ORIGINAL ARTICLES

PROPOSAL OF “THE CYCLIC PAIRED COMPARISONS”

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Abstract: The Scheffé method (including the modified methods) is a method peculiar to the sensory evaluation method and typical statistical sensory evaluation method. The experiment and method of analysis of the cyclic paired comparisons those improve on the Scheffé method for paired comparisons in cases where the experiment and data are the minimum requirement. The proposed method by cyclic paired comparisons requires the same times of paired comparisons as the number of samples and is able to order all the samples and calculate the analysis of variance. When the cyclic paired comparisons are applied, the complete paired comparisons can be divided into the same times as the number of samples. Moreover, the number of panelists and psychological and physical stress of panelists are less than those of the Scheffé method (including the modified methods).

Keywords: *Scheffé method, Paired comparison, Scheffé's paired comparison method, Sensory evaluation*

1. INTRODUCTION

Paired comparison is a method of comparing k samples by selecting them two by two, comparing the pair like a league match, putting together all the results of the kC_2 pairs as a total, and evaluating conclusively the whole of the k samples.

The paired comparison is relatively easy to perform even for a nonprofessional or an unskilled person who is not an expert. In addition, assessing the results is simple and predictable. On the other hand, there are also some disadvantages. For example the number of combinations becomes immensely greater as the number of samples increases. Moreover, when this is performed using a small number of subjects, the number of experiments required per subject becomes substantial and this results in the subject becoming exhausted and makes it difficult to conduct the experiment.

The Scheffé method for paired comparisons (Scheffé method) is a paired comparison with a grade. The Scheffé method (including the modified methods) is a method peculiar to the sensory evaluation method and typical statistical sensory evaluation method.

Comparison is performed by assigning one combination to one subject in the original method proposed by H. Scheffé in 1952 (Scheffé's original method for paired comparisons or the original method for paired comparisons). In this case, a large number of combinations are required for a large number of items to be compared. This is advantageous in cases where the number of subjects is large, such as in

market research, but it is disadvantageous in the case of experiments in a laboratory or a factory.

Three modified methods besides the original method have therefore been proposed by Japanese researchers based on the Scheffé original method for paired comparisons. These are the Haga modified method (1962), the Ura modified method (1956), and the Nakaya modified method (1970). These are methods in which one person compares all the combinations or for cases where both-way assessment is not (or cannot be) performed.

The Scheffé method for paired comparisons (including the modified methods) estimates each factorial effect and performs variance analysis by assuming a structure formula for the data related to the grade. This is based on the assumption that the paired comparisons are fully executed. Therefore paired comparisons for all combinations should be carried out in this method. Thus, no method of experiment and analysis is available for the case where the paired comparisons are consisted on the minimum requirement of experiment and the most essential paired comparisons data.

An experiment and analysis method that improves on the Scheffé method for paired comparisons in cases where the given data are cyclic will therefore be described here. A method to be proposed is “cyclic paired comparisons” of the same times paired comparisons as the number of samples is able to order all the samples and calculate the analysis of variance.

2. EXPERIMENT OF CYCLIC PAIRED COMPARISONS AND METHOD OF ANALYSIS OF DATA BASED ON A LINEAR MODEL

Firstly, in the previous paper a general method of analysis for complete paired comparisons data and that of incomplete paired comparisons data were described based on a linear model.

In contrast to the general analysis method of incomplete paired comparisons data described in the previous paper, a general method of analysis of cyclic paired comparisons data as shown in Table 1 will be described based on a linear model as a particular case of incomplete paired comparisons data.

Table 1 Data by cyclic paired comparisons					
$i \backslash j$	1	2	3	4	5
1		x_{12}			
2			x_{23}		
3				x_{34}	
4					x_{45}
5	x_{51}				

The structure formula (1) is assumed for cyclic paired comparisons data in Table 1 in the case of one panelist or the average of plural panelists.

$$x_{ij} = (\alpha_i - \alpha_j) + \delta + e_{ij} \quad (1)$$

where i : Index of a sample designated first
 $i=1, 2, \dots, k$

j : Index of a sample designated second
 $j=1, 2, \dots, k$

α_i, α_j : Average of the preferences of the whole panel for samples A_i and A_j
 $\sum \alpha_i = 0$ is assumed.

δ : Order effect

e_{ij} : Error: This is assumed to comply with a normal distribution with average 0 and variance σ^2 .

Formula (2) is for cyclic paired comparisons data in Table 1 in the case of one panelist.

$$\begin{cases} x_{12} = (\alpha_1 - \alpha_2) + \delta_{12} + e_{12} \\ \vdots \\ x_{43} = (\alpha_4 - \alpha_3) + \delta_{43} + e_{43} \end{cases} \quad (2)$$

where observation vector x for the cyclic paired

comparisons data in Table 1 is expressed as formula (3), so the design matrix A is formula (4), parameter θ is formula (5), and the error term e is formula (6).

$$x = [x_{12} \ x_{23} \ x_{34} \ x_{45} \ x_{51}]^T \quad (3)$$

$$A = [A_1, A_2] = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (4)$$

$$\theta = [\theta_1, \theta_2]^T = [\alpha_1 \ \alpha_2 \ \alpha_3 \ \alpha_4 \ \alpha_5 \ \delta]^T \quad (5)$$

$$e = [e_{12} \ e_{23} \ e_{34} \ e_{45} \ e_{51}]^T \quad (6)$$

Now formula (2) can be replaced with formula (7).

$$x = A\theta + e \quad (7)$$

The least-squares solution for parameter θ becomes as it is in formula (8) using $A^T A \theta = A^T x$ and $\sum \alpha_i = 0$. So the sum of squares for each factor and the analysis of variance are shown in Table 2 and Table 3 respectively. The degree of freedom of error is 0.

$$\theta = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \delta \end{bmatrix} = \begin{bmatrix} (2x_{12} - 2x_{51} - x_{45} + x_{23})/5 \\ (2x_{23} - 2x_{12} - x_{51} + x_{34})/5 \\ (2x_{34} - 2x_{23} - x_{12} + x_{45})/5 \\ (2x_{45} - 2x_{34} - x_{23} + x_{51})/5 \\ (2x_{51} - 2x_{45} - x_{34} + x_{12})/5 \\ (x_{12} + x_{23} + x_{34} + x_{45} + x_{51})/5 \end{bmatrix} \quad (8)$$

Table 2 The sum of the squares for each factor in the case of cyclic paired comparisons

$S_\alpha = x^T A_1 \hat{\theta}_1$	
$= 4(x_{12}^2 + x_{23}^2 + x_{34}^2 + x_{45}^2 + x_{51}^2)/5$	
$- 2(x_{12}x_{23} + x_{12}x_{34} + x_{12}x_{45} + x_{12}x_{51} + x_{23}x_{34} + x_{23}x_{45} + x_{23}x_{51} + x_{34}x_{45} + x_{34}x_{51} + x_{45}x_{51})/5$	
$S_\delta = x^T A \hat{\theta} - x^T [A_1, A_2] \hat{\theta}_{12}$	
$= (x_{12} + x_{23} + x_{34} + x_{45} + x_{51})^2 / 5 = x_{..}^2 / 5$	
$S_e = x^T x - x^T A \hat{\theta} = S_T - S_\alpha - S_\delta = 0$	
$S_T = x^T x = x_{12}^2 + x_{23}^2 + x_{34}^2 + x_{45}^2 + x_{51}^2 = \sum \sum x_{ij}^2$	

Table 3 The analysis of variance in the case of cyclic paired comparisons

Factor	Sum of Squares	Degree of freedom
Main effect	S_α	$k-1$
Order effect	S_δ	1
error	S_e	0
Total	S_T	k

Table 4 Example data by cyclic paired comparisons

$i \backslash j$	1	2	3	4	5
1		-2			
2			0		
3				0	
4					1
5	2				

Table 5 The analysis of variance in the case of cyclic paired comparisons

Factor	Sum of Squares	Degree of freedom
Main effect	8.8	4
Order effect	0.2	0
Total	9.0	5

For numerical example, the case of five samples is taken in consideration. When the data are obtained by cyclic paired comparisons as shown in Table 4, each parameter is estimated as shown below. The sum of squares for each factor and the analysis of variance are shown in Table 5.

$$\begin{cases} \alpha_1 = -1.8 \\ \alpha_2 = 0.4 \\ \alpha_3 = 0.6 \\ \alpha_4 = 0.8 \\ \delta = 0.2 \end{cases}$$

Thus, cyclic paired comparisons method is available for the case where the paired comparisons are consisted on the minimum requirement of experiment and the most essential paired comparisons data.

3 EXPERIMENT OF CYCLIC PAIRED COMPARISONS AND METHOD OF DATA ANALYSIS IN THE CASE OF PLURAL PANELISTS

The experiment and method of analysis for the cyclic paired comparisons data based on a linear model has been described above. In this section an experiment and a method of analysis for cyclic paired comparisons data will be proposed in the case of plural panelists as shown in Table 6, which is often used in paired comparisons.

Structure formula (9) is assumed for cyclic paired comparisons data in Table 5 in the case of plural panelists.

Table 6 Data by cyclic paired comparisons

(a) Data of panelist O_1

$i \backslash j$	1	2	3	4	5
1		x_{121}			
2			x_{231}		
3				x_{341}	
4					x_{451}
5	x_{511}				

(b) Data of panelist O_2

$i \backslash j$	1	2	3	4	5
1		x_{122}			
2			x_{232}		
3				x_{342}	
4					x_{452}
5	x_{512}				

(c) Data of panelist O_3

$i \backslash j$	1	2	3	4	5
1		x_{123}			
2			x_{233}		
3				x_{343}	
4					x_{453}
5	x_{513}				

(d) Total of panelists

$i \backslash j$	1	2	3	4	5
1		$x_{12\cdot}$			
2			$x_{23\cdot}$		
3				$x_{34\cdot}$	
4					$x_{45\cdot}$
5	$x_{51\cdot}$				

$$x_{ijk} = (\alpha_i - \alpha_j) + (\alpha_{il} - \alpha_{jl}) + (\delta + \delta_l) + \varepsilon_{ijl} \quad (9)$$

where k : Number of samples

i : Index of a sample designated first

$i=1, 2, \dots, k$

j : Index of a sample designated second.

$j=1, 2, \dots, k$

n : Total number of panelists (or repetition)

l : Index of a panelist. $l=1, 2, \dots, n$

α_i, α_j : Average of the preferences of the whole panel for samples A_i and A_j
 $\sum \alpha_i = 0$ is assumed.

α_{il}, α_{jl} : Individual difference in the preference of panelist O_l for samples A_i and A_j , in other words, the difference between the preference of panelist O_l and the average preference of the panel.

Therefore panelist O_l has preference $\alpha_i + \alpha_{il}$ for sample A_i . Here again $\sum \alpha_{il} = 0$ and $\sum \alpha_{jl} = 0$ are assumed.

δ : Average of the order effects.

δ_l : Individual difference of the order effect.
 $\sum \delta_l = 0$ is assumed.

e_{ijl} : Error. This is assumed to comply with a normal distribution with the average 0 and variance σ^2 .

Formula (10) is for cyclic paired comparisons data

in Table 5 in the case of three panelists.

$$\begin{cases} x_{121} = (\alpha_1 - \alpha_2) + (\alpha_{11} - \alpha_{21}) + (\delta + \delta_1) + \varepsilon_{121} \\ \vdots \\ x_{513} = (\alpha_5 - \alpha_1) + (\alpha_{53} - \alpha_{13}) + (\delta + \delta_3) + \varepsilon_{433} \end{cases} \quad (10)$$

Now this can be replaced with the following formula.

$$x = A\theta + e \quad (11)$$

Here, x is the observation vector for paired comparisons data in Table 5, A is the design matrix, θ is the parameter, and e is the error term.

$$x = [x_{121} \ x_{231} \ x_{341} \ x_{451} \ x_{511} \ x_{122} \ x_{232} \ x_{342} \ x_{452} \ x_{512} \ x_{123} \ x_{233} \ x_{343} \ x_{453} \ x_{513}]^T \quad (12)$$

$$A = [A_1, A_2, A_3, A_4]$$

$$= \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ \hline 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 1 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -0 & 0 & 1 & 1 \end{bmatrix} \quad (13)$$

$$\theta = [\theta_1, \theta_2, \theta_3, \theta_4]^T$$

$$= [\alpha_1 \ \alpha_2 \ \alpha_3 \ \alpha_4 \ \alpha_5 \ \alpha_{11} \ \alpha_{12} \ \alpha_{13} \ \alpha_{21} \ \alpha_{22} \ \alpha_{23} \ \alpha_{31} \ \alpha_{32} \ \alpha_{33} \ \alpha_{41} \ \alpha_{42} \ \alpha_{43} \ \alpha_{51} \ \alpha_{52} \ \alpha_{53} \ \delta_1 \ \delta_2 \ \delta_3 \ \delta]^T \quad (14)$$

$$e = [e_{121} \ e_{231} \ e_{341} \ e_{451} \ e_{511} \ e_{122} \ e_{232} \ e_{342} \ e_{452} \ e_{512} \ e_{123} \ e_{233} \ e_{343} \ e_{453} \ e_{513}]^T \quad (15)$$

The least-squares solution for parameter θ is derived as below by solving the simultaneous equation of the normal equation $A^T A \theta = A^T x$ and $\sum \alpha_i = 0$, $\sum \alpha_{il} = 0$, $\sum \alpha_{jl} = 0$, and $\sum \delta_l = 0$.

$$\begin{cases} \hat{\alpha}_1 = (2x_{12} - 2x_{51} - x_{45} + x_{23})/15 \\ \hat{\alpha}_2 = (2x_{23} - 2x_{12} - x_{51} + x_{34})/15 \\ \hat{\alpha}_3 = (2x_{34} - 2x_{23} - x_{12} + x_{45})/15 \\ \hat{\alpha}_4 = (2x_{45} - 2x_{34} - x_{23} + x_{51})/15 \\ \hat{\alpha}_5 = (2x_{51} - 2x_{45} - x_{34} + x_{12})/15 \\ \hat{\alpha}_{1l} = (2x_{12l} - 2x_{51l} - x_{45l} + x_{23l})/5 - \hat{\alpha}_1 \\ \hat{\alpha}_{2l} = (2x_{23l} - 2x_{12l} - x_{51l} + x_{34l})/5 - \hat{\alpha}_2 \\ \hat{\alpha}_{3l} = (2x_{34l} - 2x_{23l} - x_{12l} + x_{45l})/5 - \hat{\alpha}_3 \\ \hat{\alpha}_{4l} = (2x_{45l} - 2x_{34l} - x_{23l} + x_{51l})/5 - \hat{\alpha}_4 \\ \hat{\alpha}_{5l} = (2x_{51l} - 2x_{45l} - x_{34l} + x_{12l})/5 - \hat{\alpha}_5 \\ \hat{\delta} = (x_{12} + x_{23} + x_{34} + x_{45} + x_{51})/15 \\ \hat{\delta}_l = (x_{12l} + x_{23l} + x_{34l} + x_{45l} + x_{51l})/5 - \hat{\delta} \end{cases} \quad (16)$$

Table 7 The sum of the squares for each factor in the case of cyclic paired comparisons

$$\begin{aligned} S_\alpha &= x^T A_1 \hat{\theta}_1 = 4(x_{12}^2 + x_{23}^2 + x_{34}^2 + x_{45}^2 + x_{51}^2)/15 \\ &\quad - 2(x_{12}x_{23} + x_{12}x_{34} + x_{12}x_{45} + x_{12}x_{51} + x_{23}x_{34} \\ &\quad + x_{23}x_{45} + x_{23}x_{51} + x_{34}x_{45} + x_{34}x_{51} + x_{45}x_{51})/15 \\ S_{\alpha \emptyset} &= x^T A_1 \hat{\theta}_1 - x^T [A_1, A_2] \hat{\theta}_{12} \\ &= 4(\sum x_{12l}^2 + \sum x_{23l}^2 + \sum x_{34l}^2 + \sum x_{45l}^2 + \sum x_{51l}^2)/5 \\ &\quad - 2(\sum x_{12l}x_{23l} + \sum x_{12l}x_{34l} + \sum x_{12l}x_{45l} + \sum x_{12l}x_{51l} \\ &\quad + \sum x_{23l}x_{34l} + \sum x_{23l}x_{45l} + \sum x_{23l}x_{51l} + \sum x_{34l}x_{45l} \\ &\quad + \sum x_{34l}x_{51l} + \sum x_{45l}x_{51l})/5 - S_\alpha \\ S_{\delta \emptyset} &= x^T [A_1, A_2, A_3] [\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3] - x^T [A_1, A_2] [\hat{\theta}_1, \hat{\theta}_2] \\ &= \sum (x_{12l} + x_{23l} + x_{34l} + x_{45l} + x_{51l})^2/5 - S_\delta \\ &= \sum x_{..l}^2/5 - S_\delta \\ S_\delta &= x^T A \hat{\theta} - x^T [A_1, A_2, A_3] [\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3] \\ &= (x_{12} + x_{23} + x_{34} + x_{45} + x_{51})^2/15 = x_{..}^2/15 \\ S_e &= x^T x - x^T A \hat{\theta} = S_T - S_\alpha - S_{\alpha \emptyset} - S_{\delta \emptyset} - S_\delta = 0 \\ S_T &= x^T x = x_{121}^2 + x_{231}^2 + \dots + x_{453}^2 + x_{513}^2 = \sum \sum \sum x_{ijl}^2 \end{aligned}$$

Table 8 The analysis of variance in the case of cyclic paired comparisons

Factor	Sum of Squares	Deg. of freedom
Main effect	S_α	$k-1$
Main effect $\times$ panel	$S_{\alpha(B)}$	$(k-1)(n-1)$
Order effect	S_δ	1
Order effect $\times$ panel	$S_{\delta(B)}$	$n-1$
error	S_e	0
Total	S_T	kn

So the sum of the squares for each factor and the analysis of variance are shown in Table 7 and Table 8 respectively. Though the degree of freedom of error is 0, there is no problem when the factors with smaller sum of squares are pooled into error.

The analysis method of cyclic paired comparisons in the case of plural panelists has been described above. The proposed “cyclic paired comparisons” of less experiment than that of the Scheffé is able to order all the samples and calculate the analysis of variance.

4. RELATION BETWEEN CYCLIC PAIRED COMPARISONS AND COMPLETE PAIRED COMPARISONS

The experiment and method of analysis for the cyclic paired comparisons data has been described above. The relation between the cyclic paired comparisons and the complete paired comparisons will be discussed in this chapter.

Structure formula (9) is assumed for cyclic paired comparisons data in Table 9 in the case of plural cycle series or panelists.

Now data in Table 9 can be replaced with the formula (11). Here, x is the observation vector for paired comparisons data in Table 5, A is the design matrix, θ is the parameter, and e is the error term.

$$x = [x_{121} \ x_{231} \ x_{341} \ x_{451} \ x_{511} \ x_{132} \ x_{242} \ x_{352} \ x_{412} \ x_{522} \ x_{143} \ x_{253} \ x_{313} \ x_{423} \ x_{533} \ x_{154} \ x_{214} \ x_{324} \ x_{434} \ x_{544}]^T \quad (12)$$

$$A = [A_1, A_2, A_3, A_4]$$

$$= \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \quad (13)$$

Table 9 Data by cyclic paired comparisons

(a) Data of cycle series C_1 (panelist O_1)

$i \backslash j$	1	2	3	4	5
1		x_{121}			
2			x_{231}		
3				x_{341}	
4					x_{451}
5	x_{511}				

(b) Data of cycle series C_2 (panelist O_2)

$i \backslash j$	1	2	3	4	5
1			x_{132}		
2				x_{242}	
3					x_{352}
4	x_{412}				
5		x_{522}			

(c) Data of cycle series C_3 (panelist O_3)

$i \backslash j$	1	2	3	4	5
1				x_{143}	
2					x_{253}
3	x_{313}				
4		x_{423}			
5			x_{533}		

(d) Data of cycle series C_4 (panelist O_4)

$i \backslash j$	1	2	3	4	5
1					x_{154}
2	x_{214}				
3		x_{324}			
4			x_{434}		
5				x_{544}	

(e) Total of cycle series C_1-C_4 (panelists O_1-O_4)

$i \backslash j$	1	2	3	4	5
1		x_{21}	x_{132}	x_{143}	x_{154}
2	x_{214}		x_{231}	x_{242}	x_{253}
3	x_{313}	x_{324}		x_{341}	x_{352}
4	x_{412}	x_{423}	x_{434}		x_{451}
5	x_{511}	x_{522}	x_{533}	x_{544}	

$$\theta = [\theta_1, \theta_2, \theta_3, \theta_4]^T$$

$$= [\alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_5; \alpha_{11} \alpha_{12} \alpha_{13} \alpha_{14} \alpha_{21} \alpha_{22} \alpha_{23} \alpha_{24} \alpha_{31} \alpha_{32} \alpha_{33} \alpha_{34} \alpha_{41} \alpha_{42} \alpha_{43} \alpha_{44} \\ \alpha_{51} \alpha_{52} \alpha_{53} \alpha_{54}; \delta_1 \delta_2 \delta_3; \delta]^T \quad (14)$$

$$e = [e_{121} e_{231} e_{341} e_{451} e_{511} e_{132} e_{242} e_{352} e_{412} e_{522} e_{143} e_{253} e_{313} e_{423} e_{533} e_{154} e_{214} e_{324} e_{434} e_{544}]^T \quad (15)$$

The least-squares solution for parameter θ is derived as below by solving the simultaneous equation of the normal equation $A^T A \theta = A^T x$ and $\sum \alpha_i = 0$, $\sum \alpha_{il} = 0$, $\sum \alpha_{jl} = 0$, and $\sum \delta_l = 0$.

$$\left\{ \begin{array}{l} \hat{\alpha}_1 = (x_{1..} - x_{.1})/10 + (x_{231}x_{451} + x_{352}x_{242} + x_{423}x_{533} + x_{544}x_{324})/20 \\ \hat{\alpha}_2 = (x_{2..} - x_{.2})/10 + (x_{341}x_{511} + x_{412}x_{352} + x_{533}x_{143} + x_{154}x_{434})/20 \\ \hat{\alpha}_3 = (x_{3..} - x_{.3})/10 + (x_{451}x_{121} + x_{522}x_{412} + x_{143}x_{253} + x_{214}x_{544})/20 \\ \hat{\alpha}_4 = (x_{4..} - x_{.4})/10 + (x_{511}x_{231} + x_{132}x_{522} + x_{253}x_{313} + x_{324}x_{154})/20 \\ \hat{\alpha}_5 = (x_{5..} - x_{.5})/10 + (x_{121}x_{341} + x_{242}x_{132} + x_{313}x_{423} + x_{434}x_{214})/20 \\ \hat{\alpha}_{11} = (2x_{121} - 2x_{511} + x_{231}x_{451})/5 - \hat{\alpha}_1 \\ \hat{\alpha}_{12} = (2x_{132} - 2x_{412} + x_{352}x_{242})/5 - \hat{\alpha}_1 \\ \hat{\alpha}_{13} = (2x_{143} - 2x_{313} + x_{423}x_{533})/5 - \hat{\alpha}_1 \\ \hat{\alpha}_{14} = (2x_{154} - 2x_{214} + x_{544}x_{324})/5 - \hat{\alpha}_1 \\ \hat{\alpha}_{21} = (2x_{231} - 2x_{121} + x_{341}x_{511})/5 - \hat{\alpha}_2 \\ \hat{\alpha}_{22} = (2x_{242} - 2x_{522} + x_{412}x_{352})/5 - \hat{\alpha}_2 \\ \hat{\alpha}_{23} = (2x_{253} - 2x_{423} + x_{533}x_{143})/5 - \hat{\alpha}_2 \\ \hat{\alpha}_{24} = (2x_{214} - 2x_{324} + x_{154}x_{434})/5 - \hat{\alpha}_2 \\ \hat{\alpha}_{31} = (2x_{341} - 2x_{231} + x_{451}x_{121})/5 - \hat{\alpha}_3 \\ \hat{\alpha}_{32} = (2x_{352} - 2x_{132} + x_{522}x_{412})/5 - \hat{\alpha}_3 \\ \hat{\alpha}_{33} = (2x_{313} - 2x_{533} + x_{143}x_{253})/5 - \hat{\alpha}_3 \\ \hat{\alpha}_{34} = (2x_{324} - 2x_{434} + x_{214}x_{544})/5 - \hat{\alpha}_3 \\ \hat{\alpha}_{41} = (2x_{451} - 2x_{341} + x_{511}x_{231})/5 - \hat{\alpha}_4 \\ \hat{\alpha}_{42} = (2x_{412} - 2x_{242} + x_{132}x_{522})/5 - \hat{\alpha}_4 \\ \hat{\alpha}_{43} = (2x_{423} - 2x_{143} + x_{253}x_{313})/5 - \hat{\alpha}_4 \\ \hat{\alpha}_{44} = (2x_{434} - 2x_{544} + x_{324}x_{154})/5 - \hat{\alpha}_4 \\ \hat{\alpha}_{51} = (2x_{511} - 2x_{451} + x_{121}x_{341})/5 - \hat{\alpha}_5 \\ \hat{\alpha}_{52} = (2x_{522} - 2x_{352} + x_{242}x_{132})/5 - \hat{\alpha}_5 \\ \hat{\alpha}_{53} = (2x_{533} - 2x_{253} + x_{313}x_{423})/5 - \hat{\alpha}_5 \\ \hat{\alpha}_{54} = (2x_{544} - 2x_{154} + x_{434}x_{214})/5 - \hat{\alpha}_5 \\ \hat{\delta}_l = (x_{12l} + x_{23l} + x_{34l} + x_{45l} + x_{51l})/5 - \hat{\delta} = x_{..l}/5 - \hat{\delta} \\ \hat{\delta} = x_{..}/20 \end{array} \right. \quad (12)$$

So the sum of the squares for each factor is shown in Table 11 and the analysis of variance is the same as Table 8. Though the degree of freedom of error is 0, there is no problem when the factors with smaller sum of squares are pooled into error.

Table 10 The sum of the squares for each factor in the case of cyclic paired comparisons

$$S_\alpha = x^T A_1 \hat{\theta}_1 = \sum (x_{i..} - x_{.i})^2 - x_{..}^2/20$$

$$S_{\alpha \varpi} = x^T A_1 \hat{\theta}_1 - x^T [A_1, A_2] \hat{\theta}_{12}$$

$$= \sum (\sum (x_{i..} - x_{.il}) - x_{..}^2/5) - S_\alpha$$

$$S_{\delta \varpi} = x^T [A_1, A_2, A_3] [\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3] - x^T [A_1, A_2] [\hat{\theta}_1, \hat{\theta}_2]$$

$$= \sum (x_{12l} + x_{23l} + x_{34l} + x_{45l} + x_{51l})^2/5 - S_\delta$$

$$= \sum x_{..l}^2/5 - S_\delta$$

$$S_\delta = x^T A \hat{\theta} - x^T [A_1, A_2, A_3] [\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3]$$

$$= (x_{12.} + x_{23.} + x_{34.} + x_{45.} + x_{51.})^2/20 - x_{..}^2/20$$

$$S_e = x^T x - x^T A \hat{\theta} = S_T - S_\alpha - S_{\alpha \varpi} - S_{\delta \varpi} - S_\delta = 0$$

$$S_T = x^T x = x_{121}^2 + x_{231}^2 + \dots + x_{453}^2 + x_{513}^2 = \sum \sum \sum x_{ljl}^2$$

5. CONCLUSION

Therefore the experiment and method of analysis of the cyclic paired comparisons those improve on the Scheffé method for paired comparisons in cases where the experiment and data are the minimum requirement.

The proposed method by cyclic paired comparisons requires the same times of paired comparisons as the number of samples and is able to order all the samples and calculate the analysis of variance.

When the cyclic paired comparisons is applied, the complete paired comparisons can be divided into the same times as the number of samples. Moreover, the number of panelists and psychological and physical stress of panelists are less than those of the Scheffé method (including the modified methods).

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