THE CENTRAL VALUE OF THE TRIPLE SINE FUNCTION

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Abstract

We study the central value of the triple sine function for a general period. We give an explicit integral expression and an inequality. As an application we obtain an expression for $\zeta(3)$.

1. Introduction

The triple sine function

$$S_3(x, (\omega_1, \omega_2, \omega_3)) = \prod_{n_1, n_2, n_3 \geq 0} (n_1 \omega_1 + n_2 \omega_2 + n_3 \omega_3 + x)$$

$$\times \prod_{m_1, m_2, m_3 \geq 1} (m_1 \omega_1 + m_2 \omega_2 + m_3 \omega_3 - x)$$

constructed and studied in our previous papers [K] [KK] (cf. Manin [M]) is a generalization of the usual sine function

$$S_1(x, \omega) = \prod_{n \geq 0} (n\omega + x) \prod_{m \geq 1} (m\omega - x)$$

$$= 2 \sin \left( \frac{\pi x}{\omega} \right),$$

where we use the regularized product notation $\prod$ due to Deninger [D]:

$$\prod_{\lambda} \lambda = \exp \left( -\frac{\delta}{c} \sum_{\lambda} \lambda^{-s} \bigg|_{s=0} \right).$$

As is well-known, $S_1(x, \omega)$ is invariant under $x \leftrightarrow \omega - x$, and the central value of $S_1(x, \omega)$ is the simple value $S_1 \left( \frac{\omega}{2}, \omega \right) = 2$. Similarly, the function $S_3(x, (\omega_1, \omega_2, \omega_3))$ has the symmetry $x \leftrightarrow \omega_1 + \omega_2 + \omega_3 - x$, so the central value

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is $S_3\left(\frac{\omega_1 + \omega_2 + \omega_3}{2}, (\omega_1, \omega_2, \omega_3)\right)$. This value is quite mysterious as seen from the simplest case

$$S_3\left(\frac{3}{2}, (1, 1, 1)\right) = 2^{-1/8} \exp\left(-\frac{3\zeta(3)}{16\pi^2}\right),$$

where the zeta value $\zeta(3)$ appears; see [KK].

In this paper we investigate the central value $S_3\left(\frac{\omega_1 + \omega_2 + \omega_3}{2}, (\omega_1, \omega_2, \omega_3)\right)$ for general $\omega_1, \omega_2, \omega_3 > 0$. The first result is the explicit expression:

**Theorem 1.**

$$S_3\left(\frac{\omega_1 + \omega_2 + \omega_3}{2}, (\omega_1, \omega_2, \omega_3)\right) = \exp\left(-\int_0^\infty \left(\frac{1}{4} \prod_{k=1}^3 \left(\sinh\left(-\frac{\sqrt{2} \omega_k t}{\sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2}}\right)\right)^{-1} - \frac{(\omega_1^2 + \omega_2^2 + \omega_3^2)^{3/2}}{8\sqrt{2}\omega_1\omega_2\omega_3 t^3} \left(1 - \frac{t^2}{3}\right)\right) dt\right).$$

The second result is the following estimate.

**Theorem 2.**

$$0 < S_3\left(\frac{\omega_1 + \omega_2 + \omega_3}{2}, (\omega_1, \omega_2, \omega_3)\right) < 1.$$

We obtain an application of Theorem 2:

**Theorem 3.**

$$\prod_{i=1}^3 S_3\left(\frac{\omega_i}{2}, (\omega_1, \omega_2, \omega_3)\right) \times \prod_{i<j} S_3\left(\frac{\omega_i + \omega_j}{2}, (\omega_1, \omega_2, \omega_3)\right) > 2.$$  

Using Theorems 2 and 3 we see the behavior of the triple sine function $S_3(x, (\omega_1, \omega_2, \omega_3))$ in the fundamental domain $0 \leq x \leq \omega_1 + \omega_2 + \omega_3$:

**Theorem 4.** *The graph of $S_3(x, (\omega_1, \omega_2, \omega_3))$ is as in Fig. 1. It is symmetric with respect to the line $x = \frac{\omega_1 + \omega_2 + \omega_3}{2}$, and it has three extremal values: two*
maximal values larger than 1 at two points and the local minimal less than 1 at \( x = \frac{\omega_1 + \omega_2 + \omega_3}{2} \).

We also obtain the following integral expression for \( \zeta(3) \) from Theorem 1:

**Theorem 5.**

\[
\zeta(3) = \frac{16\pi^2}{3} \int_0^\infty \left( 2(e^{\sqrt{2/3}t} - e^{-\sqrt{2/3}t})^{-3} + \frac{3}{16} \sqrt{\frac{2}{3} \left( \frac{1}{t} - \frac{3}{t^3} \right)} \right) \frac{dt}{t} - \frac{2}{3} \pi^2 \log 2.
\]

We remark that the formula

\[
S_3\left( \frac{\omega_1 + \omega_2 + \omega_3}{2}, (\omega_1, \omega_2, \omega_3) \right) = H(\omega_1, \omega_2, \omega_3)^2
\]

with

\[
H(\omega_1, \omega_2, \omega_3) = \prod_{n_1, n_2, n_3 \geq 0} \left( n_1 + \frac{1}{2} \right) \omega_1 + \left( n_2 + \frac{1}{2} \right) \omega_2 + \left( n_3 + \frac{1}{2} \right) \omega_3
\]

reminds us the phenomenon that central values frequently become "squares" especially for zeta and \( L \)-functions. This is valid also for

\[
S_1\left( \frac{\omega}{2}, \omega \right) = H(\omega)^2
\]

with

![Figure 1. The graph of \( S_3(x, (\omega_1, \omega_2, \omega_3)) \).](attachment:graph.png)
Moreover, these $H(\omega_1, \omega_2, \omega_3)$ and $H(\omega)$ are considered as determinants of hamiltonians for harmonic oscillators in dimension 3 and 1 respectively. We refer to [KO] for studies from this viewpoint.

2. Integral expression: Proof of Theorem 1

We first recall needed facts on multiple Hurwitz zeta functions. The multiple Hurwitz zeta function $\zeta_r(s, x, (\omega_1, \ldots, \omega_r))$ is defined (for $\omega_1, \ldots, \omega_r > 0$ and $x > 0$) as

$$
\zeta_r(s, x, (\omega_1, \ldots, \omega_r)) = \sum_{n_1, \ldots, n_r \geq 0} (n_1\omega_1 + \cdots + n_r\omega_r + x)^{-s}.
$$

This converges absolutely in $\text{Re}(s) > r$, and Barnes [B] shows that $\zeta_r(s, x, (\omega_1, \ldots, \omega_r))$ has an analytic continuation to all $s \in \mathbb{C}$ as a meromorphic function. Moreover, it is holomorphic at $s = 0$. Hence we have the regularized product

$$
\prod_{n_1, \ldots, n_r \geq 0} (n_1\omega_1 + \cdots + n_r\omega_r + x) = \exp(-\zeta'_r(0, x, (\omega_1, \ldots, \omega_r))),
$$

where the differentiation concerns the first variable $s$. In particular, in our case, we have

$$
S_3\left(\frac{\omega_1 + \omega_2 + \omega_3}{2}, (\omega_1, \omega_2, \omega_3)\right) = \left(\prod_{n_1, \ldots, n_r \geq 0} \left(\left(n_1 + \frac{1}{2}\right)\omega_1 + \left(n_2 + \frac{1}{2}\right)\omega_2 + \left(n_3 + \frac{1}{2}\right)\omega_3\right)\right)^2
$$

$$
= \exp\left(-2\zeta'_3\left(0, \frac{\omega_1 + \omega_2 + \omega_3}{2}, (\omega_1, \omega_2, \omega_3)\right)\right).
$$

Thus we must look at $\zeta_3(s, x, (\omega_1, \omega_2, \omega_3))$ around $s = 0$. We use the Riemann-Mellin integral expression for the zeta function. Here, we show the analytic continuation of $\zeta_3(s, x, (\omega_1, \omega_2, \omega_3))$ in $\text{Re}(s) > -1$, which is sufficient for our purpose.

We start from the integral expression in $\text{Re}(s) > 3$:

$$
\zeta_3(s, x, (\omega_1, \omega_2, \omega_3)) = \frac{1}{\Gamma(s)} \int_0^{\infty} \left(\sum_{n_1, n_2, n_3 \geq 0} e^{-(n_1\omega_1 + n_2\omega_2 + n_3\omega_3)t} \right) e^{-tx} t^{s-1} dt
$$

$$
= \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{e^{-tx} t^{s-1}}{(1 - e^{-\omega_1 t})(1 - e^{-\omega_2 t})(1 - e^{-\omega_3 t})} dt.
$$
which follows from the integral expression for the gamma function $\Gamma(s)$. Hence we have

$$
\zeta_3\left(s, \frac{\omega_1 + \omega_2 + \omega_3}{2}, (\omega_1, \omega_2, \omega_3) \right) = \frac{1}{\Gamma(s)} \int_{1}^{\infty} \Theta(t, (\omega_1, \omega_2, \omega_3)) t^{s-1} \, dt
$$

in $\Re(s) > 3$ with

$$
\Theta(t, (\omega_1, \omega_2, \omega_3)) = \frac{e^{-((\omega_1 + \omega_2 + \omega_3)/2)t}}{(1 - e^{-\omega_1 t})(1 - e^{-\omega_2 t})(1 - e^{-\omega_3 t})}
\begin{align*}
&= \frac{1}{(e^{\omega_1 t/2} - e^{-\omega_1 t/2})(e^{\omega_2 t/2} - e^{-\omega_2 t/2})(e^{\omega_3 t/2} - e^{-\omega_3 t/2})} \\
&= \frac{1}{8} \prod_{k=1}^{3} \left( \sinh \left( \frac{\omega_k t}{2} \right) \right)^{-1}.
\end{align*}
$$

We remark that $\Theta(t, (\omega_1, \omega_2, \omega_3))$ is an odd function of $t$ with the Laurent expansion

$$
\Theta(t, (\omega_1, \omega_2, \omega_3)) = \frac{a_{-3}}{t^3} + \frac{a_{-1}}{t} + a_1 t + \cdots
$$

around $t = 0$, where $a_j = a_j(\omega_1, \omega_2, \omega_3)$ is a rational function of $\omega_1, \omega_2, \omega_3$. In particular

$$
a_{-3} = \frac{1}{\omega_1 \omega_2 \omega_3},
$$

$$
a_{-1} = -\frac{\omega_1^2 + \omega_2^2 + \omega_3^2}{24 \omega_1 \omega_2 \omega_3}.
$$

Now, the integral expression splits into three parts:

$$
\begin{align*}
\zeta_3\left(s, \frac{\omega_1 + \omega_2 + \omega_3}{2}, (\omega_1, \omega_2, \omega_3) \right) &= \frac{1}{\Gamma(s)} \int_{1}^{\infty} \Theta(t, (\omega_1, \omega_2, \omega_3)) t^{s-1} \, dt \\
&= \frac{1}{\Gamma(s)} \int_{1}^{\infty} \Theta(t, (\omega_1, \omega_2, \omega_3)) t^{s-1} \, dt \\
&\quad + \frac{1}{\Gamma(s)} \int_{0}^{1} \left( \Theta(t, (\omega_1, \omega_2, \omega_3)) - \frac{a_{-3}}{t^3} - \frac{a_{-1}}{t} \right) t^{s-1} \, dt \\
&\quad + \frac{1}{\Gamma(s)} \int_{0}^{1} \left( \frac{a_{-3}}{t^3} + \frac{a_{-1}}{t} \right) t^{s-1} \, dt.
\end{align*}
$$

Here, the first term is holomorphic for all $s \in \mathbb{C}$ since the integral converges absolutely. The second term is holomorphic in $\Re(s) > -1$ since

$$
\Theta(t, (\omega_1, \omega_2, \omega_3)) - \frac{a_{-3}}{t^3} - \frac{a_{-1}}{t} = O(t)
$$
as \( t \to 0 \). The third term is written as

\[
\frac{1}{\Gamma(s)} \left( \frac{a_{-3}}{s - 3} + \frac{a_{-1}}{s - 1} \right)
\]

and it is meromorphic in \( s \in \mathbb{C} \) with possible (simple) poles at \( s = 3, 1 \) only. Thus we have shown the analytic continuation of \( \zeta_3 \left( s, \frac{\omega_1 + \omega_2 + \omega_3}{2}, (\omega_1, \omega_2, \omega_3) \right) \) in \( \text{Re}(s) > -1 \), and it is holomorphic at \( s = 0 \); in fact the above calculation implies that \( \zeta_3 \left( 0, \frac{\omega_1 + \omega_2 + \omega_3}{2}, (\omega_1, \omega_2, \omega_3) \right) = 0. \)

Hence, remarking that \( \Gamma(s) \) has a zero at \( s = 0 \) with \( \frac{d}{ds} \Gamma(s)^{-1} \bigg|_{s=0} = 1 \), we see that

\[
\zeta_3' \left( 0, \frac{\omega_1 + \omega_2 + \omega_3}{2}, (\omega_1, \omega_2, \omega_3) \right) = \int_0^\infty \Theta(t, (\omega_1, \omega_2, \omega_3)) \frac{dt}{t} \]

\[
+ \int_0^1 \left( \Theta(t, (\omega_1, \omega_2, \omega_3)) - \frac{a_{-3}}{t^3} - \frac{a_{-1}}{t} \right) \frac{dt}{t} - \frac{a_{-3}}{3} - a_{-1}.
\]

Here we remark that

\[
\zeta_3 c \left( 0, \frac{c\omega_1 + c\omega_2 + c\omega_3}{2}, (c\omega_1, c\omega_2, c\omega_3) \right) = \zeta_3' \left( 0, \frac{\omega_1 + \omega_2 + \omega_3}{2}, (\omega_1, \omega_2, \omega_3) \right)
\]

for \( c > 0 \). This is seen as follows. The definition says that

\[
\zeta_3 \left( s, \frac{c\omega_1 + c\omega_2 + c\omega_3}{2}, (c\omega_1, c\omega_2, c\omega_3) \right) = c^{-s} \zeta_3 \left( s, \frac{\omega_1 + \omega_2 + \omega_3}{2}, (\omega_1, \omega_2, \omega_3) \right),
\]

so we have

\[
\zeta_3' \left( 0, \frac{c\omega_1 + c\omega_2 + c\omega_3}{2}, (c\omega_1, c\omega_2, c\omega_3) \right)
\]

\[
= \zeta_3' \left( 0, \frac{\omega_1 + \omega_2 + \omega_3}{2}, (\omega_1, \omega_2, \omega_3) \right)
\]

\[
- (\log c) \zeta_3 \left( 0, \frac{\omega_1 + \omega_2 + \omega_3}{2}, (\omega_1, \omega_2, \omega_3) \right)
\]

\[
= \zeta_3' \left( 0, \frac{\omega_1 + \omega_2 + \omega_3}{2}, (\omega_1, \omega_2, \omega_3) \right).
\]

from \( \zeta_3 \left( 0, \frac{\omega_1 + \omega_2 + \omega_3}{2}, (\omega_1, \omega_2, \omega_3) \right) = 0. \)
Take
\[ c = \frac{2\sqrt{2}}{\sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2}}, \]
and put
\[ (\tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3) = (c\omega_1, c\omega_2, c\omega_3). \]

Then using \( \tilde{\omega}_1^2 + \tilde{\omega}_2^2 + \tilde{\omega}_3^2 = 8 \) we have
\[
\zeta_3^f \left( 0, \frac{\omega_1 + \omega_2 + \omega_3}{2}, (\omega_1, \omega_2, \omega_3) \right) = \zeta_3^f \left( 0, \frac{\tilde{\omega}_1 + \tilde{\omega}_2 + \tilde{\omega}_3}{2}, (\tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3) \right) = \int_1^\infty \frac{\Theta(t, (\tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3))}{t} \, dt + \int_0^1 \left( \frac{\Theta(t, (\tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3))}{t} - \frac{1}{\tilde{\omega}_1 \tilde{\omega}_2 \tilde{\omega}_3 t^3} \left( 1 - \frac{t^2}{3} \right) \right) \, dt.
\]

Thus we see that
\[
\zeta_3^f \left( 0, \frac{\omega_1 + \omega_2 + \omega_3}{2}, (\omega_1, \omega_2, \omega_3) \right) = \int_1^\infty \frac{\Theta(t, (\tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3))}{t} \, dt + \int_0^1 \left( \frac{\Theta(t, (\tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3))}{t} - \frac{1}{\tilde{\omega}_1 \tilde{\omega}_2 \tilde{\omega}_3 t^3} \left( 1 - \frac{t^2}{3} \right) \right) \, dt
\]

since
\[
\int_1^\infty \frac{1}{t^3} \left( 1 - \frac{t^2}{3} \right) \, dt = 0.
\]

This proves Theorem 1. \( \blacksquare \)

3. Estimates: Proof of Theorem 2

Let
\[ \tilde{\omega}_k = \frac{2\sqrt{2}\omega_k}{\sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2}}, \]
as in the proof of Theorem 1. To prove Theorem 2 it is sufficient to show that
\[
\int_0^1 \left( \prod_{k=1}^3 (e^{\tilde{\omega}_k t/2} - e^{-\tilde{\omega}_k t/2})^{-1} - \frac{1}{\tilde{\omega}_1 \tilde{\omega}_2 \tilde{\omega}_3 t^3} \left( 1 - \frac{t^2}{3} \right) \right) \, dt > 0
\]
since
\[
\int_1^\infty \left( \prod_{k=1}^3 \left( e^{i\hat{w}_k t/2} - e^{-i\hat{w}_k t/2} \right) \right)^{-1} \frac{1}{\hat{w}_1 \hat{w}_2 \hat{w}_3 t^3} \left( 1 - \frac{t^3}{3} \right) dt / t > 0.
\]

Now, we prove the inequality

\[
(\ast) \quad \prod_{k=1}^3 \left( e^{i\hat{w}_k t/2} - e^{-i\hat{w}_k t/2} \right)^{-1} > \frac{1}{\hat{w}_1 \hat{w}_2 \hat{w}_3 t^3} \left( 1 - \frac{t^3}{3} \right)
\]

for \(0 < t \leq 1\). First we show the following two inequalities:

\[
(1) \quad \left( e^{i\omega t/2} - e^{-i\omega t/2} \right)^{-1} \geq \frac{1}{\omega t} \left( 1 - \frac{\omega^2 t^2}{24} \right)
\]

for \(0 < \omega < 2\sqrt{2}\) and \(0 < t \leq 1\).

\[
(2) \quad (1 - au)(1 - bu)(1 - cu) > 1 - u
\]

for \(a, b, c > 0\) with \(a + b + c = 1\) and \(0 < u < 1\).

**Proof of (1).** Taylor expansion shows that

\[
e^{i\omega t/2} - e^{-i\omega t/2} = i\omega t \sum_{n=0}^\infty \frac{\omega^{2n}}{(2n+1)!2^{2n}} t^{2n}
\]

\[
\leq i\omega t \sum_{n=0}^\infty \left( \frac{\omega^2}{24} \right)^n t^{2n} = \frac{i\omega t}{1 - \frac{\omega^2 t^2}{24}}
\]

where we used the easy fact

\[
(2n+1)! \geq 6^n
\]

for \(n = 0, 1, 2, \ldots\).

**Proof of (2).** Since

\[
(1 - au)(1 - bu)(1 - cu) = 1 - u + abc u^2 \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} - u \right),
\]

it is sufficient to check that

\[
\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 1.
\]
Actually, the stronger inequality
\[
\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 9
\]
follows from the famous inequality
\[
(a + b + c) \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \geq 9
\]
with \(a + b + c = 1\).

Proof of (\(*\)). By using (1) we have
\[
\prod_{k=1}^{3} \left( e^{\omega_k t/2} - e^{-\omega_k t/2} \right)^{-1} > \frac{1}{\omega_1 \omega_2 \omega_3 t^3} \left( 1 - \frac{\omega_1^2 t^2}{24} \right) \left( 1 - \frac{\omega_2^2 t^2}{24} \right) \left( 1 - \frac{\omega_3^2 t^2}{24} \right)
\]
since \(0 < \omega_k < 2\sqrt{2}\) from \(\omega_1^2 + \omega_2^2 + \omega_3^2 = 8\). On the other hand, (2) shows that
\[
\left( 1 - \frac{\omega_1^2 t^2}{24} \right) \left( 1 - \frac{\omega_2^2 t^2}{24} \right) \left( 1 - \frac{\omega_3^2 t^2}{24} \right) > 1 - \frac{t^2}{3}
\]
since
\[
\frac{\omega_1^2}{8} + \frac{\omega_2^2}{8} + \frac{\omega_3^2}{8} = 1.
\]
This proves (\(*\)). Thus we have shown Theorem 2. \(\blacksquare\)

4. An application: Proof of Theorem 3

We recall the following result proved in [KK]:
\[
\prod_{k_1, \ldots, k_r} S_r \left( \frac{k_1 \omega_1 + \cdots + k_r \omega_r}{N}, (\omega_1, \ldots, \omega_r) \right) = N
\]
for each integer \(N \geq 2\). Especially, letting \(N = 2\) and \(r = 3\), we have
\[
\prod_{i=1}^{3} S_3 \left( \frac{\omega_i}{2}, (\omega_1, \omega_2, \omega_3) \right) \prod_{i<j} S_3 \left( \frac{\omega_i + \omega_j}{2}, (\omega_1, \omega_2, \omega_3) \right)
\times S_3 \left( \frac{\omega_1 + \omega_2 + \omega_3}{2}, (\omega_1, \omega_2, \omega_3) \right) = 2.
\]
Hence we see that
\[
\prod_{i=1}^{3} S_3\left(\frac{\omega_i}{2}, (\omega_1, \omega_2, \omega_3)\right) \prod_{i<j} S_3\left(\frac{\omega_i + \omega_j}{2}, (\omega_1, \omega_2, \omega_3)\right)
\]
\[
= \frac{2}{S_3\left(\frac{\omega_1 + \omega_2 + \omega_3}{2}, (\omega_1, \omega_2, \omega_3)\right)}
\]
\[
> 2
\]
from Theorem 2.

5. Proof of Theorem 4

Put \( f(x) = S_3(x, \omega) \) for simplicity, and we restrict \( x \) to \( 0 < x < |\omega| \) hereafter. Since
\[
f(x) = \exp\left( -\left( \frac{\partial}{\partial s} \zeta_3 \right)(s, x, \omega) \right)_{s=0} - \left( \frac{\partial}{\partial s} \zeta_3 \right)(s, |\omega| - x, \omega) \right)_{s=0},
\]
we have the following formulas:

\[
\log f(x) = -\left( \frac{\partial}{\partial s} \zeta_3 \right)(s, x, \omega)\bigg|_{s=0} - \left( \frac{\partial}{\partial s} \zeta_3 \right)(s, |\omega| - x, \omega)\bigg|_{s=0},
\]
\[
\frac{f'}{f}(x) = -\left( \frac{\partial^2}{\partial s^2} \zeta_3 \right)(s, x, \omega)\bigg|_{s=0} + \left( \frac{\partial^2}{\partial s^2} \zeta_3 \right)(s, |\omega| - x, \omega)\bigg|_{s=0},
\]
\[
\left( \frac{f'}{f} \right)'(x) = -\left( \frac{\partial^3}{\partial s^3} \zeta_3 \right)(s, x, \omega)\bigg|_{s=0} - \left( \frac{\partial^3}{\partial s^3} \zeta_3 \right)(s, |\omega| - x, \omega)\bigg|_{s=0},
\]
\[
\left( \frac{f'}{f} \right)''(x) = -\left( \frac{\partial^4}{\partial s^4} \zeta_3 \right)(s, x, \omega)\bigg|_{s=0} + \left( \frac{\partial^4}{\partial s^4} \zeta_3 \right)(s, |\omega| - x, \omega)\bigg|_{s=0},
\]
\[
\left( \frac{f'}{f} \right)'''(x) = -\left( \frac{\partial^5}{\partial s^5} \zeta_3 \right)(s, x, \omega)\bigg|_{s=0} - \left( \frac{\partial^5}{\partial s^5} \zeta_3 \right)(s, |\omega| - x, \omega)\bigg|_{s=0}.
\]

Using
\[
\left( \frac{\partial}{\partial s} \zeta_3 \right)(s, x, \omega) = -s\zeta_3(s + 1, x, \omega)
\]
we have
\[
\left( \frac{\partial^4}{\partial s^4} \zeta_3 \right)(s, x, \omega) = s(s + 1)(s + 2)(s + 3)\zeta_3(s + 4, x, \omega),
\]
so we get
\[
\left( \frac{\xi^5}{\xi_3^5 4^{4/3}} \right) (s, x, \omega) \bigg|_{s=0} = 6\xi_3 (4, x, \omega) = 6 \sum_{n \geq 0} (n \cdot \omega + x)^{-4} > 0.
\]

Thus, we know that

\[
\left( \frac{f''}{f} \right)^{\prime}'(x) = -6 \left( \sum_{n \geq 0} (n \cdot \omega + x)^{-4} + \sum_{m \geq 1} (m \cdot \omega - x)^{-4} \right)
< 0.
\]

Noting

\[
\left( \frac{f''}{f} \right)^{\prime}' \left( \frac{\omega}{2} \right) = 0
\]

from the symmetry (see the above formula for \( (f'/f)^{\prime}' \)), we see the shape of the graph of \( (f'/f)^{\prime}' \) as in Fig. 2:

![Figure 2. The graph of \( (f'/f)^{\prime}' \)(x).](image)

We remark a key observation

\[
\left( \frac{f'}{f} \right)^{\prime}' \left( \frac{\omega}{2} \right) > 0.
\]

In fact, otherwise we see that

\[
\left( \frac{f'}{f} \right)^{\prime}(x) \leq 0.
\]
for $0 < x < |\omega|$ from the behavior of $(f'/f)''$. This implies that the shapes of the graphs of $f'/f$, $\log f$ and $f$ are as in Fig. 3, since we already know that $\log f \left(\frac{|\omega|}{2}\right) < 0$ from Theorem 2.

Especially, this consideration shows that $0 < f(x) < 1$. This consequence contradicts to Theorem 3, since at least one of six values $S_3 \left(\frac{\omega_i}{2}\right)$ and $S_3 \left(\frac{\omega_i + \omega_j}{2}\right)$ are larger than 1 from Theorem 3.

Thus we know that $(f'/f) \left(\frac{|\omega|}{2}\right) > 0$. Hence we see the true shapes of $(f'/f)'$, $f'/f$, $\log f$ and $f$ as in Fig. 4.

This shows Theorem 4.
6. Special case: Proof of Theorem 5

Theorem 1 says in the special case $(\omega_1, \omega_2, \omega_3) = (1, 1, 1)$ that

$$S_3\left(\frac{3}{2}, (1, 1, 1)\right) = \exp\left(-\int_0^\infty \left(\frac{1}{4} \sinh\left(\frac{2}{3t}\right) - \frac{3\sqrt{3}}{8\sqrt{2t^3}} \left(1 - \frac{t^2}{3}\right)\right) \frac{dt}{t}\right)$$

$$= \exp\left(-\int_0^\infty \left(2(e^{\sqrt{2/3t}} - e^{-\sqrt{2/3t}})^3 + \frac{3}{16} \sqrt{\frac{2}{3}} \left(1 - \frac{3}{t^3}\right)^3\right) \frac{dt}{t}\right).$$

Hence, using the result

$$S_3\left(\frac{3}{2}, (1, 1, 1)\right) = 2^{-1/8} \exp\left(-\frac{3\zeta(3)}{16\pi^2}\right)$$

proved in [KK], we obtain Theorem 5.

References


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