REPRESENTATION OF FUNCTIONS ANALYTIC IN A MULTIPLY-CONNECTED DOMAIN

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We may and do use, as a canonical domain of multiplicity \( n \) \( (n > 2) \), a concentric annular ring slit along concentric circular arcs. Let the boundary components of such a domain \( D \), laid on \( \xi \)-plane, be

\[
C_1 : |\xi| = 1; \quad C_2 : |\xi| = n \quad (n < 1);
\]

and the interior and the exterior sides of the slits \( C_j \), \( 3 \leq j \leq n \), be

\[
C_j^{(i)} : |\xi| = m_j - 0, \quad \theta_j = \arg \xi \leq \theta_j + \gamma_j;
\]

\[
C_j^{(e)} : |\xi| = m_j + 0, \quad \theta_j + \gamma_j \leq \arg \xi \leq \theta_j;
\]

respectively. The total boundary of \( D \) be denoted by

\[
C = \sum_{j=1}^{n} C_j.
\]

Any function \( U(\xi) \) regular harmonic in the domain \( D \), and continuous on the closed domain \( D + C \), is represented by Green's formula in the form

\[
U(\xi) = \frac{1}{2\pi} \int_C \frac{\partial U(\zeta)}{\partial T_\zeta} d\sigma_\zeta,
\]

\( \partial (\xi, \zeta) \) being, as usual, Green function (with variable \( \xi \)) of \( D \) with singularity at \( \zeta \), \( \zeta \) and \( \zeta \) denoting inward normal and arc-length parameter at a boundary point \( \xi \).

If we denote the equation of the boundary \( C \) by \( \zeta = \zeta(\xi) \) and the harmonic measure of a part of \( C \) from a fixed point to the point \( \zeta(\xi) \) by \( \omega(\xi, \zeta(\xi)) \), then we have

\[
\frac{1}{2\pi} \frac{\partial G(\xi, \zeta)}{\partial T_\zeta} d\sigma_\zeta = \omega(\xi, \zeta(\xi)),
\]

\[
= \omega(\xi, \zeta(\xi)),
\]

But, we use here another aggregation, namely the one corresponding to Herglotz type. Let \( \Phi(\xi) \) be an analytic function one-valued and regular in \( D \) and continuous on \( D + C \). We denote by \( G(\xi, \zeta) \) an analytic function of \( \xi \) whose real part coincides with \( \Phi(\xi) \); \( G(\xi, \zeta) \) being uniquely determined except an additive purely imaginary quantity depending possibly on \( \xi \) and possessing multivaluedness due to periodicity moduli with respect to the boundary components. We have then, by the formula mentioned above,

\[
\Phi(\xi) = \frac{1}{2\pi} \int_C \frac{\partial G(\xi, \zeta)}{\partial T_\zeta} d\sigma_\zeta + ic,
\]

\( c \) being a real constant.

We now assume that \( \mathcal{R}(\Phi) \) is of bounded variation along \( C \). Then, so is also the function \( \zeta \). We have

\[
\int_C |dG_{\xi}(\zeta)| = \int_C |\mathcal{R}(\Phi)| d\sigma_\zeta.
\]

In fact,

\[
\int_C |d\Phi_{\xi}(\zeta)| = \int_C |\mathcal{R}(\Phi)| d\sigma_\zeta.
\]

In this case, we may write the expression as in the Herglotz type which states

\[
\Phi(\xi) = \frac{1}{2\pi} \sum_{j=1}^{n} \int_C \frac{\partial G(\xi, \zeta)}{\partial T_\zeta} d\sigma_\zeta + ic,
\]

now, considering residue at point \( \zeta \), we have particularly

\[
\frac{1}{2\pi} \int_C \frac{\partial G(\xi, \zeta)}{\partial T_\zeta} d\sigma_\zeta = 1,
\]

and hence

\[
1 = \frac{1}{2\pi} \sum_{j=1}^{n} \int_C \frac{\partial G(\xi, \zeta)}{\partial T_\zeta} d\sigma_\zeta,
\]

where \( G_{\xi}(\zeta) \) is defined by

\[
G_{\xi}(\zeta) = \begin{cases} \xi & \text{on } C_1 \\ -Q \cdot \xi & \text{on } C_2 \\ m_j \cdot (\xi - \zeta) & \text{on } C_j^{(i)} \\ -m_j \cdot (\xi - \zeta) & \text{on } C_j^{(e)} \end{cases} \quad (1 \leq j \leq n).
\]

The last equation shows that an additive purely imaginary constant \( i \) is contained in the general representation vanishes out for the particular function \( \Phi(\xi) = 1 \).

2. Consider now an analytic function \( f(\xi) \) one-valued and regular in \( D \) and piecewise regular on \( D + C \).
The boundary points, finite in number, where the regularity of \( f(z) \) is broken down, be
\[
\{ z_j : j = 1, \ldots, m \} \quad (\mu = 1, 2, \ldots, m_j)
\]
The existence of limits of \( f(z) \) from both sides along \( C \) will be assumed at each of such points.

We assume further that \( f(z) \) vanishes nowhere on \( D + C \) except at these exceptiona! points \( z_j \). The image of \( D \) by mapping \( w = f(z) \) then possesses, on Riemann surface, a piecewise analytic boundary and the function \( f(z) \) can be prolonged analytically over every boundary arc containing no exceptional point. Denoting generally by \( \xi \) any exceptional point, then the image-curve of \( C \) possesses at \( f(\xi) \) an angular point. Denoting by \( \alpha \) the exterior angle at such an angular point with respect to the image-domain, the jump of \( \arg f(z) \) at \( \xi \) along \( C \) is given by
\[
\frac{f(\xi_+)}{f(\xi_-)} = (\alpha - i\pi).
\]
\( \xi_+ \) being infinitely adjacent points at both sides of \( \xi \).

The image-curve of \( C \) will moreover have angular points, in general, also at the image-points of end-points of the slits. If \( f'(\zeta) \) is regular at such an end-point \( \zeta \) and does not vanish there, then the exterior angle of the image-curve at \( f(\zeta) \) is 0 and the jump of \( \arg f(z) \) there vanishes out.

Let \( \zeta \) be an exceptional point coinciding with none of end-points of the slits and the corresponding angle \( \alpha \pi \) be different from \( 2\pi \). Then
\[
(f(z) - f(\zeta))^{1/(z - \zeta)}
\]
is regular at a vicinity of \( \zeta \) and has \( \zeta \) as a simple pole; namely, the function
\[
(f(z) - f(\zeta))^{1/(z - \zeta)}
\]
is uniformized by a local parameter \( (z - \zeta)^{-1} \). Therefore,
\[
f(z) - f(\zeta),
\]
as a function of \( (z - \zeta)^{-1} \), possesses a simple pole at \( \zeta = 0 \), in place of \( \zeta = \infty \), may be replaced by a local uniformizing parameter. In any case, the function
\[
(z - \zeta)^{-1} f(z)
\]
is regular and non-vanishing around \( \zeta \); if an exceptional point \( \xi \) coincides with an end-point \( \zeta \) of a slit, then the power \( 2 - \alpha \) in local parameter has to be replaced by \((2 - \alpha)/2\).

In the following, we suppose none of exceptional points coincide with any one of end-points of the slits, i.e., \( \zeta \neq \zeta \). But, if it happens \( \zeta = \zeta \), the only modification must be made, according to the fact stated just above, that \( \alpha \) has to be replaced by \( \alpha/2 + i \).

Now, the function defined by
\[
\Phi(\zeta) = \frac{1}{2\pi i} \sum_{j=1}^{m} \int_{C_j} \frac{dG(s, \zeta)}{s - \zeta_j} \frac{dz}{f(z)}
\]
is evidently one-valued and regular throughout \( D + C \). Hence, it is expressible in the form
\[
\Phi(\zeta) = \frac{1}{2\pi} \sum_{j=1}^{m} \int_{C_j} \frac{dG(s, \zeta)}{s - \zeta_j} \frac{dz}{f(z)} + \frac{1}{2\pi} \sum_{j=1}^{m} \frac{1}{\zeta_j - \zeta_j} \int_{C_j} \frac{dG(s, \zeta)}{s - \zeta_j} \frac{dz}{f(z)}
\]
\( C_j \) being a real constant and \( \varphi_j(\varphi) \) being a real function of \( \varphi = \arg \zeta \) given by
\[
\varphi_j(\varphi) = \int_{C_j} R_j(\zeta) d\zeta, \quad \text{for } \zeta \in C_j.
\]

The linear function \( x/(x - \zeta_j) \) behaves regularly everywhere except only at a simple pole \( \zeta_j \) and its real part is identically equal to \( 1/2 \) along \( C_j \). It will be easily seen that a representation of the same type as given above for \( \Phi(z) \) holds good also for such a function. Hence, we obtain the following representation formula with respect to \( f(z) \):
\[
\frac{z f(z)}{f'(z)} = \frac{1}{2\pi i} \sum_{j=1}^{m} \int_{C_j} \frac{G(s, \zeta)}{s - \zeta_j} m \arg f(z) d\zeta + c_j
\]
c\( c_j \) being a real constant and \( m = 1, \varphi = Q \).

On the other hand, we have seen that, for particular case \( f(z) = z \), the corresponding representation reduces to
\[
L = \frac{1}{2\pi} \sum_{j=1}^{m} \int_{C_j} \frac{G(s, \zeta)}{s - \zeta_j} m \arg \zeta,
\]
an additive constant vanishing out. Hence, remembering that the relation
\[
\arg d\zeta = \arg (z f'(z) i dz)
\]
\( \arg \zeta = \arg (z f'(z)) \)

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is valid along \( C \), we have

\[
1 + \frac{zf(z)}{f'(z)} = \frac{1}{2\pi i} \sum_{j=1}^{m} \int_{C_j} \frac{2G(z, \xi)}{\beta_0^2} \alpha_j \frac{d\arg f(\xi)}{d\xi} + c^*.
\]

The real constant \( c^* \) can be determined as follows. For any fixed point \( z_0 \) in \( D \), we put

\[
L(z, \xi) = \frac{1}{2\pi i} \int_{z_0}^{z} \frac{2G(z, \xi)}{\beta_0^2} \alpha_j \frac{d\arg f(\xi)}{d\xi}.
\]

This function has a periodicity modulus around each boundary component \( C_j \). Hence, if, introducing the uniformizing parameter

\[
\sigma z,
\]

we put

\[
L(z, \xi) = \frac{1}{2\pi i} \int_{z_0}^{z} \frac{2G(z, \xi)}{\beta_0^2} \frac{d\xi}{\sigma z},
\]

then the difference

\[
M(\log z, \xi) = L(z, \xi),
\]

remains constant, for fixed \( \xi \), along each \( C_j \), i.e., \( z \) is contained in this expression only apparently. Hence, we may put

\[
M_j(\xi) = M(\log z + 2\pi i, \xi) - M(\log z, \xi)
\]

Integrating the above obtained expression for \( f'(z)/f(z) \) with respect to \( z \), we get

\[
\log \frac{f(z)}{f'(z)} = \sum_{j=1}^{m} \int_{C_j} M_j(\log z, \xi) \alpha_j \frac{d\arg f(\xi)}{d\xi} + c^* \log z.
\]

Now, \( f'(z) \) being one-valued, the left-hand member of the last relation increases by an integral multiple of \( 2\pi i \) for substitution \( \log z \rightarrow \log z + 2\pi i \). Accordingly, the real part of this increase calculated from the right-hand member must vanish. Hence we get

\[
2\pi c^* = \sum_{j=1}^{m} \int_{C_j} \mathcal{R}M_j(\log z, \xi) \frac{d\arg f(\xi)}{d\xi},
\]

which is the relation determining \( c^* \). Since, in particular case \( f'(z) = z \), the corresponding constant becomes \( 0 \), we may write also

\[
c^* = \frac{1}{2\pi i} \sum_{j=1}^{m} \int_{C_j} \mathcal{R}M_j(\log z, \xi) \frac{d\arg f(\xi)}{d\xi}.
\]

The constant \( c^* \) having been determined, we obtain the desired representation formula

\[
1 + \frac{zf(z)}{f'(z)} = \frac{1}{2\pi i} \sum_{j=1}^{m} \int_{C_j} \frac{2G(z, \xi)}{\beta_0^2} \mathcal{R}M_j(\xi) \frac{d\arg f(\xi)}{d\xi},
\]

which, by integration, yields a representation for \( f(z) \) itself.

3. As an application of the above general formula, we consider here the case where \( w = f(z) \) maps the basic domain \( D \) onto a domain bounded by \( n \) rectilinear polygons. Then, the exceptional points \( \{ \xi \} \) are the points corresponding to vertices of the image curve of \( C \), and \( \arg f \) becomes a step function having jump with height \( \angle -\bar{\theta} \) at each \( \xi \). Hence, the general formula reduces here to a simple form without integration sign which states

\[
1 + \frac{zf(z)}{f'(z)} = \frac{1}{2\pi i} \sum_{j=1}^{m} \int_{C_j} \frac{2G(z, \xi)}{\beta_0^2} \mathcal{R}M_j(\xi) \frac{d\arg f(\xi)}{d\xi},
\]

where \( c^* \) being given by

\[
c^* = \frac{1}{2\pi i} \sum_{j=1}^{m} \int_{C_j} \mathcal{R}M_j(\xi) \frac{d\arg f(\xi)}{d\xi}.
\]

The successive integration yields then

\[
1 + \frac{zf(z)}{f'(z)} = \frac{1}{2\pi i} \sum_{j=1}^{m} \int_{C_j} \mathcal{R}M_j(\xi) \frac{d\arg f(\xi)}{d\xi} + A_1,
\]

\( A_1 \) being an integration constant, and

\[
f(z) = A_1 \zeta^c \exp \left( \sum_{j=1}^{m} \int_{C_j} \mathcal{R}M_j(\xi) \frac{d\arg f(\xi)}{d\xi} \right)
\]

\[
1 + A_1.
\]

(\( A \) and \( A' \) denoting integration constants which depend only on position and magnitude of the polygonal image domain. The last formula may be regarded as a generalization of Schwarz-Christoffel's one for simply-connected case and of a formula for doubly-connected case previously given by the present author.\(^{(1)}\))

(\( )^{(1)} \) Received October 9, 1950.
(1) Cf. Y. Komatu, Derstellungen der
in einem Kreisringe analytischen Funk-
tionen nebst den Anwendungen auf kon-
forme Abbildung über Polygonalringe-
203-215.

(2) The corresponding formulae for
simply- and doubly-connected cases
have previously been given in Y. Komatu,
Einzige Darstellungen analytischer Funk-
tionen und ihre Anwendungen auf kon-
Tokyo 20(1944), 536-541 and in the
paper cited(1), respectively.

(3) Loc. cit(1) and(2).

(4) For generalization of Schwarz-
Christoffel formula, see also Y. Komatu,
Conformal mapping of polygonal domains,
preliminary note of which has been re-
ported under the same title in these
Reports Nos. 3-4 (1949), 47-50.

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