1. In this paper some Tauberian theorems for a class of stochastic processes will be proved. We shall give the theorems in the form including also an Abelian result.

2. We state first the following

**Lemma.** Let \( \{\alpha(t); \lambda \in \Lambda\} \) be a class of complex-valued functions of bounded variation in every finite interval, and assume for every \( \lambda \in \Lambda \) that

\[
\int_0^\infty e^{-st}d\alpha(t)
\]

converges for \( s > 0 \). If \( \alpha(t) \) are uniformly bounded in every finite interval of \( t \) and if there exists a positive constant \( \gamma \) such that

\[
\frac{\text{Im} t^{-r}}{m t^{-r}} f_{\lambda}(s) = \frac{A_1}{\Gamma(\gamma + 1)}
\]

uniformly in \( \lambda \in \Lambda \), where \( A_1 \) is bounded on \( \Lambda \), then

\[
\lim_{s \to +0} s^r f_{\lambda}(s) = A_1
\]

uniformly in \( \lambda \in \Lambda \). Conversely if there exist constants \( K \) and \( \gamma > 0 \) such that for every \( \lambda \in \Lambda \) the functions \( \text{Re} \alpha(t) + Kt^r \) and \( \text{Im} \alpha(t) + Kt^r \) are non-decreasing in \( 0 \leq t < \infty \) and if (2) holds uniformly in \( \lambda \in \Lambda \) with \( A_1 \) bounded on \( \Lambda \), then (1) holds uniformly in \( \lambda \in \Lambda \).

The proof of this Lemma will not be given here, since it is similar in the main to the proof of well-known Tauberian theorem (see [1]).

3. We shall now prove the following

**Theorem 1.** Let \( \{X(t); t \geq 0\} \) be a stochastic process such that \( \int_0^T X(t)dt \) exists for every finite \( T > 0 \), and assume that there exist positive constants \( M \) and \( \gamma \) such that

\[
\sqrt{E[|X(t)|^2]} \leq Mt^{r-1}
\]

for every \( t > 0 \). Then a necessary and sufficient condition that

\[
\lim_{T \to \infty} T^{-r} \int_0^T X(t)dt = \frac{Y}{\Gamma(\gamma + 1)}
\]

is that

Received May 7, 1966.
Proof. Note that for proving our theorem it is sufficient to consider the case \( Y=0 \). Otherwise, indeed, we may consider the stochastic process \( \{X_i(t); t \geq 0\} \), where \( X_i(t) = X(t) - \langle \Gamma(\gamma) \rangle^{-1/2} Y \). We note further that by the assumption we have

\[
E[|s^{i}L(s)|^2] = s^{2r} \int_{0}^{\infty} \int_{0}^{\infty} e^{-st} e^{-s\tau} E\{X(t)X(\tau)\} dtd\tau \leq (M\Gamma(\gamma))^r,
\]

and

\[
E\left[ \left( T^{-r} \int_{0}^{T} X(t) dt \right)^2 \right] = T^{-2r} \int_{0}^{T} \int_{0}^{T} E\{X(t)X(\tau)\} dtd\tau \leq (Mr^{-2})^r.
\]

Hence by Schwarz’s inequality

\[
|E\{s^{i}L(s)X(\tau)\}| \leq M^{2r-1} \Gamma(\gamma)^{r-1},
\]

\[
E\left| T^{-r} \int_{0}^{T} X(t) dt \right| \leq E\left| T^{-r} \int_{0}^{T} X(t) dt \right|^2 = T^{-2r} \int_{0}^{T} \int_{0}^{T} E\{X(t)X(\tau)\} dtd\tau \leq (Mr^{-2})^r.
\]

First we suppose that (3) holds with \( Y=0 \) and prove (4) with \( Y=0 \). It follows from (5) and (9) that

\[
\lim_{s \to +0} E\left[ s^{i}L(s)T^{-r} \int_{0}^{T} X(t) dt \right] = \lim_{s \to +0} T^{-r} \int_{0}^{T} E\{s^{i}L(s)X(\tau)\} d\tau = 0
\]

uniformly in \( s>0 \). It can be seen from (7) that the class \( \{\alpha(t); s>0\} \) of functions \( \alpha(t) = s^{i}E\{s^{i}L(s)X(\tau)\} d\tau \) satisfies the conditions of the first part of Lemma with \( A_1=0 \). Hence we have that

\[
\lim_{s \to +0} \sigma^{i} \int_{0}^{\infty} e^{-st} E\{s^{i}L(s)X(\tau)\} d\tau = \lim_{s \to +0} E\{s^{i}L(s)\sigma^{i}L(\sigma)\} = 0
\]

uniformly in \( s>0 \), and therefore we have

\[
\lim_{s \to +0} E\{s^{i}L(s)|^2\} = 0
\]

which implies (4) with \( Y=0 \). Next we suppose that (4) holds with \( Y=0 \) and prove (3) with \( Y=0 \). From (6) and (9) we have that

\[
\lim_{s \to +0} E\left[ s^{i}L(s) \cdot T^{-r} \int_{0}^{T} X(\tau) d\tau \right] = \lim_{s \to +0} s^{i} \int_{0}^{\infty} e^{-st} E\{X(t) \cdot T^{-r} \int_{0}^{T} X(\tau) d\tau \} dt = 0
\]
uniformly in \( T > 0 \). From (8) we see that the conditions of the second part of Lemma are satisfied with \( A_1 \equiv 0 \) by the class \( \{ \alpha_T(t); T > 0 \} \) of functions

\[
\alpha_T(t) = \int_0^t E \left| T^{-r} \int_0^T X(u) \cdot T^{-r} \int_0^T X(\tau) d\tau \right| du.
\]

Hence

\[
\lim_{T' \to \infty} \int_0^T E \left| T^{-r} \int_0^T X(t) \cdot T^{-r} \int_0^T X(\tau) d\tau \right| dt = \lim_{T' \to \infty} E \left[ T^{-r} \int_0^T X(t) dt \cdot T^{-r} \int_0^T X(\tau) d\tau \right] = 0
\]

uniformly in \( T > 0 \), and therefore we have

\[
\lim_{T' \to \infty} E \left| T^{-r} \int_0^T X(t) dt \right|^2 = 0,
\]

which implies (3) with \( Y = 0 \). Thus our theorem is proved.

It follows immediately the following

**Corollary 1.** Let \( \{X(t); t > 0\} \) be a stochastic process such that \( \int \int X(t) dt \) exists for every finite \( T > 0 \), and let \( E(|X(t)|^2) \) be bounded for \( t \geq 0 \). Then a necessary and sufficient condition that

\[
\lim_{T' \to \infty} \int_0^T E \left| T^{-r} \int_0^T X(t) dt \right|^2 = 0,
\]

which implies (3) with \( Y = 0 \). Thus our theorem is proved.

It follows immediately the following

**Corollary 1.** Let \( \{X(t); t > 0\} \) be a stochastic process such that \( \int \int X(t) dt \) exists for every finite \( T > 0 \), and let \( E(|X(t)|^2) \) be bounded for \( t \geq 0 \). Then a necessary and sufficient condition that

\[
\lim_{T' \to \infty} E \left| T^{-r} \int_0^T X(t) dt \right|^2 = 0,
\]

which implies (3) with \( Y = 0 \). Thus our theorem is proved.

**Theorem 2.** Let \( \{X_n; n \geq 1\} \) be a sequence of random variables and assume that there exists a constant \( \gamma > 0 \) such that \( n^2 \gamma^2 E(|X_n|^2) \) is bounded for \( n \geq 1 \). Then a necessary and sufficient condition that

\[
\lim_{n \to \infty} \frac{1}{n^2} \sum_{n=1}^n X_n = \frac{Y}{\gamma^2+1}
\]

is that

\[
\lim_{s \to 1-0} \int_0^s e^{-n^2} e^{-rt} \rho(t, \tau) dtd\tau = 0,
\]

where

\[
\rho(t, \tau) = E(X(t)X(\tau)).
\]

**Theorem 2.** Let \( \{X_n; n \geq 1\} \) be a sequence of random variables and assume that there exists a constant \( \gamma > 0 \) such that \( n^2 \gamma^2 E(|X_n|^2) \) is bounded for \( n \geq 1 \). Then a necessary and sufficient condition that

\[
\lim_{n \to \infty} \frac{1}{n^2} \sum_{n=1}^n X_n = \frac{Y}{\gamma^2+1}
\]

is that

\[
\lim_{s \to 1-0} \int_0^s e^{-n^2} e^{-rt} \rho(t, \tau) dtd\tau = 0,
\]

where

\[
\rho(t, \tau) = E(X(t)X(\tau)).
\]

**Proof.** Define a stochastic process \( \{X(t); t \geq 0\} \) by \( X(t) = X_n \) for \( n \leq t < n+1 \), where \( X_0 \equiv 0 \), and apply Theorem 1.

**Corollary 2.** Let \( \{X_n; n \geq 1\} \) be a sequence of random variables. Suppose that \( E(|X_n|^2) \) is bounded for \( n \geq 1 \). Then a necessary and sufficient condition that

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^n X_k = 0
\]

is that
is that

\[ \lim_{s \to 1 - 0} (1 - s)^2 \sum_{k,l=1}^{\infty} \rho_{k,l} s^{k+l} = 0, \]

where

\[ \rho_{k,l} = E\{X_k \overline{X}_l\}. \]

**Remark 1.** When \( \gamma > 1 \), the assumption in Theorem 1 that \( \sqrt{E[|X(t)|^2]} \leq M t^{-\gamma} \) for every \( t \) may be weakened. In fact, we have the result of Theorem 1 under the assumption that \( \sqrt{E[|X(t)|^2]} \leq M(1 + t^{-\gamma}) \) for every \( t \).

**Remark 2.** In the case \( \gamma = 0 \), we have also theorems analogous to Theorem 1 and Theorem 2.

**Remark 3.** The weak law of large numbers for the class of weakly stationary processes follows from our theorems. In fact, let \( \{X(t); t \geq 0\} \) be a weakly stationary process, and let \( \rho(t) = E\{X(t+\tau)\overline{X}(\tau)\} \) be its covariance function with spectral representation

\[ \rho(t) = \int_{-\infty}^{\infty} e^{it\mu} dF(\lambda), \]

where \( F(\lambda) \) is the spectral distribution function of \( \{X(t); t \geq 0\} \). Then we have that

\[ s^2 \int_{-\infty}^{\infty} \int_{0}^{\infty} e^{-ut} e^{-\tau t} \rho(t-\tau) dt d\tau \]

\[ = s^2 \int_{-\infty}^{\infty} \int_{0}^{\infty} e^{-ut} e^{-\tau t} \left[ \int_{-\infty}^{\infty} e^{i\lambda(t-\tau)} dF(\lambda) \right] dt d\tau \]

\[ = s^2 \int_{-\infty}^{\infty} \left| \int_{0}^{\infty} e^{-(\tau-\lambda)t} dt \right| dF(\lambda) \]

\[ = \int_{-\infty}^{\infty} \frac{s^2}{s^2 + \lambda^2} dF(\lambda) \]

converges to zero as \( s \to +0 \) if and only if \( F(\lambda) \) is continuous at \( \lambda = 0 \). Hence by Corollary 1, (16) holds if and only if \( F(\lambda) \) is continuous at \( \lambda = 0 \). The discrete analogue is obtained in a similar way.

**Reference**