MINIMAL SLIT REGIONS AND LINEAR OPERATOR METHOD

BY KÔTARO OIKAWA

1. Let $\Omega$ be a plane region containing the point at infinity. Let $\mathfrak{F}_\varnothing$ be the family of all the univalent functions $f$ on $\Omega$ having the expansion

\begin{equation}
 f(z) = z + \frac{c}{z} + \ldots
\end{equation}

about $\infty$. The function maximizing (minimizing) $\operatorname{Re} c$ in $\mathfrak{F}_\varnothing$ exists and is determined uniquely, which we denote by $\varphi_\varnothing(\varphi_\varnothing$, resp.).

The image region $\varphi_\varnothing(\Omega)$ ($\varphi_\varnothing(\Omega)$) is a horizontal (vertical) parallel slit plane. Conversely, however, an arbitrary horizontal (vertical) parallel slit plane can not be, in general, the image of an $\Omega$ under $\varphi_\varnothing(\varphi_\varnothing)$; in fact the measure of $\varphi_\varnothing(\Omega)$ and $\varphi_\varnothing(\Omega)$ vanish. Accordingly, with Koebe, we introduce the following:

**DEFINITION.** A horizontal (vertical) parallel slit plane $\Delta$ is said to be minimal if $\Delta = \varphi_\varnothing(\Omega)$ ($\Delta = \varphi_\varnothing(\Omega)$, resp.) for an $\Omega$ containing $\infty$.

The minimality of slit regions is characterized by moduli of quadrilaterals (Grötzsch [2]) or extremal length (Jenkins [3]). From the point of view of the latter a number of interesting properties are derived in Suita's paper in these Reports [8].

The linear operator method due to Sario [6] (see also Chapter III of the book by Ahlfors-Sario [1]) gives us another approach to $\varphi_\varnothing$ and $\varphi_\varnothing$. From this a characterization of minimality is derived, which is rather similar to the original one due to Koebe [4]. It is the purpose of the present paper to show how to use this method to prove alternatively a part of Suita's results mentioned above.

2. We begin with reviewing the definition of the normal linear operators $L_0$ and $L_1$ in Ahlfors-Sario [1].

Let $W$ be an open Riemann surface, let $V$ be a regularly imbedded non-compact subregion with compact relative boundary $\alpha$. For any real analytic function $f$ on $\alpha$, consider the problem of constructing the function $u$ such that

\begin{equation}
 \text{harmonic on } V \cup \alpha, \quad u = f \text{ on } \alpha.
\end{equation}

If $V$ is the interior of a compact bordered surface we can assign the behavior of $u$ on $\beta = (\text{border of } V) - \alpha$ so that $u$ may be determined uniquely. For our purpose the following two are necessary:

Received March 22, 1965.
(L₀): \( du^*=0 \) along \( \beta \),

(\text{L₁}): \( du=0 \) along \( \beta \), \( \oint du^*=0 \) for each contour of \( \beta \);

here the correspondence \( f \rightarrow u \) is expressed by the notations in the left.

Note that the present \( L_1 \) is the \( (P)L_1 \) in Ahlfors-Sario's book with respect to the canonical partition \( P \). (See [1, p. 160].)

If \( V \) is arbitrary we may define \( L_0 \) and \( L_1 \) as the limit through an exhaustion. We can define them also as follows:

**Definition.** \( L_0 f \) is defined as the \( u \) determined uniquely by the condition (2), \( D_v(u)<\infty \), and

\[
\int_{\gamma} (du)(dv)^* = \int_{\alpha} vdu^*
\]

for every harmonic function \( v \) on \( \overline{V} \) with \( D_v(v)<\infty \). \( L_1 f \) is defined as the \( u \) determined uniquely by the condition (2), \( D_v(u)<\infty \), \( \int_{\gamma} du^*=0 \) for every dividing cycle \( \gamma \) which does not separate components of \( \alpha \), and

\[
\int_{\gamma} (du)\omega = \int_{\alpha} f\omega
\]

for every harmonic differential \( \omega \) on \( V \cup \alpha \) such that \( ||\omega\||<\infty \) and \( \int_{\gamma} \omega=0 \) for every \( \gamma \) mentioned above.

We remark the following:

(i) If \( V \) is the interior of a compact bordered surface, this definition coincides with the previous.

(ii) In (3), the harmonicity of \( v \) may be replaced by the following: \( v \) is of \( C^{(1)} \) on \( \overline{V} \). In (4) the harmonicity of \( \omega \) may be replaced by the following: \( \omega \) is of \( C^{(1)} \) and closed on \( \overline{V} \).

(iii) If \( V' \subset V \) then

\[
L_{0f'}(L_{0f})=L_{0f'}, \quad L_{1f'}(L_{1f})=L_{1f'}
\]

on \( V' \) for any \( f \) on \( \alpha \); here the subscripts \( V' \) and \( V \) express the region where the operators are considered.

(iv) Conversely, let \( V_1, \ldots, V_n \subset V \) be mutually disjoint and such that \( V-\bigcup_{k=1}^n V_k \) is relatively compact. Given \( f \) on \( \alpha \), suppose a \( u \) on \( V \) satisfy (2) and

\[
u = L_{0f'}u \quad (u = L_{1f}u)
\]

on \( V_k, k=1, \ldots, n \). Then \( u = L_0f(u = L_1f, \text{resp.}) \) on \( V \).

3. We find in Ahlfors-Sario [1, p. 176ff] that \( \varphi_\Omega \) and \( \psi_\Omega \) are characterized as functions regular on \( \Omega-\{\infty\} \), having expansion (1) about \( \infty \), and such that

\[
L_0(\text{Re} \, \varphi_\Omega) = \text{Re} \, \varphi_\Omega, \quad L_1(\text{Re} \, \varphi_\Omega) = \text{Re} \, \psi_\Omega
\]

on \( \partial \Omega \); this means the validity of (5) on \( V_1, \ldots, V_n \) with compact \( \Omega-\bigcup_{k=1}^n V_k \), which
is independent of the choice of $V_k$ because of the above remarks (iii) and (iv). Therefore

\textbf{Theorem 1.} A region $\Delta$ in the $z=x+iy$-plane with $\infty \in \Delta$ is a minimal horizontal (vertical) parallel slit plane if and only if

$$L_0x=x \quad (L_1y=y, \text{ resp.})$$
on $\partial \Delta$.

It is evident that the condition is equivalent with

$$(7) \quad L_0y=y \quad (L_1y=y, \text{ resp.}).$$

On regarding the definition of $L_0$ we see that the validity of $L_0x=x$ on a $V$ is equivalent with the following:

$$\int_V \frac{\partial h}{\partial x} dx dy = \int_V v dy.$$ 

Consequently a region $\Delta$ with $\infty \in \Delta$ is a minimal horizontal parallel slit plane if and only if

$$\int_{\partial \Delta} \frac{\partial h}{\partial x} dx dy = 0$$

for every $h$ which is of $C^{(1)}$ in $\Delta$, vanishes identically in a neighborhood of $\infty$, and has finite $D_{\Delta}(h)$. This is nothing but the original characterization of minimality due to Koebe [4].

From Theorem 1 and remarks (iii), (iv) of $2^o$, we obtain the following which is Theorem 12 of Suita [8]:

\textbf{Theorem 2.} Let $\infty \in \Delta_k (k=1, \ldots, n)$ have mutually disjoint $\Delta_k$, and let $\Delta = \bigcap_{k=1}^{n} \Delta_k$. Then $\Delta$ is a minimal horizontal (vertical) parallel slit plane if and only if so are all the $\Delta_k$.

4. Circular and radial slit planes are characterized by $L_0$ and $L_1$ in the similar way. Slit disks and annuli are the same if the outer (and inner) periphery is assumed to be isolated from other part of the boundary. For example

Let $\Delta$ be a circular slit annulus with inner and outer radius $0<Q'$ and $Q<\infty$, respectively. Let $\{ |z|=Q' \}$ and $\{ |z|=Q \}$ be isolated from $E=\bigtriangleup \cap \{ |z|<Q' \}$. Then $\Delta$ is a minimal circular slit annulus if and only if $L_1(\log |z|)=\log |z|$ on $E$.

The change of the independent variable in (4) implies the following, which is contained in Theorem 11 of Suita [8]:

\textbf{Theorem 3.} Let a circular slit annulus $\Delta$ and its slits $E$ be as above. Let $\Delta'$ be a horizontal parallel slit plane such that $E'=\bigtriangleup_{\infty}$ is contained in the interior of a vertical parallel strip with width $2\pi$. Suppose that $E$ is the image of $E'$ under the mapping $z=\exp iz$. Then $\Delta$ is minimal if and only if $\Delta'$ is minimal.

5. Characterizing minimal circular slit annuli by extremal length is easier than that of parallel slit plane. The former is found in, e.g., Reich-Warschawski [5] (for slit disk, though) or Sakai [7], and the latter is in Jenkins [3] as we have mentioned.
The former is as the following:

Let $\Delta$ be as in 4°. Let $\Gamma$ be the family of all the closed rectifiable curves in $\Delta$ separating the inner and outer peripheries. Then $\Delta$ is minimal if and only if $\log \langle Q|Q' \rangle = 2\pi \beta(\Gamma)$.

The following is derived from this:

**Theorem 4.** Let $\Delta$ be a plane region containing $\infty$. Let $R$ be a rectangle whose interior contains $\Delta^c$ and sides are parallel to the coordinate axes. Let $a$ and $b$ be respectively the width and the height of $R$. Let $\Gamma$ be the family of all the rectifiable curves in $R \cap \Delta$ joining the both vertical sides of $R$. (i) If $\Delta$ is minimal, then $\lambda(\Gamma) = a/b$ for any $R$; (ii) if there exists an $R$ with $\lambda(\Gamma) = a/b$, then $\Delta$ is minimal.

Concerning (ii), Jenkins [3] assumed the validity of $\lambda(\Gamma) = a/b$ for all sufficiently large square $R$. The present form the characterization by moduli of quadrilaterals is stated without proof by Grötzsch [2, p. 188]. The above is Theorem 8 of Suita [8].

**Proof.** (i) With the aid of linear transformation, we may assume in advance that $a = 2\pi$. Map $R$ by $\zeta = \text{const}\cdot \exp iz$ onto $1 < |\zeta| < \exp b$ and let the image of $\Delta^c$ be $\tilde{E}$. By Theorem 3 $\tilde{\Delta} = (1 < |\zeta| < \exp b) - \tilde{E}$ is minimal, so that $b = 2\pi / \lambda(\tilde{\Gamma})$, where $\tilde{\Gamma}$ is the family of all the closed curves in $\tilde{\Delta}$ separating the inner and outer peripheries. From the general theory of extremal length, it is easy to obtain $2\pi / b \leq \lambda(\tilde{\Gamma})$, $\lambda(\tilde{\Gamma}) \leq \lambda(\tilde{\Gamma})$. Thus $\lambda(\tilde{\Gamma}) = 2\pi / b$.

(ii) We may assume in advance that $a = \pi$. Let $\hat{R}$ and $\hat{E}$ be obtained from $R$ and $E$, respectively, by the reflection across the right vertical side of $R$. Let $\hat{\Gamma}$ be the family of curves obtained from $\Gamma$ by the same reflection. Map $\hat{R}$ by $\zeta = \text{const} \cdot \exp iz$ onto $1 < |\zeta| < \exp b$ and let the image of $\hat{E}$ be $\tilde{E}$. Consider $\hat{\Delta}$ and $\hat{\Gamma}$ as before. From the general theory, we have $2\pi / b \leq \lambda(\hat{\Gamma})$, $\lambda(\hat{\Gamma}) \leq \lambda(\hat{\Gamma})$, $\lambda(\hat{\Gamma}) = 2\lambda(\hat{\Gamma})$. Thus, by the assumption, $b = 2\pi / \lambda(\hat{\Gamma})$, and, therefore, $\hat{\Delta}$ is minimal. By Theorem 3 $\tilde{E}^c$ is minimal, so that, by Theorem 2, $\hat{\Delta}$ is minimal.

**References**