ON A THEOREM OF W. GUSTIN

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1. W. Gustin has recently shown that any pair of functions harmonic in respective domains of a euclidean space satisfies a certain bilinear integral identity, and then applied it obtaining systematically new proofs of a few fundamental theorems in classical harmonic function theory. His principal theorem may be stated as follows:

"Let \( \phi_1 \) be harmonic in a domain \( D_1 \) containing a point (expressed by a vector) \( \eta \), and \( \phi_2 \) be harmonic in \( D_2 \) containing \( \eta' \). Then the bilinear integral expression

\[
\int_{\Omega} \phi_1 (p, x) \phi_2 (p', x) \, d\omega_x
\]

depends only on the product \( p \, p' \), provided the closed sphere with radius \( p \) about \( \eta \) and the closed sphere with radius \( p' \) about \( \eta' \) are contained in \( D_1 \) and \( D_2 \), respectively. Here the integral is taken such that the unit vector \( x \) extends over the periphery \( \Omega \) of the unit sphere with surface element \( d\omega_x \), the dimension of the space being arbitrary."

Gustin has given two proofs of the theorem; the first being based on Poisson integral formula and the second on Green's bilinear integral identity. In this Note we shall give a brief proof of which will furthermore clarify the essential nature of the theorem.

2. Now, we may suppose, without loss of generality, that \( \eta \) and \( \eta' \) both coincide with the origin, since the harmonicity remains invariant by any translation. As well known, any function \( \Phi(y, x) \) harmonic in a closed sphere \( 0 \leq r \leq R \) can be expanded in a uniformly convergent series of the form

\[
\Phi(y, x) = \sum_{n=0}^{\infty} a_n \, Y_n(\Phi)(x),
\]

where \( Y_n(\Phi)(x) \) are spherical surface harmonics of order \( n \). Remembering the orthogonality character of spherical surface harmonics

\[
\int_{\Omega} Y_n(\Phi)(x) Y_{n'}(\Phi)(x) \, d\omega_x = 0 \quad (m \neq n),
\]

we deduce immediately the relation

\[
\int_{\Omega} \phi_1 (p, x) \phi_2 (p', x) \, d\omega_x = \sum_{n=0}^{\infty} (\Phi_{1n} \Phi_{2n}) \int_{\Omega} Y_n(\Phi)(x) Y_n(\Phi')(x) \, d\omega_x,
\]

yielding the desired result.

3. Remark 1. In Gustin's paper the dimension of basic space is assumed to be not less than two. But, if the space is one-dimensional, the bilinear integral expression may be considered to degenerate into the sum

\[
\Phi(y, x) = a \, p x + b, \quad y = x = 0,
\]

where \( a \) and \( b \) being constants. It is quite easy to see that the above expression depends on the aggregate \( p \), alone for any pair of such linear \( \Phi \) and \( \Phi' \).

Remark 2. In an \( N \)-dimensional euclidean space, the rectangular cartesian and polar coordinates, \( (y, \theta, \phi, \ldots) \) and \( (r, \phi_1, \ldots, \phi_{N-1}) \), are connected in the following manner:

\[
\begin{align*}
\xi_j &= \begin{cases}
\sum_{k=1}^{N-1} \sin \phi_k & (j = 1, \ldots, N-1), \\
y & = \phi_{N-1} & (j = N). 
\end{cases} \\
\gamma \sim \eta, & \quad 0 \leq \gamma \leq \pi, \quad (\phi_{\frac{N}{2}} \leq \xi \leq \pi - \phi_{\frac{N}{2}}).
\end{align*}
\]

The empty product being understood, in the usual way, to denote unity. The square of line element is given by

\[
d\xi_j = \sum_{j=1}^{N} d\xi_j = x_j^2 + y \sum_{j=1}^{N-1} \sin^2 \phi_k d\phi_j.
\]
On the other hand, by introducing general orthogonal curvilinear coordinates \((\varphi_i, \ldots, \varphi_N)\) with \(d^2 = \sum_{i,j} a_{ij} d\varphi_i d\varphi_j\), the Laplacian operator
\[
\Delta = \sum_{i=1}^{N} \frac{a_{ii}}{a_{ij}} \frac{\partial^2}{\partial \varphi_i^2}
\]
is transformed into
\[
\Delta = \frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left( \rho^2 \frac{\partial}{\partial \rho} \right) + \frac{1}{\rho^2 \sin^2 \theta} \frac{\partial}{\partial \theta} \left( \sin^2 \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\rho^2 \sin^2 \theta \sin \phi} \frac{\partial}{\partial \phi} \left( \sin \phi \frac{\partial}{\partial \phi} \right).
\]
which reduces, in our case of polar coordinates, to
\[
\Delta = \frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left( \rho^2 \frac{\partial}{\partial \rho} \right) \Delta^* = \frac{3}{2} \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho^2} \Delta^*.
\]
with
\[
\Delta^* = \sum_{i=1}^{N-1} \frac{1}{2i+1} \frac{\partial}{\partial \theta_i} \left( \sin \theta_i \frac{\partial}{\partial \theta_i} \right) + \frac{1}{2i+1} \frac{\partial}{\partial \phi_i} \left( \sin \phi_i \frac{\partial}{\partial \phi_i} \right).
\]
Hence, for any solid harmonics of the form \(\mathbf{Y}_\kappa(\theta_1, \ldots, \theta_{N-1})\), we have
\[
0 = \Delta (\mathbf{Y}_\kappa) = \mathbf{Y}_0^* \left( (\kappa + N - 2) \mathbf{Y}_\kappa + \kappa \Delta^* \mathbf{Y}_\kappa \right),
\]
i.e.,
\[
\Delta^* \mathbf{Y}_\kappa + \kappa (\kappa + N - 2) \mathbf{Y}_\kappa = 0.
\]

The last relation is the self-adjoint partial differential equation for spherical surface harmonics \(\mathbf{Y}_\kappa\) of order \(\kappa\), which belong to the eigenvalue \(\kappa (\kappa + N - 2)\). In case of \(N\) variables, a general homogeneous function (polynomial) of order \(\kappa\) (with respect to cartesian coordinates) possesses
\[
(\kappa + N - 1) \text{ coefficients. Hence, the maximal number of linearly independent } \mathbf{Y}_\kappa \text{ is, in general, equal to}
\]
\[
(\kappa + N - 1) = (\kappa + N - 1) - 2^{\kappa + N - 3} + \kappa + N - 3.
\]

(2) Cf., e.g., R. Courant u. D. Hilbert, Methoden der mathematischen Physik, I, Berlin (1931), p. 443, where the completeness of the system is shown for three-dimensional case.
(3) A. Dinghas, Geometrische Anwendungen der Kugelfunktionen. Göttinger Nachr. Neue Folge 1, No. 18 (1940), 213-255.

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