \\
 & - A_n (\sin \alpha_n (a+x) a \alpha_n \cos \alpha_n (a+x)) \} \\
 & + \sum_{n=n_0+1}^{n=\infty} \{ B_n (1 + a\beta_n) e^{-\beta_n a} \sinh \beta_n x \\
 & - D_n (\sin \delta_n (a+x) - a \delta_n \cos \delta_n (a+x)) \}] .
 \end{aligned} \tag{42}$$

For $x > a$,

$$\begin{aligned}
 v_1 = 2h_0 U [& -2P_1 \cdot \cos k_\beta x (\sin k_\beta a - a k_\beta \cos k_\beta a) \\
 & + \sum_{n=1}^{n=n_0} \{ P_{\gamma n} e^{-\gamma_n x} (\sinh \gamma_n a - a \gamma_n \cosh \gamma_n a) \\
 & - 2P_{\alpha n} \cos \alpha_n x (\sin \alpha_n a - a \alpha_n \cos \alpha_n a) \} \\
 & + \sum_{n=n_0+1}^{n=\infty} \{ P_{\beta n} e^{-\beta_n x} (\sinh \beta_n a - a \beta_n \cosh \beta_n a) \\
 & - 2P_{\delta n} \cos \delta_n x (\sin \delta_n a - a \delta_n \cos \delta_n a) \}] ,
 \end{aligned} \tag{43}$$

and

$$\begin{aligned}
 w_1 = 2h_0 U [& \sum_{n=1}^{n=n_0} \{ C_n e^{-\gamma_n x} (\sinh \gamma_n a - a \gamma_n \cosh \gamma_n a) \\
 & - 2A_n \cos \alpha_n x (\sin \alpha_n a - a \alpha_n \cos \alpha_n a) \} \\
 & + \sum_{n=n_0+1}^{n=\infty} \{ B_n e^{-\beta_n x} (\sinh \beta_n a - a \beta_n \cosh \beta_n a) \\
 & - 2D_n \cos \delta_n x (\sin \delta_n a - a \delta_n \cos \delta_n a) \}] ,
 \end{aligned} \tag{44}$$

where n_0 is the largest positive integer less than $k_s H / \pi$, and the following abbreviations are used:

$$\begin{aligned}
 L_n &= \frac{2\pi n}{H^2} \sin \frac{\pi}{H} n z , \\
 M_n &= k_\lambda \cdot \frac{2\pi^2}{H^3} \cos \frac{\pi}{H} n z , \\
 A_n &= L_n \cdot \frac{1}{\alpha_n^3} \cdot \frac{\alpha_n^2 - k_\lambda^2}{\alpha_n^2 - \gamma_n^2} \sim \frac{L_n}{\alpha_n^3} , \\
 B_n &= L_n \cdot \frac{1}{\beta_n^3} \cdot \frac{\beta_n^2 + k_\lambda^2}{\beta_n^2 + \delta_n^2} \sim \frac{L_n}{\beta_n^3} , \\
 C_n &= L_n \cdot \frac{1}{\gamma_n^3} \cdot \frac{\gamma_n^2 + k_\lambda^2}{\gamma_n^2 + \alpha_n^2} , \\
 D_n &= L_n \cdot \frac{1}{\delta_n^3} \cdot \frac{\delta_n^2 - k_\lambda^2}{\delta_n^2 + \beta_n^2} , \\
 P_0 &= \frac{k_s}{k_\beta} \frac{\cos(k_s k_\beta (H-z)/k_\lambda)}{\sin(k_s k_\beta H/k_\lambda)} \sim \frac{k_\lambda}{k_\beta^2 H} ,
 \end{aligned}$$

$$\begin{aligned}
P_1 &= k_\lambda / k_\beta^3 H, \\
P_2 &= - \frac{\sin(k_s k_\beta (H-z)/k_\lambda)}{\sin(k_s k_\beta H/k_\lambda)} \sim - \frac{H-z}{H}, \\
P_{\alpha n} &= \frac{M_n}{L_n} \cdot \frac{A_n}{(k_\beta^2 - \alpha_n^2)} \sim - \frac{M_n}{\alpha_n^5}, \\
P_{\beta n} &= \frac{M_n}{L_n} \cdot \frac{B_n}{(k_\beta^2 + \beta_n^2)} \sim \frac{M_n}{\beta_n^5}, \\
P_{\gamma n} &= \frac{M_n}{L_n} \cdot \frac{C_n}{(k_\beta^2 + \gamma_n^2)}, \\
P_{\delta n} &= \frac{M_n}{L_n} \cdot \frac{D_n}{(k_\beta^2 - \delta_n^2)}.
\end{aligned}$$

From these solutions, we can conclude that α_n and δ_n , which exist only when $k_s H > \pi$, contribute to establish the stationary internal waves in the down-stream side of the deformation of the bottom, and no conspicuous disturbances occur in the up-stream side. And these internal waves prevail in the current pattern when the scale of deformation, a , is sufficiently small ($ak_\lambda \ll 1$), on the other hand, they may be negligible compared with the horizontal perturbations when the scale of deformation is large. These features can be expected already from the discussion of perturbations for the sinusoidal bottom profile.

For $ak_\lambda \ll 1$, the terms containing γ_n and δ_n are negligibly small compared with those of α_n and β_n , and, at a sufficiently large distance downstream, the terms of α_n for n is near n_0 predominate, since the terms of β_n will die out. The leading term when $n_0=1$ is

$$w_1 = - \frac{4}{3} h_0 U a^3 \left(\frac{2\pi}{H^2} \right) \sin \frac{\pi}{H} z \cos \alpha_1 x, \quad (45)$$

and the stream lines are given by

$$\varphi = - \frac{4}{3} \frac{h_0 a^3}{\alpha_1} \left(\frac{2H}{H^2} \right) \sin \frac{\pi}{H} z \sin \alpha_1 x. \quad (46)$$

For $ak_\lambda \sim 1$, the terms of α_n and β_n are rather small compared with the terms of γ_n and δ_n , so that in this case, the whole terms in the equations must be taken into account.

For $ak_\lambda \gg 1$, we may reasonably neglect the terms containing α_n , β_n , γ_n and δ_n and the solutions reduce to:

$$v' = 0, \quad \text{for } x < -a, \quad (47)$$

$$v' = - \frac{2h_0 U}{H} \cdot \frac{k_\lambda}{k_\beta^3} \{ \sin k_\beta (a+x) - ak_\beta \cos k_\beta (a+x) - k_\beta x \}, \quad \text{for } -a < x < a \quad (48)$$

and

$$v' = -\frac{2h_0U}{H} \cdot \frac{2k_\lambda}{k_\beta^3} \cos k_\beta x (\sin k_\beta a - ak_\beta \cos k_\beta a), \text{ for } x > a. \quad (49)$$

These solutions can be derived from the original equations if the several assumptions are introduced such that the water is homogeneous and the variation of u_1 and v_1 in the vertical direction is negligible. In these solutions, we can see at once that in the northern hemisphere, the stream lines deviate to the right by the planetary effect and it is followed by Rossby waves. The deviation of stream line is illustrated qualitatively in Fig. 2.

Fig. 2. Deflection of stream line by a large parabolic deformation of the bottom, ($U=\text{const.}$) $a=200$ Km. $U=40$ cm-sec $^{-1}$, $k_\beta=5 \times 10^{-3}$ cm $^{-1}$.

5. The Effect of horizontal shear of a basic current.

When the scale to be considered is large, and the basic current is constant in a vertical, we can assume from the above discussion that the horizontal velocities of perturbations are uniform in the vertical direction and the vertical velocity is a linear function of z . Therefore, on the assumption that the water is homogeneous and $\frac{\partial w_1}{\partial z} = -\frac{U}{H} \frac{dh}{dx}$, equation (8) becomes:

$$\frac{\partial^2 v_1}{\partial x^2} + \frac{\partial^2 v_1}{\partial y^2} + \frac{1}{U} \left(\beta - \frac{d^2 U}{dy^2} \right) v_1 = -\frac{1}{H} \frac{dh}{dx} \left(\lambda - 2 \frac{dU}{dy} \right) \quad (50)$$

provided that h is independent of y .

In a region of strong currents, it may be justified that β is negligible compared with $d^2 U/dy^2$. On these assumptions, we will discuss a very simple case, such as $U=U_0 \cos k_1 y$ and $h=h_0 \cos k_2 x$. A particular solution of the equation is then:

$$v_1 = \frac{h_0}{H} \left\{ \frac{\lambda k_2}{k_1^2 - k_2^2} - \frac{2k_1 U_0}{k_2} \sin k_1 y \right\} \sin k_2 x, \quad (51)$$

where H is the depth of water.

In case when k_1 is several times greater than k_2 , ($k_1 = mk_2$), k_2^2 may be neglected compared with k_1^2 , and (51) reduces to:

$$v_1 = \frac{h_0}{mHk_1} (\lambda - 2m^2 U_0 k_1 \sin k_1 y) \sin k_2 x. \quad (52)$$

From the above solution, we can see that it is very important to take the horizontal

shear of a current into account, if it is comparable with Coriolis parameter. The larger the ratio of the scale of the bottom deformation to that of currents is, the greater the effect of the shear is.

For example, in case of $U_0 \sim 100$ cm/sec, $k_1 \sim 10^{-7}$ cm $^{-1}$ ($L_1 \sim 600$ km), and $k_2 \sim 3.3 \times 10^{-8}$ ($L_2 \sim 2000$ km), m becomes 3 and the order of magnitude of $m^2 U_0 k_1$ is almost comparable with that of λ ($\sim 10^{-4}$ sec $^{-1}$). The current pattern is shown qualitatively in Fig. 3.

Fig. 3. Deflection of stream lines by a sinusoidal bottom deformation in the case of $U = U_0 \cos k_1 y$, $h = h_0 \cos k_2 y$, and $\lambda = m^2 U_0 k_1$.

6. Conclusions.

We have discussed above that when we are investigating the effect of bottom topography on a uniform basic current, it is important to consider the relation of the scale of the bottom configuration to various parameters such as k_s , k_λ and k_β , each of which is the representative scale of the motion of the stratified water on the rotating earth. When the scale of bottom deformations are very small, such that $k_\lambda a \ll 1$, the parameter k_s , which is related to the vertical gradient of the density, plays a dominant role and the perturbation may be almost two dimensional in the vertical plane, and, moreover, if $k_s H > \pi$, the internal gravity waves occur downstream, the wave length of which is $2H^0/n$ in the vertical direction and $2\pi/\alpha_n$ in the horizontal direction.

When the scale of bottom deformations is very large such that $k_\lambda a \gg 1$, the parameter k_β , which is related to the latitudinal change of Coriolis parameter, plays a dominant role and the conclusions are the same as those obtained on the assumption of homogeneous water, such that the stream-lines deviate to the right in the northern hemisphere, accompanied by the Rossby waves.

In the case of an intermediate scale of phenomena, the parameter k_λ , which is related to Coriolis parameter itself, must be taken into account and the perturbations are considered as the combined effects of various factors.

On the other hand, it was shown that, if the basic current is not uniform, the horizontal shear of velocity may sometimes play an important role in the discussion of the topographic effect upon the currents. Thus, the deflection of

currents may be related to the horizontal gradient of velocity and Coriolis parameter.

It must be remembered, however, that the whole discussions are made under the assumption that the steady perturbations always exist, regardless of values of various parameters, so that the discussions of stability and instability of these flow patterns are entirely omitted.

7. Acknowledgements.

The author is indebted to Mr. H. Kusunoki, of the Low Temperature Science Institute, Hokkaido Univ., who generously communicated him the papers of V. W. Stockman. He also wishes to thank Dr. M. Nakano, who kindly read the manuscript before printing.

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