HARMONIC MAPS FROM THE RIEMANN SPHERE INTO THE
COMPLEX PROJECTIVE SPACE AND THE HARMONIC
SEQUENCES

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Abstract

When harmonic maps from the Riemann sphere into the complex projective
space are energy bounded, it contains a subsequence converging to a bubble tree
map \( f^I : T^I \rightarrow \mathbb{C}P^n \). We show that their \( \hat{\epsilon} \)-transforms and \( \hat{\delta} \)-transforms are also
energy bounded. Hence their subsequences converge to harmonic bubble tree maps
\( f^h_i : T^h_i \rightarrow \mathbb{C}P^n \) and \( f^{I,1}_j : T^{I,1} \rightarrow \mathbb{C}P^n \) respectively. In this paper, we show relations
between \( f^I, f^h_i \) and \( f^{I,1}_j \).

1. Introduction

In [12], Sacks & Uhlenbeck have shown that any harmonic maps defined on
a closed surface with bounded energy contains a subsequence weakly converging
to a set of harmonic maps and that a bubbling phenomenon may occur in the
convergence. Gromov ([6]) also noticed a bubbling phenomenon in the study of
pseudo holomorphic maps.

In this paper, we concentrate on harmonic maps from the Riemann sphere
\( S^2, g_0 \) into the complex projective space \( \mathbb{C}P^n, g \). Here we identify \( S^2, g_0 \) with
\( \mathbb{C}P^1, g \) and consider it as the complex manifold. Combining the resuls by Eells &
Wood in [4, §6] with Wolfson in [14], for each full harmonic map \( f : S^2 \rightarrow \mathbb{C}P^n \),
we get a harmonic sequence

\[
\text{seq}(f, r) : 0 \leftarrow f_0 \rightarrow f_1 \rightarrow \cdots \rightarrow f_r \rightarrow \cdots \rightarrow f_n \rightarrow 0
\]

with \( f_r = f \).

Let \( \mathcal{Harm}(\mathbb{C}P^n) \) be the set of harmonic maps in a Banach manifold
\( W^{1,p}(S^2, \mathbb{C}P^n) \) for \( p > 2 \). Refining the “Sacks-Uhlenbeck” limit, Parker & Wolf-
son ([11]) give a definition of “converging to a harmonic bubble tree map”. Though
their definition in [11] is for pseudo-holomorphic maps, as mentioned in it, the definition is applicable for harmonic maps. In [11] and [10], they have

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shown that, in this sense, harmonic maps with bounded energy contain a sequence converging to a harmonic bubble tree map satisfying appropriate conditions. Our main result is the following. As for details of notations or terminologies, we will define in the following sections.

Main Theorem. Let $S^2, g_0$ be the Riemann sphere and $\mathbb{C}P^n, g$ be the complex projective space. Take a sequence $\{f^k\}_k$ in $\mathcal{H}\text{arm}(\mathbb{C}P^n)$ which are energy bounded. Then both $\{\partial f^k\}_k$ and $\{\bar{\partial} f^k\}_k$ are also energy bounded. Passing through subsequences, $\{f^k\}_k, \{\partial f^k\}_k$ and $\{\bar{\partial} f^k\}_k$ converge to either trivial maps or harmonic bubble tree maps

$$f^I = \bigvee_{\ell \in I} f^{(\ell)} : T^I \to \mathbb{C}P^n$$

$$f^I_1 = \bigvee_{\ell' \in I_1} f^{(\ell')} : T^I_1 \to \mathbb{C}P^n$$

$$f^{I}_{-1} = \bigvee_{\ell'' \in I_{-1}} f^{(\ell'')} : T^{I}_{-1} \to \mathbb{C}P^n$$

respectively satisfying the followings:

(1) If $\partial f^{(\ell)}$ is non-trivial, it is equivalent to $f^{(\ell')}_1$ for some $\ell' \in I_1$; $f^{(\ell')}_1 = \partial f^{(\ell)} \circ \sigma_\ell$ satisfying $\sigma_\ell(B_{f^{(\ell')}_1}) \subset B_{f^{(\ell')}}$.

(2) When $f^{(\ell')}_1$ is not equivalent to any $\partial f^{(\ell)}$, $f^{(\ell')}_1$ is a holomorphic map of the length no greater than $n - r - 1$.

(3) If $\bar{\partial} f^{(\ell)}$ is non-trivial, it is equivalent to $f^{(\ell'')}_{-1}$ for some $\ell'' \in I_{-1}$; $f^{(\ell'')}_{-1} = \bar{\partial} f^{(\ell)} \circ \bar{\sigma}_\ell$ with $\bar{\sigma}_\ell(B_{f^{(\ell'')}_{-1}}) \subset B_{f^{(\ell'')}}$.

(4) When $f^{(\ell'')}_{-1}$ is not equivalent to any $\bar{\partial} f^{(\ell)}$, $f^{(\ell'')}_{-1}$ is an anti-holomorphic map of the length no greater than $r - 1$.

Here $r + 1$ is the $\bar{\partial}$-order of $f$.

Here and throughout this paper, to simplify notation, we adopt the convention of immediately renaming subsequences and so a subsequence of $\{f^k\}$ is still denoted by the same way.

Contents are as follows. In §2, we begin to introduce harmonic maps defined on $S^2, g_0$ into $\mathbb{C}P^n, g$. Associated to each harmonic map, we consider its harmonic sequence. We refer related results. In §3, we define a harmonic bubble tree map introduced by Parker & Wolfson in [11]. Then we show Main Theorem. In §4, we consider when harmonic maps into either $\mathbb{C}P^1$ or $\mathbb{C}P^2$ are gluable. Lastly, in §5, we consider examples of gluable or non-gluable harmonic bubble tree maps and their harmonic sequences.

2. A harmonic map and a harmonic sequence

Let $\mathbb{C}^{n+1}$ be the complex $(n + 1)$-dimensional space equipped with the standard Hermitian inner product defined by

$$X \cdot Y = \sum_j x_j \bar{y}_j \text{ where } X = (x_j)_{0 \leq j \leq n}, Y = (y_j)_{0 \leq j \leq n} \in \mathbb{C}^{n+1}.$$
Put $|X| = \sqrt{X \cdot X}$. We equip the Fubini-Study metric $g$ on $\mathbb{C}P^n$ of constant holomorphic sectional curvature 4. As for the geometry of $\mathbb{C}P^n$, refer [7, IX. 6, Example 6.3]. When $n = 1$, we get an isomorphism $S^2 \approx \mathbb{C}P^1$ through a stereographic projection

$$S^2 - \{ \infty \} \to \mathbb{C} \cong U_0 = \{ [z_0 : z_1] \in \mathbb{C}P^1 | z_0 \neq 0 \} = \mathbb{C}P^1 - \{ [0 : 1] \}$$

which takes the north pole to the origin, the south pole $\infty$ to infinity, and the equator to the unit circle. Here $[z_0 : z_1]$ is the homogeneous coordinate system of $\mathbb{C}P^1$. Let $S^2$, $g_0$ be the sphere with the Reimann metric $g_0$ induced from $\mathbb{C}P^1$. As mentioned in §1, we also equip the complex structure on $S^2$ induced from $\mathbb{C}P^1$. On a coordinate neighbourhood $U_0$, the metric $g_0$ is customary represented by $ds^2_0 = \varphi \overline{\varphi} = \frac{dz d\overline{z}}{(1 + |z|^2)^2}$ for $z = \frac{z_1}{z_0} \in \mathbb{C} \approx U_0$. Here $\varphi$ is determined up to a complex factor of absolute value 1.

Throughout this paper, take and fix a real $p > 2$. As $1 \leq \frac{2}{p}$, we can get a Banach manifold $W^{1,p}(S^2, \mathbb{C}P^n)$ consisting of maps $f : S^2 \to \mathbb{C}P^n$ whose derivatives of order $\leq 1$ are $L_p$ integrable. A map $f \in W^{1,p}(S^2, \mathbb{C}P^n)$ is harmonic if it is a critical point of the energy functional $E : W^{1,p}(S^2, \mathbb{C}P^n) \to \mathbb{R}$ defined by

$$E(f) = \int_{S^2} |df|^2 \frac{\sqrt{-1}}{2} \varphi \wedge \overline{\varphi}$$

where $|df|^2$ is the Hilbert-Schmidt norm $\langle g_0, f^* g \rangle_{HS}$. Thus we consider the set $\mathcal{Harm}(\mathbb{C}P^n)$ of harmonic maps as a subspace of $W^{1,p}(S^2, \mathbb{C}P^n)$. Because of the regularity and the Sobolev embedding theorem $C^0 \to W^{1,p}$, $\mathcal{Harm}(\mathbb{C}P^n)$ is contained in the set $C^s(S^2, \mathbb{C}P^n)$ of all $C^s$ maps for any $s \geq 0$. Since $f$ is defined between Kähler manifolds, any holomorphic or anti-holomorphic map is harmonic. Refer [9] and also [3, (8.15) Corollary]. Denote by $\mathcal{Hol}(\mathbb{C}P^n)$ the subspace of $\mathcal{Harm}(\mathbb{C}P^n)$ consisting of holomorphic maps.

Now we introduce a $\partial$ transform and a $\overline{\partial}$ transform in [1] which is the same correspondence given in [4, §3]. For a smooth map $f : S^2 \to \mathbb{C}P^n$, let $\pi_f : V(f) \to S^2$ be the tautological complex line bundle whose fiber at $z \in S^2$ is $f(z)$. For a C-line $X$ in $\mathbb{C}^{n+1}$, denote by $X_\perp$ the orthogonal complement of $X$ in $\mathbb{C}^{n+1}$. Define a smooth map $f^\perp : S^2 \to G(n, n + 1)$ by $f^\perp(z) = f(z)_\perp$. Here $G(n, n + 1)$ is the complex Grassmann manifold consisting of $n$-dimensional subspaces in $\mathbb{C}^{n+1}$. We equip the standard Riemann metric $g_n$ and the complex structure on it. Refer [7, IX, Example 6.4]. $f^\perp$ also defines the tautological bundle $V(f^\perp) \to S^2$. By [1, §2], both $V(f)$ and $V(f^\perp)$ are holomorphic bundles over $S^2$.

Take a unitary frame $Z_0, Z_1, \ldots, Z_n$ of $\mathbb{C}^{n+1}$ so that $Z_0$ defines $f$. Then put

$$dZ_0 = \omega_0 Z_0 + \sum_{r \geq 1} \omega_r Z_r, \quad f^* \omega_r = a_r \varphi + b_r \overline{\varphi}$$
and define maps

$$\delta : V(f) \to V(f^\perp) \otimes T^{(1,0)}, \quad \delta(\xi^0 Z_0) = \left(\xi^0 \sum_r a_r Z_r\right) \otimes \varphi,$$

$$\bar{\delta} : V(f) \to V(f^\perp) \otimes T^{(0,1)}, \quad \bar{\delta}(\xi^0 Z_0) = \left(\xi^0 \sum_r b_r Z_r\right) \otimes \bar{\varphi}.$$ 

Here $T^{(1,0)}$ (resp. $T^{(0,1)}$) is the cotangent bundle on $S^2$ of type $(1,0)$ (resp. $(0,1)$). We get the followings.

**Theorem 1** ([1], §2). If $f \in Harm(CP^n)$, $\delta$ is a holomorphic bundle map and $\bar{\delta}$ is an anti-holomorphic bundle map.

Denote by $[V(f)]$ the projectivization of $V(f)$. Though $\varphi$ is determined only up to a complex factor of absolute value 1, we get the fundamental collinination of $f$

$$[V(f)] \ni [f(z)] \to [\delta f(z)] \in [V(f^\perp)]$$

if $\delta f(z) \neq 0$. As mentioned in [1, §2], when $f$ is harmonic, by Theorem 1, we can get a well-defined non-trivial map $\delta f : S^2 \to CP^n$ as far as $f$ is not anti-holomorphic. We call it the $\delta$ transform of $f$. When $f$ is anti-holomorphic, we define the $\delta$ transform of $f$ as a zero map. Similarly we also get the fundamental collinination

$$[V(f)] \ni [f(z)] \to [\bar{\delta} f(z)] \in [V(f^\perp)]$$

if $\bar{\delta} f(z) \neq 0$. If $f$ is not holomorphic, this defines a non-trivial map $\bar{\delta} f : S^2 \to CP^n$ which we call the $\bar{\delta}$ transform of $f$. When $f$ is holomorphic, the $\bar{\delta}$ transform of $f$ is defined as a zero map.

**Theorem 2** ([1], Theorem 2.2). Take $f \in Harm(CP^n)$. Then we get the followings.

1. $f^\perp : S^2 \to G(n, n+1)$ is harmonic.
2. Both the $\delta$ transform of $f$ and its $\bar{\delta}$ transform are harmonic.
3. If $\delta f$ is non-trivial, $\delta \delta f = f$.
4. If $\bar{\delta} f$ is non-trivial, $\bar{\delta} \bar{\delta} f = f$.

We say that $f \in Harm(CP^n)$ is full if its image lies in no proper projective subspace of $CP^n$. Associated to a full map $f \in Hol(CP^n)$, take a lift $Z : S^2 \supset U \to C^{n+1} - \{0\}$ over a chart $U$. Classically we get the Frenet frame $\{Z_r\}_{r \geq 0}$ of $f$ which is obtained by the Gram-Schmidt’s orthogonalization of $\{\frac{x^r}{r!} Z\}_r$ except at finite point of $S^2$. Since the zeros of

$$Z \wedge \frac{\partial}{\partial z} Z \wedge \cdots \wedge \frac{\partial^n}{\partial z^n} Z$$
are finite and are removable, this frame can be uniquely extended over \( S^2 \). Refer [15, §4]. We get
\[
dZ_r = -a_{r-1}gZ_{r-1} + \omega_r Z_r + a_r \phi Z_{r+1}
\]
for \( 0 \leq r \leq n \) where \( a_{-1} = a_n = 0 \). For \( 0 \leq r \leq n \), let \( f_r : S^2 \to CP^n \) be the non-
trivial map defined by \( Z_r \). By definition, \( f_{r+1} \) is the \( \bar{\partial} \) transform of \( f_r \) and \( f_{r-1} \) is the \( \partial \) transform of \( f_r \). Hence, by Theorem 2, \( f_r \) is harmonic for any \( r \). We

call the sequence of harmonic maps
\[
\text{seq}(f, 0) : 0 \overset{\partial}{\longrightarrow} f_0 \overset{\partial}{\longrightarrow} f_1 \overset{\partial}{\longrightarrow} \cdots \overset{\partial}{\longrightarrow} f_r \overset{\partial}{\longrightarrow} \cdots \overset{\partial}{\longrightarrow} f_n \overset{\partial}{\longrightarrow} 0
\]
a harmonic sequence of \( f \) with the length \( n \).

When \( f \in Hol(CP^n) \) is not full, we can choose an isometry \( T^A : CP^n \to CP^n \)
induced from a unitary transformation \( A : C^{n+1} \to C^{n+1} \) so that
\[
f = T^A \circ i \circ f^A : S^2 \to CP^n \subseteq CP^n \overset{T^A}{\to} CP^n
\]
by a full \( f^A \in Hol(CP^n) \) and the inclusion \( i \). We define a harmonic sequence of
\( f \) of the length \( n_0 \)
\[
\text{seq}(f, 0) : 0 \overset{\partial}{\longrightarrow} f_0 \overset{\partial}{\longrightarrow} f_1 \overset{\partial}{\longrightarrow} \cdots \overset{\partial}{\longrightarrow} f_r \overset{\partial}{\longrightarrow} \cdots \overset{\partial}{\longrightarrow} f_{n_0} \overset{\partial}{\longrightarrow} 0
\]
by compositions \( f_r = T^A \circ i \circ f^A \);
\[
\text{seq}(f^A, 0) : 0 \overset{\partial}{\longrightarrow} f^A_0 \overset{\partial}{\longrightarrow} f^A_1 \overset{\partial}{\longrightarrow} \cdots \overset{\partial}{\longrightarrow} f^A_r \overset{\partial}{\longrightarrow} \cdots \overset{\partial}{\longrightarrow} f^A_{n_0} \overset{\partial}{\longrightarrow} 0.
\]
Here \( \text{seq}(f, 0) \) is defined independently on the choice of a unitary matrix \( A \). Following to [4, Definition 5.1], we define the \( \partial \)-order of \( f \in Harm(CP^n) \) by
\[
\max_U \max_z \dim \text{span}\{\partial^\alpha Z_U(z) \mid 0 \leq \alpha\}
\]
and also the \( \bar{\partial} \)-order of \( f \) by
\[
\max_U \max_z \dim \text{span}\{\overline{\partial^\beta} Z_U(z) \mid 0 \leq \beta\}.
\]
Here \( Z_U : S^2 \supseteq U \to C^{n+1} \) is a lift of \( f \) over a chart \( U \) and \( \text{span}\{v^k\}_z \) is the
subspace of \( C^{n+1} \) spanned by vectors \( \{v^k\}_z \). These orders are determined inde-
pendently on the choice of a lift \( Z_U \). By [14, Theorem 3.1 & Theorem 3.4], we
get the following.

**Theorem 3.** For any non-trivial \( f \in Harm(CP^n) \), we get \( f_0 \in Hol(CP^n) \) so
that the harmonic sequence of \( f_0 \) contains \( f \);
\[
\text{seq}(f, r) : 0 \overset{\partial}{\longrightarrow} f_0 \overset{\partial}{\longrightarrow} f_1 \overset{\partial}{\longrightarrow} \cdots \overset{\partial}{\longrightarrow} f_r \overset{\partial}{\longrightarrow} \cdots \overset{\partial}{\longrightarrow} f_{n_0} \overset{\partial}{\longrightarrow} 0
\]
where \( 1 \leq n_0 \leq n \), \( r + 1 \) is the \( \partial \)-order of \( f \) and \( n_0 - r + 1 \) is its \( \partial \)-order.

We also call \( \text{seq}(f, r) \) the harmonic sequence of \( f \) with the length \( n_0 \). Obviously \( f \in Harm(CP^n) \) is full exactly when the length of \( \text{seq}(f, r) \) is \( n \). Let \( Harm^*(CP^n) \) be the subspace of \( Harm(CP^n) \) consisting of full maps. Correspondingly \( Hol^*(CP^n) \) is denoted for the space of all full maps in \( Hol(CP^n) \).
Essentially, by [4, Theorem 6.9], we get the following. Theorem 3 gives the correspondence of the following theorem.

**Theorem 4.** There is a bijective correspondence between \( f \in \mathcal{H}arm^*(\mathbb{C}P^n) \) and pairs \((f_0, r)\) where \( f_0 \in \mathcal{H}ol^*(\mathbb{C}P^n) \) and \( r \) is an integer with \( 0 \leq r \leq n \).

For a smooth map \( f \in W^{1,p}(S^2, \mathbb{C}P^n) \), we denote by \( c_1(f) \) the first Chern number of the tautological bundle \( V(f) \to S^2 \). By [14], we get the followings.

**Lemma 2.1 [14, §2 & §3].** For \( f \in \mathcal{H}ol^*(\mathbb{C}P^n) \), choose the Frenet frame \( \{Z_r\}_r \) of \( f \) and put

\[
\begin{aligned}
dZ_r &= -\alpha_{r-1}\phi Z_{r-1} + \omega_r Z_r + \alpha_r \phi Z_{r+1} \\
E(f_r) &= \int (|\alpha_{r-1}|^2 + |\alpha_r|^2) \frac{\sqrt{-1}}{2} \phi \wedge \bar{\phi}, \\
c_1(f_r) &= \frac{1}{\pi} \int (|\alpha_{r-1}|^2 - |\alpha_r|^2) \frac{\sqrt{-1}}{2} \phi \wedge \bar{\phi}.
\end{aligned}
\]

Denote by \( R_\partial(f) \) the ramification index of \( \partial : V(f) \to V(\partial f) \otimes T^{(1,0)} \) which is the number of zeros of \( \partial \) counted according to multiplicity. Similarly \( R_\partial(f) \) is the ramification index of \( \partial : V(f) \to V(\partial f) \otimes T^{(0,1)} \). As for the following lemma, we refer [5] and also [14, §3].

**Lemma 2.2.** For \( f \in \mathcal{H}arm(\mathbb{C}P^n) \), if \( \partial f \) is non-trivial, we get

\[
c_1(\partial f) = c_1(f) + R_\partial(f) + 2.
\]

When \( \partial f \) is non-trivial, we also get

\[
c_1(\partial f) = c_1(f) - R_\partial(f) - 2.
\]

By Lemma 2.1 and 2.2, we get the following inequalities.

**Lemma 2.3 [14, Theorem 3.1].** For \( f \in \mathcal{H}ol^*(\mathbb{C}P^n) \), choose the Frenet frame \( \{Z_r\}_r \) as in Lemma 2.1. Then we get the followings for any \( r \).

\[
\begin{align*}
(1) \sum_{r+1 \leq q \leq n} \sum_{r \leq s \leq q-1} R_\partial(f_s) &< \frac{1}{2} E(f_r) + (n + 1) \cdot |c_1(f_r)|, \\
(2) \sum_{q \leq r-1} \sum_{q \leq s \leq r-1} R_\partial(f_s) &< \frac{1}{2} E(f_r) + (n + 1) \cdot |c_1(f_r)|.
\end{align*}
\]

3. Harmonic bubble tree maps

It is well-known that \( \mathcal{H}arm(\mathbb{C}P^n) \) may be non-compact with respect to \( W^{1,2} \)-topology. To consider this bubbling phenomenon, we refer Parker & Wolfson ([11]) and Parker ([10]).

Let \( TS^2 \to S^2 \) be the complex tangent bundle over the complex manifold \( S^2 \), \( g_0 \). Compactifying each vertical fiber, we get a bundle \( \Sigma(S^2) \to S^2 \) with fibers
$S_2 = S^2$ where we identify $z$ of $S_2$ with the south pole $\infty$ of $S^2$ and equip the complex structure on $\Sigma (S^2)$. By the induction on $k \geq 1$, we define a bundle

$$\Sigma^k(S^2) := \Sigma (\Sigma^{k-1}(S^2)) \rightarrow \Sigma^{k-1}(S^2).$$

A bubble domain at level $k$ is a fiber $S^k_z = S^2$ of $\Sigma^k(S^2) \rightarrow \Sigma^{k-1}(S^2)$ and a bubble domain tower is a union $T^I = \bigcup_{\ell \in I} S^{(\ell)}$ of the base space $S^{(0)}$ of $\Sigma(S^2) \rightarrow S^2$ and finite number of bubble domains $S^{(\ell)}$ ($\ell \in I$, $\ell \geq 1$) with

$$\pi_{\ell} : \Sigma S^{(\ell)} \supset S^{k_{\ell}} = \pi_{\ell}^{-1} (z_\ell) \rightarrow z_\ell \in S^{(\ell)}.$$

We denote by $\infty_\ell$ the south pole of $S^{(\ell)}$. Motivated by Parker [10], if a map $f^I = \bigcup_{\ell \in I} f^{(\ell)} : T^I \rightarrow \mathbb{C}P^n$ consists of non-trivial maps $f^{(\ell)}$ satisfying $f^{(\ell)}(\infty_\ell) = f^{(\ell)}(z_\ell)$ when $\pi_{\ell}^{-1} (z_\ell) = S^{(\ell)}$, we call $f^I$ a bubble tree map, $f^{(0)}$ a base map, $f^{(\ell)}$ a bubble map for $\ell \in I - \{0\}$ and $z_\ell \in S^{(\ell)}$ a bubble point of $f^{(\ell)}$. Denote by $B_{f^{(\ell)}}$ the set of bubble points of $f^{(\ell)}$.

We call $f^I$ a harmonic bubble tree map if $f^{(\ell)}$ is a harmonic map for each $\ell \in I$. Similarly we call $f^I$ a holomorphic (resp. an anti-holomorphic) bubble tree map if $f^{(\ell)}$ is holomorphic (resp. anti-holomorphic) for any $\ell \in I$.

We say that a sequence $\{f^k\}_{k \geq 1}$ in $\mathbb{H}arm(\mathbb{C}P^n)$ converges to a harmonic bubble tree map $f^I : T^I \rightarrow \mathbb{C}P^n$ if each $f^k$ defines a bubble tree map $f^{k,I} = \bigcup_{\ell \in I} f^{k,\ell} : T^I \rightarrow \mathbb{C}P^n$ by the iterated renormalization procedure and if $\{f^{k,I}\}_k$ converges to $f^I$ uniformly in $C^0 \cap W^{1,2}$ and uniformly in $C^r$ ($r \geq 1$) on any compact set of $T^I - \bigcup_{\ell} \{\infty_\ell\} \cup B_f^{(\ell)}$. Here $f^{k,\ell} = f^k \circ \sigma_{k,\ell}$ by a fractional linear transformation $\sigma_{k,\ell}$ of $S^{(\ell)} = S^2$ fixing the south pole on a compact set of $S^{(\ell)} - \{\infty_\ell\} \cup B_f^{(\ell)}$ for $k$ large enough. For details, refer [11, §4]. By [10, Theorem 2.2 & Corollary 2.3], we get the following.

**Theorem 5.** Let $\{f^k\}_k$ be a sequence in $\mathbb{H}arm(\mathbb{C}P^n)$ with $\sup_k E(f^k) < \infty$. Then a subsequence converges to a harmonic bubble tree map $f^I = \bigcup_{\ell} f^{(\ell)} : T^I \rightarrow \mathbb{C}P^n$ satisfying

$$\lim_k E(f^k) = \sum_{\ell} E(f^{(\ell)})$$

and $\alpha = \lim_k c_1(f^k) = \sum_{\ell} c_1(f^{(\ell)})$.

For $\mathbb{C}P^n$, $g$, we get a constant $B_0$ so that any $f \in \mathbb{H}arm(\mathbb{C}P^n)$ with $E(f) < 2B_0$ is trivial (refer [12]). We choose $B_0$ as a scaling constant. Put $H^+ = \{z \mid |z| \geq 1\} \subset \mathbb{C}$. By the choice of the translation and the rescaling in the renormalization, if a sequence of harmonic maps converges to a harmonic bubble tree map $f^I = \bigcup_{\ell} f^{(\ell)} : T^I \rightarrow \mathbb{C}P^n$, each bubble map $f^{(\ell)}$ is parametrized satisfying

$$(BC) \quad \int_{H^+} |df^{(\ell)}|^2 \frac{\sqrt{-1}}{2} d\bar{z}dz = B_0$$

and $B_{f^{(\ell)}}$ is contained in the northern hemisphere of $f^{(\ell)}$ when $\ell \neq 0$. 

In the case of $CP^n$, the map
\[ c_1: \pi_2(CP^n) \simeq H_2(CP^n; \mathbb{Z}) \rightarrow \mathbb{Z} \]
defined by $c_1([f]) = c_1(f)$ is an isomorphism. Let $\mathcal{H}arm_a(CP^n)$ be the subspace of $\mathcal{H}arm(CP^n)$ consisting of $f$ with $c_1(f) = a$. For each $a \in \mathbb{Z}$, $\mathcal{H}arm_a(CP^n)$ is non-empty. By Theorem 5, if $\{f^k\}_k$ in $\mathcal{H}arm(CP^n)$ converges to a harmonic bubble tree map, $f^k \in \mathcal{H}arm_a(CP^n)$ for any $k$ large enough.

**Lemma 3.1.** Let $\{f^k\}_k$ be a sequence in $\mathcal{H}arm_a(CP^n)$ with $E(f^k) \leq E$ for any $k$. Then we get
\[ E(\hat{\partial}f^k) + E(\hat{\partial}\bar{f}^k) \leq 4E + 2\pi(2 + (n + 3) \cdot |a|) \]
for any $k$.

**Proof.** By Lemma 2.2, $c_1(\hat{\partial}f^k) = c_1(f^k) + R_c(f^k) + 2$ and, by Lemma 2.3,
\[ R_c(f^k) < \frac{1}{\pi}E(f^k) + (n + 1)|c_1(f^k)|. \]
Hence
\[ |c_1(\hat{\partial}f^k)| \leq |c_1(f^k)|(n + 2) + 2 + \frac{E}{\pi}. \]
As $c_1(f^k) = a$, by Lemma 2.1, we get
\[ E(\hat{\partial}f^k) = E(f^k) - \pi c_1(f^k) - \pi c_1(\hat{\partial}f^k) \leq 2E + \pi(2 + (n + 3)|a|). \]
As for $E(\hat{\partial}\bar{f}^k)$, we can show similarly.

We say that $f_0$ is equivalent to $f_1$ in $W^{1,p}(S^2, CP^n)$ if $f_1 = f_0 \circ \sigma$ by some linear fractional transformation $\sigma: S^2 \rightarrow S^2$ fixing the south pole. We also say that $\tilde{f}^I = \bigwedge_{\ell \in I} \tilde{f}^{(\ell)}: \tilde{T}^I \rightarrow CP^n$ is equivalent to $f^I = \bigwedge_{\ell \in I} f^{(\ell)}: T^I \rightarrow CP^n$ if $\tilde{f}^{(\ell)}$ is equivalent to $f^{(\ell)}$;
\[ \tilde{f}^{(\ell)} = f^{(\ell)} \circ \sigma_\ell : \tilde{S}^{(\ell)} = \tilde{S}^{(\ell)}_1 \overset{\sigma_\ell}{\rightarrow} S^{(\ell)} = S^{(\ell)}_1 \overset{f^{(\ell)}}{\rightarrow} CP^n \]
with $\sigma_\ell(z_\ell) = z_\ell$ for each $\ell \in I$. Here $\tilde{T}^I = \bigwedge_{\ell \in I} \tilde{S}^{(\ell)}$ and $\sigma_0$ is necessary the identity.

Now we begin to show Main Theorem. As $E(f^k) \leq E$ for any $k$, a subsequence of $\{f^k\}_k$ converges and so we can assume that $f^k \in \mathcal{H}arm_a(CP^n)$ for any $k$. Hence, by Lemma 3.1, we get $E(\hat{\partial}f^k) \leq E_1$ for any $k$. Therefore, passing through subsequences, both $\{f^k\}_k$ and $\{\hat{\partial}f^k\}_k$ converge to $f^I = \bigwedge_{\ell \in I} f^{(\ell)}: T^I \rightarrow CP^n$ and $f^I_{\ell_0} = \bigwedge_{\ell \in I_0} f_{\ell_0}^{(\ell)}: T_{\ell_0} \rightarrow CP^n$ respectively. More precisely, consider the renormalization $f^{k, I} = \bigwedge_{\ell} f_{\ell}^{k, \ell}: T^I = \bigwedge_{\ell \in I} S^{(\ell)} \rightarrow CP^n$ of $f^k$ converging to $f^I = \bigwedge_{\ell} f^{(\ell)}$. Put $f_{k, I} = \hat{\partial}f^k$ and consider again its renormalization
\[ f_{k, I}^{1, I_0} = \bigwedge_{\ell_0 \in I_0} f_{k, I}^{1, \ell_0}: T_{\ell_0} = \bigwedge_{\ell_0 \in I_0} S^{(\ell_0)} \rightarrow CP^n \]
Hence we can get f on a geodesic disc D whose subsequence converges to f(\ell) which is either equal to non-trivial \( \partial f (0) \) or equivalent to non-trivial \( \partial f (\ell) \) for some \( \ell \in I - \{0\} \). On the other hand, if \( \partial f (\ell) \) is non-trivial, we can get f(\ell) equivalent to \( \partial f (\ell) \). As the convergence of \( \{ f_k(\ell) \} \) is with respect to C\( ^s \)-norm for any \( s \geq 0 \), if \( f_k(\ell) = \partial f (\ell) \circ \sigma_{\ell} \), we get

\[
\sigma_{\ell}(B_{f_k(\ell)}) < B_{f_1(\ell)}.
\]

Now suppose that \( \{ f_k(\ell) \} \) converges to f(\ell) which is not equivalent to any \( \partial f (\ell) \) for \( \ell \in I \). As \( \sigma_{\ell} f_k(\ell) : S^2 \to S^2 \) is a holomorphic map given by

\[
z = \sigma_{\ell} f_k(\ell)(w) = \sigma_{\ell} f_k(\ell) w + \beta_{\ell} f_k(\ell),
\]

\( \overline{\sigma_{\ell} z} \) is a non-zero constant \( \sigma_{\ell} f_k(\ell) \). Hence

\[
\overline{\partial f_{\ell} f_k(\ell)} = \overline{\partial (\partial f_k(\ell) \circ \sigma_{\ell} f_k(\ell))} = \overline{\partial \partial f_k(\ell) \circ \sigma_{\ell} f_k(\ell)} = f_k(\ell) \circ \sigma_{\ell} f_k(\ell)
\]

on D' for any k large enough where the constant \( \sigma_{\ell} f_k(\ell) \) vanishes because of the homogeneous coordinate. A subsequence of \( \{ \overline{\partial f_{\ell} f_k(\ell)} \} \) converges to zero on D'. By the uniqueness continuation theorem ([13]), this means the holomorphicity of f_1(\ell). The length of f_1(\ell) is obvious.

By replacing \( \partial \) transform with \( \overline{\partial} \) transform, we can show the corresponding assertion. This completes the proof of Main Theorem.

4. \( \text{Harm}_2(\mathbb{C}P^1) \) and \( \text{Harm}_2(\mathbb{C}P^2) \)

We say that a harmonic bubble tree map \( f^I : T^I \to \mathbb{C}P^n \) is gluable if a sequence of harmonic maps converges to a harmonic bubble tree map \( \tilde{f}^I : T^I \to \mathbb{C}P^n \) equivalent to \( f^I : T^I \to \mathbb{C}P^n \). Firstly we consider the case when \( n = 1 \). Note that any map in \( \text{Harm}(\mathbb{C}P^1) \) is either holomorphic or anti-holomorphic.

**Lemma 4.1.** Let \( f^I = \bigcup_{\ell \in I} f(\ell) : T^I \to \mathbb{C}P^1 \) be a holomorphic bubble tree map. Then \( \partial f^I = \bigcup_{\ell \in I} \partial f(\ell) \) is a well-defined anti-holomorphic bubble tree map defined on \( T^I \). If \( f^I \) is gluable, so is \( \partial f^I \).

**Proof.** Let \( f = [p_0 : p_1] \in \text{Hol}(\mathbb{C}P^1) \) be non-trivial where \( p_0 \) and \( p_1 \) have no common zero. Then, by calculations,

\[
\partial f = [(p_1 p_0' - p_0' p_0)p_1 : -(p_1 p_0' - p_0' p_0)p_0].
\]

If \( p_1 p_0' - p_0' p_0 = 0 \) on a domain, \( p_0 \equiv K \cdot p_1 \) and so we deduce a contradiction. Hence \( \partial f = [p_1 : -p_0] \).
Now take a holomorphic bubble tree map \( f^I = \bigvee_{\ell \in I} f^{(\ell)} : T^I \to \mathbb{C}P^1 \). As shown above, when \( f^{(\ell)}(\infty) = f^{(\ell')}(\infty), \partial f^{(\ell)}(\infty) = \partial f^{(\ell')}(\infty) \). This shows the first assertion.

When a sequence \( \{f^k\}_k \) in \( \mathcal{H}ol(\mathbb{C}P^1) \) converges to \( f^I \), by calculations, \( \{\partial f^k\}_k \) converges to \( \partial f^I \).

Now we consider the case when \( n = 2 \). We start to refer results of existence theorems. Denote by \( \mathcal{H}ol_{2,r}(\mathbb{C}P^2) \) the subspace of \( \mathcal{H}ol(\mathbb{C}P^2) \) consisting of \( f \) with \( c_1(f) = \pi \) and \( R_2(f) = r \). We also put \( \mathcal{H}ol^r_{2,r}(\mathbb{C}P^2) = \mathcal{H}ol_{2,r}(\mathbb{C}P^2) \cap \mathcal{H}ol^r(\mathbb{C}P^2) \). Obviously \( \mathcal{H}ol^r_{2,r}(\mathbb{C}P^2) = \mathcal{H}ol_{2,r}(\mathbb{C}P^2) \) if \( 2 \pi + r + 2 < 0 \).

We also consider the subspace \( \mathcal{H}arm_{2,E}(\mathbb{C}P^2) \) of \( \mathcal{H}arm(\mathbb{C}P^2) \) consisting of \( f \) with \( E(f) = \pi E \). Note that any map in \( \mathcal{H}arm_{2,E}(\mathbb{C}P^2) \) is full when \( E \neq 0 \), \(|\pi| \). We get the followings.

**Theorem 6** ([2], Lemma 1.3 & Theorem 1.4). For \( 0 \leq r \leq -\frac{3}{2} \pi - 3 \), \( \mathcal{H}ol_{2,r}(\mathbb{C}P^2) \) is a smooth connected complex submanifold of \( \mathcal{H}ol(\mathbb{C}P^2) \) of complex dimension \( 2 - 3 \pi - r \). Moreover there is a homeomorphism

\[
\mathcal{H}ol_{2,r}(\mathbb{C}P^2) \ni f \to \partial f \in \mathcal{H}arm_{2+2+r,-(3\pi+r+2)}(\mathbb{C}P^2).
\]

**Remark 4.1.** By [8, Proposition 2.7], \( \mathcal{H}ol_{2,r}(\mathbb{C}P^2) \) is non-empty exactly when \( 0 \leq r \leq -\frac{3}{2} \pi - 3 \).

As for the gluing, we get the following.

**Proposition 4.2.** Let \( f^I = \bigvee_{\ell \in I} f^{(\ell)} : T^I \to \mathbb{C}P^2 \) be a harmonic bubble tree map with \( f^{(\ell)} \in \mathcal{H}arm_{2,E}(\mathbb{C}P^2) \) and \( E \neq |\pi| \) for any \( \ell \in I \). If \( f^I \) is gluable, both \( \partial f^I = \bigvee_{\ell \in I} \partial f^{(\ell)} : T^I \to \mathbb{C}P^2 \) and \( \tilde{\partial} f^I = \bigvee_{\ell \in I} \tilde{\partial} f^{(\ell)} : T^I \to \mathbb{C}P^2 \) are well-defined gluable bubble tree maps.

**Proof.** If necessary, replace \( f^I \) by an equivalent harmonic bubble tree map (which we denote by the same way) and take a sequence \( \{f^k\}_k \) in \( \mathcal{H}arm(\mathbb{C}P^2) \) converging to \( f^I \). Without loss of generality, we can assume that \( f^k \in \mathcal{H}arm_{2,E}(\mathbb{C}P^2) \) with \( E \neq |\pi| \) for any \( k \). We get a harmonic sequence

\[
\text{seq}(f^k, 1) : 0 \overset{\partial}{\leftarrow} \partial f^k \overset{\partial}{\leftarrow} f^k \overset{\partial}{\leftarrow} \partial f^k \overset{\partial}{\leftarrow} 0.
\]

Passing through a subsequence, \( \{\partial f^k\}_k \) converges to \( f^I_1 = \bigvee_{\ell \in I} f^{(\ell)} : T^I \to \mathbb{C}P^2 \). When \( f^{(\ell')} \) is not equivalent to any \( f^{(\ell)} \), by Main Theorem, \( f^{(\ell')} \) is a holomorphic map whose \( \partial \) transform is trivial. Then \( f^{(\ell')} \) becomes trivial and we deduce a contradiction. By the assumption, \( \partial f^{(\ell)} \) is non-trivial. Hence \( I_1 = I \) and each \( f^{(\ell)} \) is equivalent to \( \partial f^{(\ell)} \). This implies that \( \partial f^I = \bigvee_{\ell \in I} \partial f^{(\ell)} \) is a well-defined anti-holomorphic bubble tree map. Moreover, by Main Theorem again, \( \partial f^I \) is defined on \( T^I \). Similarly we can show the corresponding result for \( \tilde{\partial} f^I \).
For any $l$, $\sum_{\ell} R_{\ell}(f^{(\ell)}) = r - 2 \times (|I| - 1)$.

Here $|I|$ is denoted for the number of elements of $I$.

**Proof.** Since $E(\partial f^k) = (-3x - 2 - r) \pi > 0$, by Main Theorem, a subsequence of $\{\partial f^k\}_k$ converges to $f^i_1 : T^i_1 \to \mathbb{C}P^2$.

Firstly we note that $\partial f^{(\ell)}_k$ is non-trivial. As $E(f^k) = \sum_{\ell} E(f^{(\ell)})$, by Lemma 2.1 and Lemma 2.2, we get

$$E(\partial f^k) - \sum_{\ell} E(\partial f^{(\ell)}) = \pi \left\{ \sum_{\ell} R_{\ell}(f^{(\ell)}) - R_{\ell}(f^k) + 2 \cdot |I| - 2 \right\} \geq 0.$$  

Here this is equal to zero exactly when $\{\partial f^k\}_k$ converges to a bubble tree map equivalent to a well-defined bubble tree map $\partial f^i_1$. 

5. **Example**

In this section, we show examples to consider relations between a harmonic bubble tree map $f^I$ and its $\partial$ transform. We consider the case when $n = 2$.

For any $f \in \mathcal{H}ol(CP^2)$, put $f = [p_0 : p_1 : p_2]$ where $[p_0 : p_1 : p_2]$ are homogeneous coordinates of $CP^2$. Put $h_f = [h_0 : h_1 : h_2]$ where

$$(h_0, h_1, h_2) = (p_1^0 p_2 - p_1 p_2^0, -p_0^0 p_2 + p_0 p_2^0, p_0^0 p_1 - p_0 p_1^0).$$

When $p_0, p_1, p_2$ have no common zeros, $R_3(f)$ is the number of common zeros of three holomorphic maps $h_0, h_1, h_2$ as far as $2 \cdot \max_j \deg p_j - 2 = \max_j \deg h_j$.

For details, refer [2, §2].

From now on, we denote by $T^I = S^{(0)} \vee S^{(1)}$ the bubble domain tower defined by the base space $S^{(0)} = S^3$ and a bubble domain $S^{(1)} = \pi^{-1} \circ \Sigma S^{(0)}$. Denote by

$$\mathcal{H}ol_{-2,0}(CP^2) \ast \mathcal{H}ol_{-2,0}(CP^2)$$

the set of holomorphic bubble tree maps $f^I = f^{(0)} \vee f^{(1)} : T^I \to \mathbb{C}P^2$ with $f^{(\ell)} \in \mathcal{H}ol_{-2,0}(CP^2)$ for $\ell = 0, 1$. Since $\mathcal{H}ol_{-2,0}(CP^2) = \mathcal{H}ol_{-2,0}(CP^2)$, by Theorem 6, $\mathcal{H}ol_{-2,0}(CP^2)$ is a complex manifold of the complex dimension 8.

**Example 5.1.** Take $f^I \in \mathcal{H}ol_{-2,0}(CP^2) \ast \mathcal{H}ol_{-2,0}(CP^2)$. By Proposition 4.3, if a sequence of $f_k \in \mathcal{H}ol_{-4,2}(CP^2)$ converges to $f^I$, a subsequence of $\{\partial f_k\}_k$...
converges to a harmonic bubble tree map equivalent to \( \partial f^I := \partial f^{(0)} \lor \partial f^{(1)} : T^I \to \mathbb{C}P^2 \).

In this case, we also get \( E(\partial^2 f_k) = 4\pi \) and \( E(\partial^2 f^{(\ell)}(\cdot)) = 2\pi \) for \( \ell = 0, 1 \). Hence, by Main Theorem, passing through a subsequence, \( \{ \partial^2 f_k \}_k \) converges to a harmonic bubble tree map equivalent to \( \partial^2 f^I := \partial^2 f^{(0)} \lor \partial^2 f^{(1)} : T^I \to \mathbb{C}P^2 \).

**Example 5.2.** Let \( f^I = f^{(0)} \lor f^{(1)} : T^I \to \mathbb{C}P^2 \) be the holomorphic bubble tree map defined by

\[
f^{(0)}(z) = [1 : z : z^2], \quad f^{(1)}(z) = [z^2 : z : 1].
\]

As \( R_\ell(f^{(\ell)}) = 0 \) for \( \ell = 0, 1 \), \( f^I \in \mathcal{H}ol_{-2,0}(\mathbb{C}P^2) \ast \mathcal{H}ol_{-2,0}(\mathbb{C}P^2) \).

A sequence of harmonic maps

\[
f_R(z) = \left[ 1 : z + \frac{1}{R^2} : z^2 + \frac{1}{R^2z^2} \right]
\]

converges to a holomorphic bubble tree map equivalent to \( f^I : T^I \to \mathbb{C}P^2 \). By calculations, we also get \( R_\ell(f_R) = 2 \). Hence, by Proposition 4.3, \( \{ \partial f_R \}_R \) converges to a harmonic bubble tree map equivalent to a well-defined harmonic bubble tree map \( \partial f^I : T^I \to \mathbb{C}P^2 \). Moreover

\[
\partial^2 f_R(z) = \left[ z^2 + \frac{1}{R^2z^2} + \frac{4}{R} : -2z - \frac{2}{Rz} : 1 \right]
\]

converge to an anti-holomorphic bubble tree map equivalent to a well-defined \( \partial^2 f^I : T^I \to \mathbb{C}P^2 \). In fact, by using “Mathematica Ver.6.0”, we can calculate

\[
\partial f^{(0)}(z) = [-z - 2zz^2 : 1 - z^2z^2 : 2z + z^2z], \quad \partial f^{(1)}(z) = [2z + z^2 : 1 - z^2z^2 : -z - z^2z^2]
\]

and

\[
\partial^2 f^{(0)}(z) = [z^2 : -2z : 1], \quad \partial^2 f^{(1)}(z) = [1 : -2z : z^2].
\]

Hence both \( \partial f^I = \partial f^{(0)} \lor \partial f^{(1)} : T^I \to \mathbb{C}P^2 \) and \( \partial^2 f^I = \partial^2 f^{(0)} \lor \partial^2 f^{(1)} : T^I \to \mathbb{C}P^2 \) are well-defined.

**Example 5.3.** We consider a bubble tree map \( f^I = f^{(0)} \lor f^{(1)} : T^I \to \mathbb{C}P^2 \) defined by

\[
f^{(0)}(z) = [1 : z^2 : z], \quad f^{(1)}(z) = [z^2 : z : 1]
\]

which is contained in \( \mathcal{H}ol_{-2,0}(\mathbb{C}P^2) \ast \mathcal{H}ol_{-2,0}(\mathbb{C}P^2) \). For \( R > 1 \) large enough, we define holomorphic maps \( f_R \in W^{1,p}(S^2, \mathbb{C}P^2) \) by

\[
f_R(z) = \left[ 1 : z^2 + \frac{1}{R^2} : z + \frac{1}{R^2z^2} \right]
\]
which converge to a holomorphic bubble tree map equivalent to \( f^I : T^I \to \mathbb{C}P^2 \) if \( R \to \infty \). Here \( R_i(f_R) = 0 \) and so \( \{ \partial f_R \}_R \) does not converge to a harmonic map equivalent to \( \partial f^I \). In fact, we calculate \( \partial f^I(\cdot) \) to get

\[
\partial f^{(0)}(z) = [-z - 2zz^2 : 2z + z^2 z : 1 - z^2 z^2], \\
\partial f^{(1)}(z) = [2z + z^2 z : 1 - z^2 z^2 : -\bar{z} - 2zz^2]
\]

where

\[
\partial f^{(0)}(0) = [0 : 0 : 1], \quad \partial f^{(1)}(\infty) = [0 : 1 : 0].
\]

Hence these cannot define a bubble tree map on \( T^I \). In fact, when \( R \to +\infty \), \( \partial f_R \) converge to a harmonic bubble tree map

\[
f^{I}_{1} = \partial f^{(0)} \vee f^{(0)}_{1} \vee f^{(1)}_{1} : T^{I} = S^{(0)} \vee S^{(0)} \vee S^{(1)} \to \mathbb{C}P^2
\]

where \( f^{(1)}_{1} \) is equivalent to \( \partial f^{(1)} \) and the map \( f^{(0)}_{1} : S^{(0)} \to \mathbb{C}P^2 \) is equivalent to \( f_{1}^{(0)} \),

\[
f_{1}^{(0)}(z) = [0 : 1 : -z^2].
\]

Since the center of mass of \( f^{(0)}_{1} \) is the north pole, we can define \( T^{I} \) by

\[
S^{(0)} = S^{(0)}_0 \subset \Sigma S^{(0)}, \quad S^{(1)} = S^{(0)}_0 \subset \Sigma S^{(0)}
\]

and choose \( f^{(0)}_{1} \) with \( f^{(0)}_{1}(0) = f_{1}^{(0)}(0) \). We have

\[
E(\partial f_R) = E(\partial f^{(0)}) + E(\partial f^{(1)}) + E(f^{(0)}_{1}) = 10\pi.
\]

When \( R \to +\infty \), \( \partial^2 f_R \) given by

\[
\partial^2 f_R(z) = [1 - 2R^2z^3 + 4Rz^3 + R^3z^6 : -2Rz + R^3z^4 : R^2z^2 - 2R^3z^5]
\]

\[
= \left[ \frac{z^3}{R^2} - \frac{2}{R} + \frac{1}{R^3z^3} : z - \frac{2}{R^2z^2} : -2z^2 + \frac{1}{Rz} \right]
\]

\[
= \left[ \frac{1}{R} + \frac{4}{R^3z^3} - \frac{2}{R^2z^3} + \frac{1}{R^4z^6} : \frac{1}{R^2z^2} - \frac{2}{Rz} : \frac{1}{R^3z^5} - \frac{2}{R^2z^4} + \frac{1}{Rz} \right]
\]

also converges to an anti-holomorphic bubble tree map

\[
f^{I}_{2} = \partial^2 f^{(0)} \vee f^{(0)}_{2} \vee f^{(1)}_{2} : T^{I} \to \mathbb{C}P^2
\]

where \( f^{(1)}_{2} \) and \( f^{(0)}_{2} \) are equivalent to \( \partial^2 f^{(1)} \) and \( \partial f^{(0)}_{1} \) respectively;

\[
\partial^2 f^{(0)}(z) = [\bar{z}^2 : 1 : -2\bar{z}], \\
\partial f^{(0)}_{1}(z) = [0 : \bar{z}^2 : 1], \\
\partial^2 f^{(1)}(z) = [1 : -2\bar{z} : \bar{z}^2].
\]

These satisfy \( E(\partial^2 f_R) = E(\partial^2 f^{(0)}) + E(\partial f^{(0)}_{1}) + E(\partial^2 f^{(1)}) = 6\pi. \)
Example 5.4. Let $f^I = f^{(0)} \vee f^{(1)} \in \mathcal{H}ol_{-2,0}(\mathbb{C}P^2) \ast \mathcal{H}ol_{-2,0}(\mathbb{C}P^2)$ be defined by
\[
 f^{(0)}(z) = [p_0 : p_1 : p_2] = [1 : z^2 : 1 + z] \quad \text{and} \quad f^{(1)}(z) = [q_0 : q_1 : q_2] = [z^2 : 1 + z + z^2].
\]
We can get $f_R \in \mathcal{H}ol_{-4}^+(\mathbb{C}P^2)$ defined by
\[
 f_R(z) = \left[ 1 : \frac{1}{R}z^2 + z^2 : 1 + z + \frac{1}{Rz} \right]
\]
converging to a harmonic bubble tree map equivalent to $f^I$ when $R \to +\infty$. Since $R^2(f_R) = 2$, by Proposition 4.3, both $\{\partial f_R\}_R$ and $\{\partial^2 f_R\}_R$ converge to harmonic bubble tree maps equivalent to well-defined bubble tree maps $\partial f^I : T^I \to \mathbb{C}P^2$ and $\partial^2 f^I : T^I \to \mathbb{C}P^2$ respectively.

For $p(z) = a_0 + a_1z + a_2z^2$ and $q(z) = b_0 + b_1z + b_2z^2$, put $|p - q| = \sum_k |a_k - b_k|$.

Lemma 5.1. Let $f^I = f^{(0)} \vee f^{(1)} : T^I \to \mathbb{C}P^2$ be a holomorphic bubble tree map in Example 5.4 with
\[
 f^{(0)} = [p_0 : p_1 : p_2], \quad f^{(1)} = [q_0 : q_1 : q_2].
\]
Then, for any $\epsilon > 0$ small enough, we can choose a holomorphic bubble tree map $\tilde{f}^I = \tilde{f}^{(0)} \vee \tilde{f}^{(1)} : T^I \to \mathbb{C}P^2$ in $\mathcal{H}ol_{-2,0}(\mathbb{C}P^2) \ast \mathcal{H}ol_{-2,0}(\mathbb{C}P^2)$ with
\[
 \tilde{f}^{(0)} = [\tilde{p}_0 : \tilde{p}_1 : \tilde{p}_2], \quad \tilde{f}^{(1)} = [\tilde{q}_0 : \tilde{q}_1 : \tilde{q}_2]
\]
so that $\sum_{\ell} (|\tilde{p}_\ell - p_\ell| + |\tilde{q}_\ell - q_\ell|) < \epsilon$ and that $\partial \tilde{f}^I = \partial^2 \tilde{f}^{(0)} \vee \partial^2 \tilde{f}^{(1)} : T^I \to \mathbb{C}P^2$ is well-defined but non-gluable.

Proof. When the degrees of polynomials $p$ and $q$ are no greater than 2, we can choose $\varepsilon_0 > 0$ so that $p$ and $q$ have no common zeros as far as $|p - p_0| + |q - q_0| < \varepsilon_0$. Here $p_0(z) = 1$ and $q_0(z) = z^2$. Hence we can choose $\varepsilon > 0$ so that $\tilde{f}^{(\ell)} \in \mathcal{H}ol_{-2,0}(\mathbb{C}P^2)$ for $\ell = 0, 1$ if
\[
 \sum_{0 \leq j \leq 2} (|\tilde{p}_j - p_j| + |\tilde{q}_j - q_j|) < \varepsilon.
\]
Put
\[
 \tilde{p}_j(z) = x_{j0} + x_{j1}z + x_{j2}z^2, \quad \tilde{q}_j(z) = \beta_{j0} + \beta_{j1}z + \beta_{j2}z^2
\]
with $x_{00} = 1$ and $x_{02} = 1$. Since $x_{00} = x_{02} = 1$, $\tilde{f}^{(0)}(0) = \tilde{f}^{(1)}(\infty)$ exactly when $x_{\ell0} = 0$. Moreover the complex conjugates of $h^{(\ell)}$ is equal to $\bar{\partial}^2 h^{(\ell)}$. Since $\tilde{f}^{(\ell)}$ is full, $\bar{\partial}^2 \tilde{f}^{(\ell)}$ is non-trivial with $c_1(\partial \tilde{f}^{(\ell)}) = 2$. Moreover $\bar{\partial}^2 \tilde{f}^{(0)}(0) = \bar{\partial}^2 \tilde{f}^{(1)}(\infty)$ exactly when $x_{\ell k}$ and $\beta_{\ell k}$ additionally satisfy
\[
 \beta_{10} (x_{01} x_{20} - x_{21}) - \beta_{20} (x_{02} x_{10} - x_{11}) - \beta_{01} (x_{10} x_{21} - x_{11} x_{20}).
\]
If necessary, we rechoose \( \varepsilon > 0 \) so small that
\[
x_{20} > \frac{1}{2} \quad \text{and} \quad -\frac{1}{2} > x_{01} x_{20} - x_{21}.
\]
Denote by \( \hat{U}_e \) the set of all \((\hat{f}^{(0)}, \hat{f}^{(1)})\) whose polynomials have coefficients \(\alpha_{jk}, \beta_{jk}\) satisfying above conditions. By definition, both \(\hat{f}^I = \hat{f}^{(0)} \vee \hat{f}^{(1)}\) and \(\partial^2 \hat{f}^I = \partial^2 \hat{f}^{(0)} \vee \partial^2 \hat{f}^{(1)}\) are well-defined bubble tree maps defined on \(T^I\). Moreover, in such a case, we can calculate to show that \(\partial^2 \hat{f}^I = \partial^2 \hat{f}^{(0)} \vee \partial^2 \hat{f}^{(1)}\) are well-defined harmonic bubble tree map defined on \(T^I\).

As the complex dimension of \(\hat{U}_e\) is equal to 13 and that of \(\mathcal{H} \text{ol}_{-1/2}(\mathbb{C}P^2)\) is 12 by Theorem 6, there is \(\hat{f}^I \in \hat{U}_e\) so that \(\partial^2 \hat{f}^I\) is well-defined but not gluable. \(\square\)

**Example 5.5.** We consider a holomorphic bubble tree map which contains a non-full map. Let \( f^I = f^{(0)} \vee f^{(1)} : T^I \rightarrow \mathbb{C}P^2 \) be the holomorphic bubble tree map defined by
\[
f^{(0)}(z) = [1 : z : 0], \quad f^{(1)}(z) = [z : 1 : 1].
\]
Then \( f_R \in \mathcal{H} \text{ol}_{-2,0}(\mathbb{C}P^2) \) defined by
\[
f_R(z) = \left[ 1 : z + \frac{1}{R^2 z} : \frac{1}{R^2 z} \right]
\]
converge to \( f^I \) when \( R \rightarrow +\infty \). We get \( E(f_R) = E(f^{(0)}) + E(f^{(1)}) = 2\pi \). By calculations, we get
\[
\partial f_R(z) = \left[ \frac{-z - \frac{1}{R^2 z} + \frac{2}{R^4 z^2} + \frac{z}{R^4 z^2}}{R^4 z^2} : 1 - \frac{1}{R^2 z^2} + \frac{2}{R^6 z^2} - \frac{1}{R^4 z^2} - \frac{2}{R^2 z^2} - \frac{2\pi}{R^2 z^2} \right]
\]
which converge to
\[
f^I_1 = \partial f^{(0)} \vee f^{(0)}_1 \vee f^{(1)}_1 : T^I = S^{(0)} \vee S^{(0)}_1 \vee S^{(1)} \rightarrow \mathbb{C}P^2
\]
if \( R \rightarrow +\infty \). Here \( T^I_1 \) is the same bubble domain tower in Example 5.3 and, \( f^{(0)}_1 \) and \( f^{(1)}_1 \) are equivalent to \( f^{(0)}_1 \) and \( f^{(1)} \) respectively;
\[
\partial f^{(0)}(z) = [-z : 1 : 0], \quad f^{(0)}_1(z) = [0 : 1 - z^2 : 1], \quad \partial f^{(1)}(z) = [2 : -z : -z].
\]
In fact, making calculations, we can show that the center of mass of \( f^{(0)}_1 \) is the north pole. We also get
\[
E(\partial f_R) = E(\partial f^{(0)}) + E(\partial f^{(1)}) + E(f^{(0)}_1) = \pi + \pi + 2\pi.
\]
Moreover
\[
\partial^2 f_R(z) = \left[ -\frac{2}{R^2 z} : \frac{1}{R^2 z^2} : 1 - \frac{1}{R^2 z^2} \right]
\]
converges to \( f_{z^I} = \partial \hat{f}_1^{(01)} : T_{z^I} = S^{(01)} \to \mathbb{C}P^2; \)
\[
\partial \hat{f}_1^{(01)}(z) = [0 : 1 : -1 + z^2].
\]
In this case, \( \partial \hat{f}_1^{(01)} \) is the base map and \( E(\partial^2 f_R) = E(\partial \hat{f}_1^{(01)}) = 2\pi. \)

REFERENCES