Optimum Mesh Generation for the Finite Element Method (1st Report)*

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This paper proposes a method of r-type mesh optimization for the finite element analysis of two dimensional elasticity problem. The objective function of the optimization is the global error energy norm based on stress discontinuity on the element boundaries.

The mesh generation is based on the method to generate boundary-fitted coordinate systems as solutions of biharmonic equation in the physical plane.

Computer codes were developed and some examples were investigated. It is shown that the approach to the optimized mesh in this paper is effective and useful for practical use.

1. Introduction

The finite element method has been used for a long time in a variety of fields in engineering analysis. On the other hand, owing to remarkable progress in engineering workstation and personal computer, the finite element method has won wide popularity in recent years.

However, because the quality of the finite element solution greatly depends on the discretized model, the analyst must have advanced experience and knowledge in the finite element analysis. As the finite element method gets popularity, it has been realized to be necessary that the mesh which gives a solution of satisfactory accuracy with the minimum total cost can be automatically generated. For that reason, many approaches to mesh optimization by the adaptive method have been presented.1-11)

In the adaptive method, it is usual that the process of analysis and mesh refinement is continued until a prespecified accuracy is achieved. There are three types of refinement, i.e. r-method, h-method and p-method.2,11)

The object of this paper is practical mesh optimization. Therefore, we adopted r-method in which the number of degrees of freedom does not increase. The mesh generation is based on the method to generate boundary-fitted coordinate systems proposed by Thompson et al.12,13) As the coordinate generating system, the authors chose biharmonic equation which governs stress functions in the two dimensional elasticity problem. As stated later, this method can generate various smooth meshes efficiently on various boundary conditions. In the r-method studied up to now, we had to decide direction and distance to move each grid and repeat it many times. However, by the method proposed in this paper, the mesh is decided by only two coefficients in the whole model. Therefore, the mesh optimization becomes the problem with two independent variables, and can be performed very efficiently.

The error is estimated in Otsubo's method simplified for computational usefulness, which uses complementary energy based on discontinuity of stress between adjacent elements.15) It should be avoided in terms of total cost to perform analysis many times in the process of mesh optimization. Therefore the authors developed the method to estimate error of any mesh according to the first one finite element solution.

Some examples of two dimensional elasticity problem are studied in chapter 4 to demonstrate the usefulness and effectiveness of these methods.

2. Mesh Generation by Biharmonic Equation

The method to generate boundary-fitted coordinate systems by transformation between the

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fixed rectangular field and the physical plane has been studied in the field of the finite difference method.\textsuperscript{12-14} Thompson et al, took the coordinates in the transformed plane to be solutions of Laplace’s equation in the physical plane. This method enables flexible control of the spacing of the coordinate lines in the physical region by adding inhomogeneous terms to the right sides of Laplace’s equations.

\((\xi, \eta)\) denoting coordinate in the transformed plane and \((x, y)\) denoting coordinate in the physical plane, the mesh generating system becomes

\[
\begin{aligned}
\frac{\partial^2\xi}{\partial x^2} + \frac{\partial^2\xi}{\partial y^2} = P(\xi, \eta) \\
\frac{\partial^2\eta}{\partial x^2} + \frac{\partial^2\eta}{\partial y^2} = Q(\xi, \eta)
\end{aligned}
\]

\(\cdots \cdots (1)\)

Our problem is to generate the value of the physical cartesian coordinate \((x, y)\) corresponding to known \((\xi, \eta)\) in the fixed square grid in the transformed region. And so, exchanging independent variables and dependent variables, we get following equation system on the transformed plane. (subscripts indicate derivatives)

\[
\begin{aligned}
\alpha x_{\xi\xi} - 2\beta x_{\xi\eta} + \gamma x_{\eta\eta} &= -f(x_P + Qx_{\eta}) \\
\alpha y_{\xi\eta} - 2\beta y_{\xi\eta} + \gamma y_{\eta\eta} &= -f(y_P + Qy_{\eta})
\end{aligned}
\]

\(\cdots \cdots (2)\)

where

\[
\alpha = x^2 + y^2, \quad \beta = x_\xi x_\eta + y_\xi y_\eta, \quad \gamma = x_\xi y_\eta - x_\eta y_\xi
\]

\(\cdots \cdots (3)\)

The control functions \(P\) and \(Q\) can be chosen arbitrarily. For example, in 12) those functions were taken as sums of exponential terms. In this method it is easy to control the density of grids around arbitrary points and lines by attracting lines to the specified points and lines. It is widely used especially in the field of computational fluid dynamics.

However, in the process of practical mesh generation, it mainly depends on the analyst’s insight into engineering analysis and experience to determine the points and lines where to attract lines or to determine the value of coefficients. This method also requires that attraction is performed to several different points and lines simultaneously, and the number of independent variables for the mesh optimization tends to increase. For those reasons, this approach becomes intractable with regard to efficient automation of adaptive mesh optimization.

We took \(P\) and \(Q\) to be solutions of Laplace’s equation in the physical region.

\[
\begin{aligned}
\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} &= 0 \\
\frac{\partial^2 Q}{\partial x^2} + \frac{\partial^2 Q}{\partial y^2} &= 0
\end{aligned}
\]

\(\cdots \cdots (4)\)

Substituting (1) to (4), we get the following biharmonic equations as the system for mesh generation.

\[
\begin{aligned}
\frac{\partial^4 \xi}{\partial x^4} + 2\frac{\partial^2 \xi}{\partial x^2 \partial y^2} + \frac{\partial^2 \xi}{\partial y^4} &= 0 \\
\frac{\partial^4 \eta}{\partial x^4} + 2\frac{\partial^2 \eta}{\partial x^2 \partial y^2} + \frac{\partial^2 \eta}{\partial y^4} &= 0
\end{aligned}
\]

\(\cdots \cdots (5)\)

This has the same form as stress function in two dimensional elasticity problem. Various meshes can be got in accordance with various boundary conditions.

In the transformed plane, (4) becomes

\[
\begin{aligned}
\alpha \xi_{\xi\xi\xi} - 2\beta \xi_{\xi\xi\eta} + \gamma \xi_{\eta\eta\eta} &= -f(x_P + Qx_{\eta}) \\
\alpha \xi_{\eta\eta\eta} - 2\beta \xi_{\eta\eta\eta} + \gamma \xi_{\eta\eta\eta} &= -f(y_P + Qy_{\eta})
\end{aligned}
\]

\(\cdots \cdots (2)\)

where

\[
\begin{aligned}
A_1 &= x_\xi x_{\xi\xi} - x_\eta y_{\eta\eta} + y_\xi y_{\eta\eta} \\
A_2 &= x_\xi x_{\xi\eta} - x_\eta y_{\eta\eta} - y_\xi y_{\eta\eta} \\
B_1 &= x_\xi J_\xi - y_\xi J_\eta \\
B_2 &= y_\xi J_\xi - y_\eta J_\eta \\
J_\xi &= x_\xi y_{\eta\eta} + x_\eta y_{\xi\xi} - x_\xi y_{\eta\xi} - x_\eta y_{\xi\eta} \\
J_\eta &= x_\xi y_{\eta\eta} + x_\eta y_{\xi\xi} - x_\xi y_{\eta\eta} - x_\eta y_{\xi\eta}
\end{aligned}
\]

\(\cdots \cdots (7)\)

\(P\) and \(Q\) are obtained by solving the equations (6). Substituting them into the equations (2), we can get \((x, y)\) as the solution. These equations are solved in finite difference approximation by SOR iteration.

Boundary conditions for \(P, Q\) computation must be given appropriately. Because \(\xi\) and \(\eta\) are biharmonic functions as seen in (5), they coincide with the stress function in a certain boundary condition. Stress function \(F\) satisfies

\[
\begin{aligned}
\frac{\partial^2 F}{\partial x^2} + \sigma_y &= 0 \\
\frac{\partial^2 F}{\partial y^2} + \sigma_x &= 0
\end{aligned}
\]

\(\cdots \cdots (8)\)

Now, we define \(c_1\) and \(c_2\) as follows.

\[\quad\]


\[ \xi = c_1 \xi \]

\[ \eta = c_2 \eta \]

Then from (1)

\[ P = c_1 (\sigma_x + \sigma_y) \]

\[ Q = c_2 (\sigma_x + \sigma_y) \]

The mesh is defined by giving these values as the boundary conditions on the perimeter of the region. When \( c_1 = 0 \) and \( c_2 = 0 \), the mesh coincides with that generated by Laplace's equation. In this paper, we optimized global error estimate with two independent variables \( c_1 \) and \( c_2 \).

### 3. Error Estimation and Optimization

The error energy norm was obtained as complementary energy based on stress discontinuity between adjacent elements. This norm is the objective function for the mesh optimization.

\[
\|e\| = \sqrt{\int_B \{\Delta \sigma\}^T \{\Delta \sigma\} d\Omega} 
\]

\[
= \sqrt{\int_B \{\Delta \sigma\}^T \{\Delta \sigma\} d\Omega} 
\]

where

\[
B = (\Delta \sigma_x + \Delta \sigma_y)^2 + 2(1+\nu)(\Delta \sigma_x \Delta \sigma_y) 
\]

Stress of each element is obtained from the displacement of the corners in the following way. In Fig. 1, \((x_i, y_i)\) is the coordinate and \((u_i, v_i)\) is the displacement of the corner \(i\).

![Fig. 1 Coordinate in the element](image)

\[
\frac{\partial u}{\partial x} = [G][H] \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} 
\]

\[
\frac{\partial u}{\partial y} = [G][H] \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} 
\]

where

\[
[G] = \begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} \end{bmatrix} 
\]

\[
= \begin{bmatrix} \frac{\partial u}{\partial \xi} & \frac{\partial u}{\partial \eta} \\ \frac{\partial v}{\partial \xi} & \frac{\partial v}{\partial \eta} \end{bmatrix} 
\]

\[
[J] = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} a_1 + a_2 \xi & a_2 + a_3 \xi \\ b_1 + b_2 \eta & b_3 + b_4 \eta \end{bmatrix} 
\]

\[
[H] = \begin{bmatrix} \frac{\partial \xi}{\partial \eta} & \frac{\partial \eta}{\partial \eta} \\ \frac{\partial \xi}{\partial \xi} & \frac{\partial \eta}{\partial \xi} \end{bmatrix} 
\]

\[
[D] = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{bmatrix} 
\]

where

\[
[D] = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{\nu} \end{bmatrix} 
\]

The stress of each corner can be obtained substituting \((\xi, \eta) = (0, 0), (1, 0), (1, 1)\) and \((0, 1)\) respectively. In order to split the stress jump on the element boundary into both sides, the mean value of the corner stress of the elements.
around the grid was computed, and \( \Delta \sigma \) was taken to be the difference between the mean value and the stress at the corner of the element. The mean stress at the Gaussian integration point was computed by quadratic interpolation using nine grid points.

Mesh generation by boundary-fitted coordinate system enables the flexible control of the grid density without losing smoothness, and it motivates us to adopt \( r \)-method as the strategy of mesh refinement.

However, it is difficult to improve the mesh to the satisfactory extent in once or a few times of refinement by \( r \)-method. Error estimation must be continued many times in the process of optimization. Because reanalyzing each refined mesh must be avoided considering the cost of computation, it is desirable that error estimation of the refined mesh can be performed in a satisfactory accuracy using the results of analysis performed only once for the original mesh.

As mentioned before, error estimation is performed based only on the displacement of the grids. Therefore, we calculated the displacement of the grids of the refined mesh from the displacement of original grids by quadratic interpolation, using nine grid points around the nearest grid from the point to calculate displacement.

The error estimates in this method gave a good agreement with the error estimates using the results of reanalysis.

4. Examples

4.1 Square Plate with Parabolic Loading

We consider a two dimensional elasticity problem with boundary conditions given in Fig. 2. The quadrant of the plate was considered because both structure and loads are symmetri-

Timoshenko gave the following solution for this problem,,15,16) and we used it for the computation of exact error energy norm \( \| e \|_{\text{ex}} \).

Stress function \( F \):

\[
F = \frac{1}{2} S y^2 \left( 1 - \frac{y^2}{a^2} \right) + (x^2 - a^2)(y^2 - a^2)(\alpha_1 + \alpha_2 x + \alpha_3 y^2) \quad \cdots \cdots \quad (21)
\]

where

\[
\begin{align*}
S &= 10 \\
a &= 100 \\
\alpha_1 &= 0.0404045724 \frac{S}{a^6} \\
\alpha_2 &= 0.0117185240 \frac{S}{a^8} \\
\alpha_3 &= \alpha_2
\end{align*}
\]

Results of error estimation are given in Table 1. First three cases are the results of square meshes before optimization of \( 4 \times 4 \), \( 8 \times 8 \), and \( 16 \times 16 \) division respectively. Although error

| case | div. | \( ||e||_{\text{ex}} \) | \( ||e||_{\text{ex}} \) | \( ||e||_{\text{ex}} \) | \( ||e||_{\text{ex}} \) | \( ||e||_{\text{ex}} \) |
|------|-----|-----------------|-----------------|-----------------|-----------------|-----------------|
| 1    | 4   | 4.95            | 0.389           | 0.419           | 0.930           |
| 2    | 8   | 4.93            | 0.200           | 0.208           | 0.961           | 0.200           |
| 3    | 16  | 4.92            | 0.102           | 0.106           | 0.964           |
| 4    | 8   | 4.93            | 0.171           | 0.177           | 0.967           | 0.172           |

Fig. 3 Error Energy Norm Distribution \((x = 10^{-2})\)
Table 2 Result for Example 2

<table>
<thead>
<tr>
<th>Case</th>
<th>div</th>
<th>$| |_{ext}$</th>
<th>$| |_{est}$</th>
<th>$| |_{est}$'</th>
<th>$\lambda_{est}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>8</td>
<td>34.3</td>
<td>2.19</td>
<td>2.27</td>
<td>0.946</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>34.3</td>
<td>1.91</td>
<td>1.92</td>
<td>0.971</td>
</tr>
<tr>
<td>3</td>
<td>12</td>
<td>34.2</td>
<td>1.69</td>
<td>1.68</td>
<td>0.985</td>
</tr>
<tr>
<td>4</td>
<td>8</td>
<td>34.3</td>
<td>1.81</td>
<td>1.82</td>
<td>0.978</td>
</tr>
<tr>
<td>5</td>
<td>8</td>
<td>34.3</td>
<td>1.73</td>
<td>1.74</td>
<td>0.982</td>
</tr>
</tbody>
</table>

Error estimates for several meshes are shown in Table 2. Cases 1 – 3 are evenly divided meshes and cases 4, 5 are meshes generated by boundary-fitted coordinate system. Case 4 is the mesh of $c_1 = 0$ and $c_2 = 0$, that is, the mesh generated by Laplace's equation, and finite element analysis was performed for this mesh to get the basis value of displacement interpolation. Case 5 is the optimized mesh. Fig. 6 – 8 show the meshes for case 1, 4, 5 respectively. Fig. 9 – 11 show the contour of equivalent stress for each mesh.

The error estimate for the optimized $8 \times 8$ mesh is by 20% less than the error estimate for

Fig. 4 Optimized Mesh

Fig. 5 Example 2

estimates $\| \|_{est}$ give a lower estimate to $\| \|_{est}$, $\| \|_{est}$ / $\| \|_{est}$ is satisfactorily close to 1.0. The distribution of error energy norm for $8 \times 8$ square mesh is shown in Fig. 3. As shown in the figure, elements in the right corner have higher error estimates, and it is obvious that the grids are attracted to the right corner through mesh optimization.

In the course of the mesh generation, grids are left free to move along the boundary.

The optimized mesh is shown in Fig. 4 and the results for the optimized mesh are given in Table 1, Case 4. For Case 4, $\| \|_{est}$ is an error estimate based on the displacement of grids obtained by interpolation from the displacement of the initial grids, and $\| \|_{est}$' is based on the displacement obtained by analysis of each mesh. Three values $\| \|_{est}$, $\| \|_{est}$ and $\| \|_{est}$' agreed well, and the mesh which minimizes $\| \|_{est}$ also nearly minimizes $\| \|_{est}$ and $\| \|_{est}$'.

4.2. Square Plate with a Circular Hole

The configuration of the problem is shown in Fig. 5.

\[ \sigma_0 = 1.0 \]

\[ \alpha = 4.68 \]

Fig. 6 Evenly Divided Mesh

Fig. 7 Mesh by Laplace's Eq.

Fig. 8 Optimized Mesh

Fig. 9 Contour (Fine Mesh)

Fig. 10 Contour (Evenly Divided)

Fig. 11 Contour (Optimized)
the evenly divided mesh of the same number of degrees of freedom, and corresponds nearly to the evenly divided 12 × 12 mesh. This means that the optimized mesh is twice as efficient as an evenly divided mesh in terms of the number of degrees of freedom to achieve the same accuracy.

References 17, 18) give \( \sigma_0 / \sigma_0 = 3.35 \) at the point A in case of \( a/\rho = 4.73 \) as in this problem. We used this value for numerical comparison. The ratio between \( \sigma_A \) (finite element solution with each mesh) and \( \sigma_A(=33.5) \) is shown in Table 2. The stress concentration is better described by the optimized mesh which provides the ratio \( \sigma_A / \sigma_0 = 0.982 \) and is almost equivalent to the evenly divided 12 × 12 mesh.

This problem also gave good agreement of \( \| e \|_{err} \), \( \| e \|_{err}' \) and \( \| e \|_{e} \). Because the exact solution is not available with this problem, we calculated displacement of each grid from the finite element solution of a fine mesh, and regarding it as the exact solution, we calculated \( \| e \|_{e} \).

4.3 Plate with Face Plates on the Edge

The last example is a plate with face plates on the edge shown in Fig. 12.

The error energy norm in a bar element was computed in the same way as the quadrilateral element. That is

\[
\| e \| = \sqrt{\frac{A}{EJ} \int_{\Gamma} \Delta \sigma^2 d \Gamma}
\]

where

\( A \) is the sectional area.

\( \Delta \sigma \) is the difference between stress of the element and its linear interpolation (see Fig. 13)

Results are shown in Table 3. Case 1 is the evenly divided mesh. Case 2 is the mesh generated by Laplace's equation with grids on the edge free to move along the boundary. Displacement of this mesh was used as the basis value of interpolation. Case 3 is the optimized mesh. Case 4 is the optimized mesh without face plates. Fig. 14

Table 3 Result for Example 3

<table>
<thead>
<tr>
<th>case</th>
<th>| e |_{err}</th>
<th>| e |_{err}'</th>
<th>| e |_{e}</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>101.3</td>
<td>104.1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>82.82</td>
<td>87.00</td>
<td>82.82</td>
</tr>
<tr>
<td>3</td>
<td>82.15</td>
<td>86.13</td>
<td>82.38</td>
</tr>
</tbody>
</table>

Fig. 13 \( \Delta \sigma \) in the bar element

Fig. 14 Evenly Divided Mesh

Fig. 15 Mesh by Laplace's Eq.

Fig. 16 Optimized Mesh

Fig. 17 Optimized Mesh without Face Plates

Fig. 18 Contour (Fine Mesh)

Fig. 19 Contour (Evenly Divided)
17 show the meshes of case 1 - 4 respectively. Fig. 18 - 21 show the contour of equivalent stress for each mesh.

Because face plates reduce stress concentration, Fig. 17 gives more drastic change from the original mesh than Fig. 16. It is shown that for the analysis of plates with face plates, the mesh by Laplace's equation is already close to the optimum mesh. This is a very important feature for practical interest because the same mesh is available for several different loading cases.

5. Conclusions

In this paper, the r-type adaptive method by mesh generation using biharmonic equation is numerically investigated. The conclusions obtained are summarized as follows:
1) Using biharmonic equation for the mesh generation, we can generate the mesh which gives less error in the results of analysis by applying appropriate boundary conditions. In the examples studied in this paper, the error estimates decreased by 20% compared with the evenly divided mesh, which needs almost twice number of degrees of freedom to achieve the same accuracy as the optimized mesh.
2) We could estimate the error in energy norm in satisfactory accuracy using complementary energy based on the stress discontinuity on the element boundary.
3) We could estimate the error of each mesh in the process of optimization without reanalysis, using the results of the original mesh. This enables remarkably efficient optimization.
4) In the problem with high stress concentration, there is a large difference between the mesh generated by Laplace's equation and the optimized mesh. However, in the case of plates in ship structure which have face plates on the edge, the mesh by Laplace's equation is near optimum. This is very important for practical interest because the same mesh is available for the analysis of several different loading cases.
5) This method can be applied to plates of complicated shape by dividing them into the units, to each of which the method in this paper can be applied.

References
13) J. F. Thompson(ed.): Numerical Grid Gener-

Discussion

[Discussion]

In the optimal mesh analysis, the estimation of errors is very important. In this paper, we present a method to estimate errors in the finite element method. The method is based on the assumption that the errors are distributed uniformly over the elements.

In the case study, we apply the method to a simple structure and compare the results with the exact solution. The results show that the method is able to accurately estimate the errors.

[Question]

You mentioned that the optimal mesh is important. Can you explain more about the method you used to estimate the errors?

[Answer]

The method we used is based on the idea that the errors are distributed uniformly over the elements. We assume that the errors are proportional to the size of the elements. This assumption simplifies the calculations and makes it possible to estimate the errors analytically.

In the case study, we compare the results with the exact solution to verify the accuracy of the method. The results show that the method is able to accurately estimate the errors.