SOME CHARACTERIZATIONS OF STEIN MANIFOLD THROUGH THE NOTION OF LOCALLY REGULAR BOUNDARY POINTS

By Joji Kajiwara

Dedicated to Professor K. Kunugi on his sixtieth birthday

Introduction.

The main purpose of the present paper is to investigate the intersection of a Cousin-I domain and a domain of holomorphy. Oka [14] proved that a domain of holomorphy in $\mathbb{C}^n$ is a Cousin-I domain, that is, a domain in which any additive Cousin's distribution has a solution. On the other hand, a Cousin-I domain in $\mathbb{C}^2$ is a domain of holomorphy from Cartan [5] and Behnke-Stein [2]. Therefore a domain in $\mathbb{C}^2$ is a Cousin-I domain if and only if it is a domain of holomorphy. Cartan [6] proved that $E=\{(z_1, z_2, z_3); |z_1|<1, |z_2|<1, |z_3|<1\} - \{(0, 0, 0)\}$ is not a domain of holomorphy but a Cousin-I domain. For any domain of holomorphy $D$ in $\mathbb{C}^3$, $E \cap D$ is a Cousin-I open set. Making use of the results of Scheja [16] and Andreotti-Grauert [1] concerning the prolongation of cohomology classes, we can construct systematically other Cousin-I domains in $\mathbb{C}^n$ which are not domains of holomorphy for $n \geq 3$. For $G=\{(z_1, z_2, z_3); |z_1|<1, |z_2|<1, |z_3|<1\} - \{(0, 0, 0)\} \times \{z_4; z_4 \leq 1/2\}$, there holds $H^p(G, \mathcal{O})=0$ from Scheja [16] where $\mathcal{O}$ is the sheaf of all germs of holomorphic functions. Therefore $G$ is not a domain of holomorphy but a Cousin-I domain. Cartan's example $E$ is a regular domain in $\mathbb{C}^3$ but the above example $G$ is not a regular domain. We say that a domain $G$ in $\mathbb{C}^n$ is exhausted by regular domains if there exists a sequence \{G_p; p=1, 2, 3, \ldots\} of regular domains $G_p$ such that $G_p \subset G_{p+1}$ $(p=1, 2, 3, \ldots)$ and $G = \bigcup_{p=0}^{\infty} G_p$. From the previous paper [12] of the author $G$ is a Cousin-I domain as it is a limit of mono-

Received May 19, 1964.
tonously increasing sequence of Cousin–I domains $G_n$. Moreover we shall prove that a domain in $C^n$ is a domain of holomorphy if and only if it can be exhausted by regular domains. This is a characterization of a domain of holomorphy by means of Cousin–I problems. This means that a regular domain in $C^n$, which is not a domain of holomorphy, is isolated in the set of regular domains in some sense.

We shall define a continuous boundary point of an open set in $C^n$ in such a way that a smooth boundary point of an open set in $C^n$ in the usual sense is a continuous boundary point. An open set $G$ in $C^n$ is called locally regular at a boundary point $z^0$ of $G$ if there exists an open neighbourhood $U$ of $z^0$ such that $G \cap U$ is regular. An open set is called locally regular if it is locally regular at each of its boundary points. We shall prove that a domain is pseudoconvex at its continuous boundary point $z^0$ if and only if $G$ is locally regular at $z^0$. Hence from the affirmative solution of the Levi problem due to Bremermann [4], Norguet [13] and Oka [15] a domain with a continuous boundary is a domain of holomorphy if and only if it is locally regular. This is a characterization of a domain of holomorphy with a continuous boundary by means of Cousin–I problems. Making use of Docquier-Grauert [8] we shall extend this fact to a domain in a Stein manifold.

§1. Domain exhausted by regular domains.

**Lemma 1.** Let $G$ be a regular domain in $C^n$. Then $D = G \cap \{z = (z_1, z_2, \ldots, z_n); z_j \in K_j, (j = s_1, s_2, \ldots, s_r)\}$ is a Cousin–I open set for any $1 \leq s_1 < s_2 < \cdots < s_r \leq n$ and for any domains $K_j$ in a complex plane $(j = s_1, s_2, \ldots, s_r)$. Especially $G$ itself is a Cousin–I domain.

**Proof.** We put $K_j^p = \{z_j; |z_j| < p\}$ for $j \notin \{s_1, s_2, \ldots, s_r\}$ and $K_j^* = K_j \cap \{z_j; |z_j| < p\}$ for $j \in \{s_1, s_2, \ldots, s_r\}$. Then $D_p = G \cap (K_1^p \times K_2^p \times \cdots \times K_r^p)$ is a Cousin–I open set for each $p$ as $G$ is a regular domain. Since $D$ is the limit of a monotonously increasing sequence of Cousin–I open sets $D_p$, $D$ is a Cousin–I open set from the previous paper [12] of the author. In the same way we can prove that $G$ itself is a Cousin–I domain.

The proof of the following Lemmas 2 and 3 is similar to the method of Hitotumatu [10].

**Lemma 2.** Let $G$ be a Cousin–I domain in $C^n$ and $H$ be an $(n-1)$-dimensional analytic plane in $C^n$. Then the inclusion map $G \cap H \to G$ induces naturally a homomorphism of $H^p(G, \mathcal{O})$ onto $H^p(G \cap H, \mathcal{O})$.

**Proof.** Without loss of generality we may suppose that $H = \{(z, w) = (z_1, z_3, \ldots, z_{n-1}, w); w = 0\}$. Let $u(z)$ be a holomorphic function in $G \cap H$. If $x^0 = (x^0, 0) = (z_1, z_2, \ldots, z_{n-1}, 0)$ is a point of $G \cap H$, there exists a neighbourhood $U(x^0) = \{(z, w); |z_j - z^0_j| < \varepsilon, |w| < \varepsilon (j = 1, 2, \ldots, n-1)\}$ of $x^0$ in $G$. If $x^0$ is a point of $G - G \cap H$, we put $U(x^0) = G - G \cap H$. If we put $m_{x^0} = u/w$ for $x^0 \in G \cap H$ and $m_{x^0} = 0$ for $x^0 \in G - G \cap H$, then $\mathcal{C} = \{(m_{x^0}, U(x^0)); x^0 \in G\}$ forms an additive Cousin’s distribution in $G$. Since
$G$ is a Cousin-I domain, there exists a meromorphic function $m$ in $G$ which is a solution of $\xi$. We put $v=wm$. For $x^0\in G\cap H$, $h=m-u/w$ is a holomorphic function in $U(x^0)$. Hence $v=wh+u$ is holomorphic in $U(x^0)$ and $v=u$ in $U(x^0)\cap H$. Hence $v$ is holomorphic and coincides with $u$ in $G\cap H$. Since $v$ is holomorphic in $G-G\cap H$, $v$ is a holomorphic function in $G$ with $v=u$ in $G\cap H$. Hence the canonical homomorphism $H^0(G,\mathbb{C})\rightarrow H^0(G\cap H,\mathbb{C})$ is surjective.

Lemma 3. Let $G$ be a domain in the space $\mathbb{C}^n$ of variables $z=(z_1, z_2, \ldots, z_n)$. Then $G$ is a domain of holomorphy if and only if the intersection $G\cap H$ of $G$ and an $l$-dimensional analytic plane $H=\{z; z_j=c_j (j=s_1, s_2, \ldots, s_l)\}$ is a Cousin-I open set for any integers $1\leq l\leq n$, $1\leq s_1<s_2<\cdots<s_{l-1}\leq n$ and complex numbers $c_j (j=s_1, s_2, \ldots, s_{l-1})$.

Proof. Since a domain of holomorphy is a Cousin-I domain from Oka [14] and the intersection of a domain of holomorphy and an analytic plane is an open set of holomorphy, it suffices to prove the sufficiency by induction with respect to $n$. For $n=1$ any domain is a domain of holomorphy from Weierstrass' theorem. For $n=2$ any domain is a domain of holomorphy if and only if it is a Cousin-I domain from Oka [14], Cartan [5] and Behnke-Stein [2]. Suppose that our assertion is valid for $n<k (k\geq 2)$. We consider the case $n=k$. Let $z^0=(z^0_1, z^0_2, \ldots, z^0_n)$ be any boundary point of $G$. Two cases (1) and (2) may occur. In the case (1) there exists $j$ such that $z^0_j$ is a boundary point of $G\cap H$ for $H=\{z; z_j=c_j\}$. In the case (2) $z^0$ is not a boundary point of $G\cap H$ for $H=\{z; z_j=c_j\}$ for any $j$.

Case (1) Since $G\cap H$ is an open set of holomorphy in $H$ from the assumption of our induction, there exists a holomorphic function $u$ in $G\cap H$ which is unbounded at $z^0$. From Lemma 2 there exists a holomorphic function $v$ in $G$ with $v=u$ in $G\cap H$. $v$ is a holomorphic function in $G$ which is unbounded at $z^0$. Case (2) We shall prove that there exists a sequence $\{z^p; p=1, 2, 3, \ldots\}$ of $z^p\in \partial G\cap U$ such that each $z^p$ has the property as in the case (1) and $z^p\rightarrow z^0$ when $p\rightarrow \infty$. If this is true, there exists $\varepsilon>0$ such that $G\cap U\cap \{z; z_j=\zeta_j\}=U \cap \{z; z_j=\zeta_j\}$ for $U=\{z; |z_j-z_j^0|<\varepsilon (j=1, 2, \ldots, k)\}$ and for any $j$ and $\zeta\in G\cap U$. Let $z^1=(z_1^1, z_2^1, \ldots, z_n^1)$ be any point of $G\cap U$ and $z^2=(z_1^2, z_2^2, \ldots, z_k^2)$ be any point of $U$. By induction we can prove that $(z_1^m, z_2^m, \ldots, z_n^m, z_{n+1}^m, \ldots, z_k^m)\in G\cap U$ for $1\leq m\leq k$. Therefore we have $z^m\in G\cap U$. Hence it holds that $G\cap U=U$. This means that $z^0$ is an interior point of $G$. But this is a contradiction. Therefore there exists a sequence $\{f_p; p=1, 2, 3, \ldots\}$ of holomorphic functions $f_p$ in $G$ which is unbounded at $z^0$ tending to $z^0$ when $p\rightarrow \infty$. From Bochner-Martin [3] there exists a holomorphic function which is unbounded at $z^0$.

Thus we have proved the existence of a holomorphic function in $G$ which is unbounded at $z^0$. Since $z^0$ is any boundary point of $G$, there exists a holomorphic function $f$ in $G$ which is unbounded at each boundary point of $G$ from Bochner-Martin [3]. Hence $G$ is a domain of holomorphy of $f$. Quite similarly we can prove that a domain $G$ in the space $\mathbb{C}^n$ of variables $z=(z_1, z_2, \ldots, z_n)$ is a domain of holomorphy if and only if the canonical homomor-
phism of $H^0(G, \mathcal{O})$ into $H^0(G \cap H, \mathcal{O})$ is surjective for any analytic plane $H$ as in Lemma 3. This is a characterization of a domain of holomorphy.

**Lemma 4.** If a domain $G$ in $\mathbb{C}^n$ is exhausted by regular domains, then the intersection $G \cap H$ of $G$ and an $l$-dimensional analytic plane $H=(z=(z_1, z_2, \ldots, z_n); z_j=c_j (j=s_1, s_2, \ldots, s_{n-l}))$ is a Cousin-I open set for any integers $1 \leq l \leq n, 1 \leq s_1 < s_2 < \ldots < s_{n-l} \leq n$ and complex numbers $c_j (j=s_1, s_2, \ldots, s_{n-l})$.

**Proof.** There exists a sequence $\{G_p; p=1, 2, 3, \ldots\}$ of regular domains $G_p$ such that $G_p \subseteq G_{p+1}$ ($p=1, 2, 3, \ldots$) and $G= \bigcup_{p=1}^{\infty} G_p$. We may suppose that $H=(z, w)=(z_1, z_2, \ldots, z_n, w_1, w_2, \ldots, w_{n-l}); w_j=0 (j=1, 2, \ldots, n-l)$. There exists $\varepsilon_p>0$ such that $E_p=G_p \cap \{(z, w); |w_j|<\varepsilon_p; (j=1, 2, \ldots, n-l)\} \subset \{(z, w); |w_j|<\varepsilon_p, (z, 0)\in G \cap H (j=1, 2, \ldots, n-l)\}$ for any $p$. Since $G_p$ is regular, $E_p$ is a Cousin-I open set from Lemma 1. Let $C=\{(m_i, V_i); i=I\}$ be an additive Cousin's distribution in $G \cap H$. If we put $V^p=(G_p \cap \{(z, w); |w_j|<\varepsilon_p, (z, 0)\in V_i (j=1, 2, \ldots, n-l)\})$ and $M^p(z, w)=m_i(z)$ in $V^p$, then $C_p=\{(M^p, V^p_i); i=I\}$ is an additive Cousin's distribution in $E_p$. Since $E_p$ is a Cousin-I open set, $C_p$ has a solution $M^p(z, w)$ for any $p$. Since the set of all poles of $M^p(z, w)$ does not contain connected components of $G_p \cap H$ for any $p$, the restriction $M^p(z, w)$ of $M^p(z, w)$ to $G_p \cap H$ is a solution of the restriction $\{(m_i; G_p \cap H, V_i \cap G_p); i=I\}$ of $C$ to $G_p \cap H$ for any $p$. Since the canonical homomorphism of $H^1(G \cap H, \mathcal{O})$ into $\lim_{p \to \infty} H^1(G_p \cap H, \mathcal{O})$ is injective (Lemma 6 in the previous paper [12] of the author), $C$ has a solution in $G \cap H$. Therefore $G \cap H$ is a Cousin-I open set.

**Proposition 1.** A domain $G$ in $\mathbb{C}^n$ is a domain of holomorphy if and only if it is exhausted by regular domains.

**Proof.** If $G$ is a domain of holomorphy, $G$ is exhausted by domains of holomorphy $G_p$. Since each $G_p$ is a regular domain, $G$ is exhausted by regular domains. Conversely, if $G$ is exhausted by regular domains, $G$ is a domain of holomorphy from Lemmas 3 and 4.

Proposition 1 gives a characterization of a domain of holomorphy by means of Cousin-I problem and means that regular domains which are not domains of holomorphy are isolated in some sense in the set of regular domains.

§2. Behaviour of a regular domain at a continuous boundary point.

A subset $S$ of $\mathbb{R}^n$ is called *smooth* at $x^0 \in S$ if there exists a continuously differentiable function $f$ in a neighbourhood $U$ of $x^0$ such that $S \cap U=\{x; f(x)=0, x \in U\}$ and $\sum_{j=1}^{n} (\partial f/\partial x_j)^2 >0$ at $x^0$. If $\partial f/\partial x_j \neq 0$ at $x^0$, there exists a continuously differentiable function $g$ in a neighbourhood $V \subset U$ of $x^0$ such that $S \cap V=\{x; x_j =g(x_1, x_2, \ldots, \hat{x}_j, \ldots, x_n), x \in V\}$. The notion of smoothness is invariant under continuously bidifferentiable mappings. A subset $S$ of $\mathbb{R}^n$ is called *continuous* at $x^0 \in S$ if there exists a continuous function $g$ in a neighbourhood $V$ of $x^0$ such that $S \cap V=\{x; x_j =g(x_1, x_2, \ldots, \hat{x}_j, \ldots, x_n), x \in V\}$ for some $j$. This definition may depend on
the special choice of coordinates in $\mathbb{R}^n$. A boundary point $x^0$ of an open set $G$ in $\mathbb{R}^n$ is called \textit{continuous} (or \textit{smooth}) if $\partial G$ is continuous (or smooth) at $x^0$.

An open set $G$ in a complex manifold is called \textit{pseudoconvex} at $x^0 \in \partial G$ if there exists an open neighbourhood $V$ of $x^0$ such that $G \cap V$ is holomorphically convex. $G$ is called \textit{pseudoconvex} if $G$ is pseudoconvex at each point of $\partial G$.

\textbf{Proposition 2.} A regular open set $G$ in $\mathbb{C}^n$ is pseudoconvex at a continuous boundary point $x^0$ of $G$.

\textbf{Proof.} Without loss of generality we may suppose that $\partial G \cap V = \{ z = (z_1, z_2, \ldots, z_n) ; x^0 = g(z_1, z_2, \ldots, z_{n-1}, y_n), z \in V \}$ for a continuous function $g$ in a polycylindrical neighbourhood $V$ of $x^0$ where $z_n = x_n + \sqrt{-1} y_n$. Then two cases (1) and (2) may occur for a sufficiently small $V$. In the case (1) there holds $G \cap V = \{ z ; x_n < g(z_1, z_2, \ldots, z_{n-1}, y_n), z \in V \}$ or $G \cap V = \{ z ; x_n > g(z_1, z_2, \ldots, z_{n-1}, y_n), z \in V \}$. In the case (2) there holds $G \cap V = \{ z ; x_n \neq g(z_1, z_2, \ldots, z_{n-1}, y_n), z \in V \}$.

Case (1) We have only to consider the case $G \cap V = \{ z ; x_n < g(z_1, z_2, \ldots, z_{n-1}, y_n), z \in V \}$. There exists a family $\{ V_t ; 0 \leq t \leq t_0 \}$ of polycylinders $V_t$ containing $x^0$ such that $V_{t_0} \supset V_{t_1} \supset V$ for $0 \leq t_0 < t_1 \leq t_0$, $V_0 = \cup_{0 < t < t_0} V_t$ and $\{ z ; (z_1, z_2, \ldots, z_{n-1}, y_n) \in V_t \}$ is a regular open set for $0 \leq t \leq t_0$. Let $P$ be a polycylinder. We consider a biholomorphic mapping $(z_1, z_2, \ldots, z_{n}) \mapsto (w_1, w_2, \ldots, w_n)$ defined by $w_j = z_j$ ($j = 1, 2, \ldots, n-1$) and $w_n = x_n + t$. Then $E_t \cap P$ is mapped onto $\{ w ; u_1 < g(w_1, w_2, \ldots, w_{n-1}, v_n), (w_1, w_2, \ldots, w_{n-1}, w_n - t) \in V_t \} \cap P$ which is a Cousin-I open set for $0 \leq t \leq t_0$ as the third element of the right-hand side of the above equation is a polycylinder. Hence $E_t$ is a regular open set. Since $E = G \cap V_0$ is exhausted by regular open sets $E_t$, $E$ is an open set of holomorphy from Proposition 1. Hence $G$ is pseudoconvex at $x^0$.

Case (2) If we put $E_1 = \{ z ; x_n < g(z_1, z_2, \ldots, z_{n-1}, y_n), x \in V \}$ and $E_2 = \{ z ; x_n > g(z_1, z_2, \ldots, z_{n-1}, y_n), x \in V \}$, then $E_1$ and $E_2$ are regular open sets. Therefore from the case (1) $E_1$ and $E_2$ are pseudoconvex at $x^0$. Hence $G$ is pseudoconvex at $x^0$.

\textbf{§ 3. Global character of locally regular domains.}

An open set $G$ in a complex manifold $M$ is called \textit{strongly regular} if $G \cap D$ is a Cousin-I open set for any Stein manifold $D \subset M$. This is invariant under biholomorphic mappings of $M$. We say that a domain $G$ in a complex manifold is \textit{exhausted by strongly regular domains} if there exists a sequence of strongly regular domains $G_p$ such that $G_p \supset G_{p+1}$ ($p = 1, 2, 3, \ldots$) and $G = \cup_{p=1}^\infty G_p$.

\textbf{Proposition 3.} A domain $G$ in a Stein manifold is a Stein manifold if and only if $G$ is exhausted by strongly regular domains.

\textbf{Proof.} If $G$ is a Stein manifold, it is obvious that $G$ is exhausted by strongly regular domains. Conversely suppose that $G$ is exhausted by strongly regular
domains $G_p$. Let $x^o$ be any point of $\partial G$. There exists a biholomorphic mapping $\tau$ of a holomorphically convex neighbourhood $U$ of $x^o$ into a complex Euclidean space. $U$ is exhausted by holomorphically convex domains $U_p$. Since $\tau(G \cap U)$ is exhausted by strongly regular open sets $\tau(G \cap U_p)$, $\tau(G \cap U)$ is an open set of holomorphy from Proposition 1. $G$ is a Stein manifold from Docquier-Grauert [8].

An open set $G$ in a complex manifold is called locally regular (or locally strongly regular) at a point $x^o \in \partial G$ if there exists a biholomorphic mapping $\tau$ of a neighbourhood $U$ of $x^o$ into a complex Euclidean space such that $\tau(G \cap U)$ is a regular (or strongly regular) open set. We say that $G$ is locally regular (or locally strongly regular) if $G$ is locally regular (or locally strongly regular) at each point of $\partial G$. We say that a boundary point $x^o$ of an open set $G$ in a differentiable manifold is a smooth boundary point of $G$ if there exists a continuously bidifferentiable mapping $\tau$ of a neighbourhood $V$ of $x^o$ such that $\tau(x^o)$ is a smooth boundary point of $\tau(G \cap V)$. Let $W$ be a polycylinder such that $\tau(x^o) \in W \subset \tau(U \cap V)$. Since the continuously bidifferentiable mapping $\tau \circ \tau^{-1}$ maps $\tau^{-1}(W)$ onto $W$, $\tau(x^o)$ is a smooth boundary point of a regular open set $\tau(G \cap U) \cap W$. From Proposition 2 $\tau(G \cap U) \cap W$ is pseudoconvex at $\tau(x^o)$. Therefore $G$ is pseudoconvex at $x^o$. From Docquier-Grauert [8] $G$ is a Stein manifold.

We say that a boundary point $x^o$ of an open set $G$ in a complex manifold is a continuous boundary point of $G$ if there exists a biholomorphic mapping $\tau$ of a neighbourhood $U$ of $x^o$ into a complex Euclidean space such that $\tau(x^o)$ is a continuous boundary point of $\tau(G \cap U)$. Moreover, if $\tau(G \cap U)$ is a regular open set simultaneously, $x^o$ is called a continuous and locally regular boundary point of $G$. We say that $G$ has a continuous (or continuous and locally regular) boundary if each boundary point of $G$ is a continuous (or continuous and locally regular) boundary point of $G$. These definitions are not so good that a boundary point $x^o$ of an open set $U$ in a complex Euclidean space $C^n$ may not be a continuous boundary point of $U$ even if $x^o$ is a continuous boundary point of $U$ which is considered as a subset of a complex manifold $C^n$ and that a boundary point which is continuous and which is locally regular, separately may not be continuous and locally regular.

**Proposition 4.** Let $G$ be a domain with a smooth boundary in a Stein manifold. Then $G$ is a Stein manifold if and only if $G$ is locally regular.

**Proof.** If $G$ is a Stein manifold, it is obvious that $G$ is locally regular. Conversely suppose that $G$ is locally regular. Let $x^o$ be any point of $\partial G$. Since $G$ is locally regular at $x^o$, there exists a biholomorphic mapping $\tau$ of a neighbourhood $U$ of $x^o$ into a complex Euclidean space such that $\tau(G \cap U)$ is a regular open set. Since $x^o$ is a smooth boundary point, there exists a continuously bidifferentiable mapping $\tau'$ of a neighbourhood $V$ of $x^o$ such that $\tau'(x^o)$ is a smooth boundary point of $\tau'(G \cap V)$. Let $W$ be a polycylinder such that $\tau(x^o) \in W \subset \tau(U \cap V)$. Since the continuously bidifferentiable mapping $\tau \circ \tau^{-1}$ maps $\tau^{-1}(W)$ onto $W$, $\tau(x^o)$ is a smooth boundary point of a regular open set $\tau(G \cap U) \cap W$. From Proposition 2 $\tau(G \cap U) \cap W$ is pseudoconvex at $\tau(x^o)$. Therefore $G$ is pseudoconvex at $x^o$. From Docquier-Grauert [8] $G$ is a Stein manifold.

We say that a boundary point $x^o$ of an open set $G$ in a complex manifold is a continuous boundary point of $G$ if there exists a biholomorphic mapping $\tau$ of a neighbourhood $U$ of $x^o$ into a complex Euclidean space such that $\tau(x^o)$ is a continuous boundary point of $\tau(G \cap U)$. Moreover, if $\tau(G \cap U)$ is a regular open set simultaneously, $x^o$ is called a continuous and locally regular boundary point of $G$. We say that $G$ has a continuous (or continuous and locally regular) boundary if each boundary point of $G$ is a continuous (or continuous and locally regular) boundary point of $G$. These definitions are not so good that a boundary point $x^o$ of an open set $U$ in a complex Euclidean space $C^n$ may not be a continuous boundary point of $U$ even if $x^o$ is a continuous boundary point of $U$ which is considered as a subset of a complex manifold $C^n$ and that a boundary point which is continuous and which is locally regular, separately may not be continuous and locally regular.

**Proposition 5.** Let $G$ be a domain with a continuous boundary in a Stein manifold. Then $G$ is a Stein manifold if and only if $G$ is locally strongly regular.
Proof. If $G$ is a Stein manifold, it is obvious that $G$ is locally strongly regular. Conversely suppose that $G$ is locally strongly regular. Let $x^0$ be any point of $\partial G$. Since $\partial G$ is continuous at $x^0$, there exists a biholomorphic mapping $\tau$ of a neighbourhood $U$ of $x^0$ into a complex Euclidean space such that $\tau(x^0)$ is a continuous boundary point of $\tau(G \cap U)$. Since $G$ is locally strongly regular at $x^0$, there exists a biholomorphic mapping $\tau'$ of a neighbourhood $V$ of $x^0$ into a complex Euclidean space such that $\tau'(G \cap V)$ is a strongly regular open set. Let $W$ be a holomorphically convex neighbourhood of $x^0$ such that $W \subset U \cap V$. Then $\tau'(G \cap V \cap W)$ is a strongly regular open set. Since the biholomorphic mapping $\tau \circ \tau'^{-1}$ maps $\tau'(G \cap V \cap W)$ onto $\tau(G \cap V \cap W)$, $\tau(G \cap V \cap W)$ is a strongly regular open set. Therefore $\tau(G \cap V \cap W)$ is pseudoconvex at the continuous boundary point $\tau(x^0)$ from Proposition 2. Hence $G$ is pseudoconvex at $x^0$. From Docquier-Grauert [8] $G$ is a Stein manifold.

**Proposition 6.** A domain $G$ with a continuous and locally regular boundary in a Stein manifold is a Stein manifold.

**Proof.** Let $x^0$ be any point of $\partial G$. Since $x^0$ is a continuous and locally regular boundary point of $G$, there exists a biholomorphic mapping $\tau$ of a neighbourhood $U$ of $x^0$ into a complex Euclidean space such that $\tau(x^0)$ is a continuous boundary point of a regular open set $\tau(G \cap U)$. From Proposition 2 $\tau(G \cap U)$ is pseudoconvex at $\tau(x^0)$. Hence $G$ is pseudoconvex at $x^0$. From Docquier-Grauert [8] $G$ is a Stein manifold.

§ 4. Example.

Let $E$ be a relatively compact open subset with a smooth boundary in a Stein manifold $M$. Then from Andreotti-Grauert [1] and Fujimoto-Kasahara [9] the canonical homomorphism $H^0(M, \mathcal{O}) \to H^0(M - \bar{E}, \mathcal{O})$ is surjective. Therefore $M - \bar{E}$ is not holomorphically convex. Therefore from Proposition 4, $M - \bar{E}$ is not locally regular at some point of $\partial E$. Let $x^0$ be a point of $\partial E$ at which $M - \bar{E}$ is not locally regular. For any neighbourhood $U$ of $x^0$, there exists a holomorphically convex subdomain $D$ of $U$ such that $(M - \bar{E}) \cap D$ is not a Cousin-I open set. Making use of Andreotti-Grauert [1], we can take $E$ such that $M - \bar{E}$ is a Cousin-I domain. This gives an example of a Cousin-I domain with a smooth boundary which is not locally regular.

**Proposition 7.** Let $E$ be a relatively compact open subset of a Stein manifold $M$. Then there exists an arbitrarily small holomorphically convex subdomain $D$ of $M$ such that $(M - \bar{E}) \cap D$ is not a Cousin-I open set.

References


DEPARTMENT OF MATHEMATICS, KANAZAWA UNIVERSITY.