ON CONTINUITY OF EXTREMAL DISTANCE AND ITS APPLICATIONS TO CONFORMAL MAPPINGS

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§ 1. Introduction.

1. In the present paper we shall first discuss the continuity of the extremal distance between two sets of the boundary of a plane region with respect to its exhaustion. Here the continuity of the extremal length means the convergence of a sequence of extremal lengths of curve families to that of a well-defined family. The continuity of a sequence of increasing curve families was shown by the author [14] and Ziemer [17]. In the problem for extremal distance, the corresponding is regarded as a decreasing sequence. The problem was dealt with by Wolontis [16] for two compact sets in a region, later by Strebel [11] for two compact sets of boundary components of a Riemann surface and recently by the author [15] for two boundary parts on a boundary component of a plane region. Another generalization for curves was given by Marden and Rodin [6].

In this paper we shall define a new boundary part of a plane region $\Omega$ which is considered as a sort of element in the research of ideal boundary by means of a filter. Let $\{\Omega_n\}$ be an exhaustion of $\Omega$ towards two disjoint boundary parts. Then we shall show the continuity, if the extremal distance of the two sets of the relative boundary components of $\Omega_i$ is positive. The proofs due to Wolontis and Strebel are based on the semicontinuity of the distance measured by an admissible metric and the second proof in [15] on a conformal representation. In the present proof we shall make use of an auxiliary metric so that the former method may be available.

After the continuity is established, canonical conformal mappings as in [14] and [15] can be easily constructed, which will be discussed in § 4. The images of boundary parts will be investigated by making use of the method of extremal metrics. The continuity of the extremal distance will be effective in such conformal mappings.

In the final section we shall give an extension of the notion of prime ends, first introduced by Carathéodory [2], to an arbitrary plane region in such a way that it gives a compactification in a suitable topology. Every conformal mapping of the region is extended to the compactification and it introduces a well-defined boundary correspondence.

Received November 7, 1968.
§ 2. Preliminaries.

2. Definition of boundary part. Let $\Omega$ be a plane region which is not the Riemann sphere. We may assume that $\Omega$ does not contain the point at infinity throughout this paper. We shall mean a boundary part by a sequence of open subset $\{\Delta_n\}$ of $\Omega$ satisfying:

I) the relative boundary of $\Delta_n$ consists of a finite number of Jordan curves either closed or open, both of whose end arcs tend to a common boundary component in the latter case,

II) the open set $\Omega - \overline{\Delta_n}$ is connected, where $\overline{\Delta_n}$ denotes the closure of $\Delta_n$ with respect to the relative topology in $\Omega$,

III) $\Delta_n \supset \Delta_{n+1}$, and

IV) $\cap \Delta_n = \phi$.

Two boundary parts $\{\Delta_n\}$ and $\{\Delta'_n\}$ are said to be equivalent, if every $\Delta_n$ contains a $\Delta'_n$ and vice versa. The sequence $\{\Delta_n\}$ is termed a defining sequence of the boundary part. A boundary part $A' = \{\Delta'_n\}$ is said to lie on $A = \{\Delta_n\}$, if the first half of the conditions of the above equivalence is satisfied.

A sequence $\{\Omega_n\}$ with $\Omega_n = \Omega - \overline{\Delta_n}$ is called an exhaustion of $\Omega$ towards $A = \{\Delta_n\}$. We say that two boundary parts $A = \{\Delta_n\}$ and $B = \{\Delta'_n\}$ are disjoint, if $\overline{\Delta_n} \cap \overline{\Delta'_n} = \phi$ for some $n$. Then we can construct an exhaustion $\{\Omega_n\}$ of $\Omega$ towards $A$ and $B$ with $\Omega_n = \Omega - \overline{\Delta_n} \cup \overline{\Delta'_n}$.

For a boundary part $A = \{\Delta_n\}$, the set $I(A) = \cap \text{Cl}(\Delta_n)$ is called the realization of $A$, where Cl(*) denotes the closure on the Riemann sphere.

3. Prime ends. We mention prime ends of a simply connected region which are needed to establish an important lemma in the proof of our continuity theorem of extremal distance. Let $\Delta$ be a bounded simply connected region. A prime end is a boundary part $\xi = \{\Delta_n\}$ with the following properties: the relative boundary of $\Delta_n$, denoted by $q_n$, is a Jordan arc with end points on the boundary of $\Delta$ in the Riemann sphere, no two of $q_n$ have any point, including their ends points in common and the diameter of $q_n$ tends to zero. Following Collingwood and Lohwater [3], we call the sequence $\{q_n\}$ a chain. It is known that every prime end can be defined by a chain consisting of concentric circular arcs [3].

4. Let $U$ be the unit disc $|z|<1$ and let $c_0$ be a Jordan curve $z = z_0(t)$, $0 < t < 1$, in $U$ such that $|z_0(t)| \rightarrow 1$ as $t \rightarrow 0$ and $t \rightarrow 1$ respectively. The $c_0$ divides $U$ into two simply connected subregions, say $V_1$ and $V_2$. Let $z_1$ be a fixed point in $V_1$ and let $c$ be an arbitrary curve in $U$ joining $z_1$ and a point of $V_2$. Then we state

**Lemma 1.** There exist at most two points $\zeta$, on the circle $|z|=1$ and a sequence of discs $|z - \zeta_n| < r^n$ with $r^n \rightarrow 0$ such that the first intersection of $c$ with $c_0$ lies on a fixed compact subarc $c_n$ of $c_0$, unless it runs through the two discs for fixed $n$.

The author expresses his warmest thanks to Mr. M. Tsuzuki for his suggestions in the proof of this lemma.
Proof. We set
\[ C(z_0, 0) = \limsup_{\varepsilon \to 0} \overline{\text{Cl}(\{z_0(t) | t < \varepsilon\})} \]
and
\[ C(z_0, 1) = \liminf_{\varepsilon \to 0} \overline{\text{Cl}(\{z_0(t) | t > 1-\varepsilon\})} \]
which are the cluster sets of \( z_0(t) \) at 0 and 1 and write \( C_0 \) and \( C_1 \) for them, respectively. We first see that there exist one or two prime ends of \( V \) with nonvoid intersection of their realizations with the union \( C_0 \cup C_1 \). In fact, at least one of the end points of \( q_n \) of a chain of such a prime end \( \xi \) lies on \( c_0 \) except for a finite number of \( n \). Indeed, if both the end points lie on the circle \( |z|=1 \) for infinitely many \( n \), we have \( J(\xi) \cap (C_0 \cup C_1) = \emptyset \), which is a contradiction. Thus we may assume that one of the end points is on \( c_0 \) for all \( n \). Then there are possible two cases where both the end points lie on \( c_0 \) for infinitely many \( n \) or only one of them on the unit circle for almost all \( n \). If one of the end points is on \( c_0 \) and the other is on the circle, then all the \( q_m \) with \( m > n \) are the same in their end points. For, if not, both the end points are on the subarc of \( c_0 \) on \( \mathcal{A}_n \), which contradicts the fact that \( J(\xi) \cap (C_0 \cup C_1) = \emptyset \). Hence, in the first case, we have a unique prime end \( \xi \), since \( V \mid \mathcal{A}_n \) is relatively compact and there are no such prime ends. In the second case we may assume that one of the end points of \( q_n \) is on \( c_0 \) and the other is on the circle for all \( n \). The sequence of the end points of \( q_n \) on \( c_0 \) has accumulation points on one of \( C_0 \) and \( C_1 \), say \( C_0 \). Then \( V \mid \mathcal{A}_n \) has another prime end \( \xi' = \{A_n\} \) such that \( J(\xi') \cap C_1 \neq \emptyset \) and each simply connected subregion \( V \mid (\mathcal{A}_n \cup A_n') \) has no desired prime ends, since the intersection of its relative boundary in \( U \) with \( c_0 \) is on a compact subarc.

As is remarked in No. 3, we can take as chains of these prime ends a sequence of concentric circular arcs with centers at \( z_i \) on the unit circle and with radii \( r_j \) tending to zero. These points are so-called principal points [3] which are the desired. To see this, it is sufficient to show it in the case where there exist two prime ends, since otherwise the proof is easier. If the point \( z_i \) is contained in the subregion \( V \mid (\mathcal{A}_n \cup A_n) \) for sufficiently large \( n \) so that \( \mathcal{A}_n \cap A_n' = \emptyset \), then the first intersection of \( c \) with the relative boundary of this region is either on \( q_n \cup q_n' \) or on \( c_n = \text{Cl}(c_0 - (\mathcal{A}_n \cup A_n)) \), which implies the assertion.

§ 3. Extremal distance and its continuity.

5. Extremal length. Let \( \Gamma \) be a family of locally rectifiable curves in \( \Omega \). We denote by \( P(\Gamma) \) the family of Borel measurable metrics \( \rho(z) |dz| \) defined in \( \Omega \) and satisfying
\[ \int_{c} \rho(z) |dz| \geq 1 \]
for all \( c \in \Gamma \), which is called the class of admissible metrics for \( \Gamma \). The module of the curve family is defined by
and the extremal length of $\Gamma$, denoted by $\lambda(\Gamma)$, is its reciprocal. The extremal length of the family of curves joining two sets is called the extremal distance between them.

We prepare a lemma for the later use.

**Lemma 2.** (Hersch [5]). Let $\{\Gamma_n\}$ be at most a countable number of curve families. Then

$$\text{mod } (\cup \Gamma_n) \leq \sum \text{mod } \Gamma_n.$$  

6. The closure of the intersection of $P(\Gamma)$ with the space of $L_2$-metrics is called the class of $L_2$-admissible metrics denoted by $P^*(\Gamma)$. There exists always a unique metric $\rho_0$ in $P^*(\Gamma)$ such that $\|\rho_0\|^2 = \text{mod } \Gamma$, if $P^*(\Gamma) \neq \emptyset$ [10]. The metric $\rho_0$ is termed the extremal metric for $\Gamma$. The following inequality is known [13]:

$$||\rho - \rho_0||^2 \leq ||\rho||^2 - ||\rho_0||^2.$$  

We quote

**Lemma 3.** (Suita [14] and Ziemer [17]). Let $\{\Gamma_n\}_{n=1}^{\infty}$ be a sequence of curve families such that $\Gamma_n \subseteq \Gamma_{n+1}$ for all $n$. Setting $\Gamma_0 = \cup \Gamma_n$, we have

$$\lambda(\Gamma_0) = \lim_{n \to \infty} \lambda(\Gamma_n).$$  

Furthermore, the extremal metrics for $\Gamma_n$ tend strongly to the extremal metric for $\Gamma_0$, if $\lambda(\Gamma_0) > 0$.

A curve family with vanishing module is called an exceptional family. By Lemma 2 the union of at most a countable number of exceptional families is also exceptional. The extremal length of a curve family remains unchanged, if an exceptional family is added or subtracted. A proposition about a curve family is said to be true for almost all curves, if it holds except for an exceptional family.

7. **Continuity theorem.** Let $A$ and $B$ be two disjoint boundary parts with respective defining sequences $\{D_n\}$ and $\{D'_n\}$ such that $\bar{D}_n \cap \bar{D}'_n = \emptyset$ and let $\{\Omega_n\}$ be an exhaustion of $\Omega$ towards $A$ and $B$. We denote by $A_n$ and $B_n$ the sets of components of the relative boundary of $D_n$ which are those of $D'_n$ and $D''_n$ respectively. Let $\Gamma_n$ be the family of curves joining $A_n$ and $B_n$ in $\Omega$. We define the curve family $\Gamma_0$ joining $A$ and $B$ by the family of curves running through $\Omega$, intersecting all the members of $D'_n$ and $D''_n$ and clustering at no point of $\Omega$. This family was called the clustering curve family in [15]. We state

**Theorem 1.** If $\lambda(\Gamma_0) > 0$, we have

$$\lambda(\Gamma_0) = \lim_{n \to \infty} \lambda(\Gamma_n).$$
Moreover the extremal metrics $\rho_n$ for $\Gamma_n$ tend strongly to the extremal $\rho_0$ for $\Gamma_0$.

The condition $\lambda(\Gamma_1)>0$ is indispensable, which was shown in [15]. After an auxiliary metric is constructed so that Strebel's proof [11] may be available, we shall need his proof and restate it here for completeness.

8. Proof. Let $\{D_n\}$ be a normal exhaustion of $\Omega$, that is an exhaustion of $\Omega$ towards the whole boundary. We may assume that $D_\nu$ is disjoint from $\mathcal{D}_\nu^1 \cup \mathcal{D}_\nu^2$ with $\nu$ fixed. We first show the continuity for the subfamilies of curves intersecting $D_\nu$. The relative boundary of $\Omega_n$ consists of a finite number of open or closed Jordan curves and each of the former is a curve whose end arcs tend to a common boundary component $\alpha$ by definition. Suppose $\alpha$ is a point. We may assume that $\alpha$ is the point at the origin. Since the extremal distance of the $\alpha$ from $D_\nu$ is infinite in the Riemann sphere, we can construct a metric $\rho^*_n$ such that $||\theta^*||_2^2<\varepsilon$ and $\int_\alpha \rho^*_n|dz|\geq 1$, where $c$ is an arbitrary curve joining $D_\nu$ and the disc $|z|\leq \delta$ for sufficiently small $\delta$ within $\Omega$. Next if $\alpha$ is not a point, by Riemann’s mapping theorem to the component of $\alpha$ containing $\Omega$, we may assume that $\alpha$ is the unit circle $|z|=1$ which is its outer boundary. Then, by the same reason, there exists an admissible metric $\rho^*_n$ for the family of curves joining $D_\nu$ and the one or two discs $|z-\zeta_j|\leq \rho^*_n$ in Lemma 1 for so large $n$ that $||\rho^*_n||^2<\varepsilon$, where each relative boundary component of $\Delta_n$ tending to $\alpha$ is taken as $c_0$ in the lemma.

Since there are at most a countable number of such boundary components over all the $\Omega_n$, by arranging them, we can construct a sequence of metrics $\rho^*_n=k+1(k\geq 1)$ which are admissible for the families of curves joining $D_\nu$ and $E_k$, where $E_k$ is the inverse image of the subset of the closed discs constructed above in the image region by the Riemann's theorem. We set $\rho'_n(z)=\sup_{k=1}^j \{\rho^*_n(z)\}$ which is clearly admissible for the union of these curve families. We have $||\rho'_n||^2\geq \varepsilon/2$.

We construct another auxiliary metric $\rho''_n$ which is continuous, positive and such that $||\rho''_n||^2<\varepsilon/2$. Indeed, setting $\rho''_n=\kappa \min(|z-a|^{-\gamma}, |z-b|^{-\gamma})$ with different $a$ and $b$, we get a desired metric for sufficiently small $\kappa$.

9. We shall see how Strebel's proof [11] is applied. Let $\Gamma_{nv}$ be the subfamily of $\Gamma_n$ which consists of curves of $\Gamma_n$ intersecting $D_\nu$. It is easily verified that $\text{mod } \Gamma_{nv}$ is decreasing with $n$ and $\text{mod } \Gamma_0=\lim \text{mod } \Gamma_{nv}$. To show the opposite inequality, setting $\rho_0=\max(\rho, \rho', \rho'')$ with $||\rho||^2<\infty$ and $\rho(P(\Gamma_0))$ and

$$L_n=\inf_{c\in \Gamma_{nv}} \int_c \rho_0|dz|,$$

we show

$$\lim_{n \to \infty} L_n \geq 1. \quad (2)$$

Contrary to the assertion, suppose $\lim L_n=l<1$. Then there exists a sequence of curves $c_n \in \Gamma_{nv}$ such that

$$\int_{c_n} \rho_0|dz|<l'=\frac{l+1}{2}<1,$$
since $L_n$ is increasing. Let $m$ be arbitrarily fixed. The $c_n$ with its initial point on $A_n$ intersects $A_m$ and $B_m$ for $n \geq m$ and its last and first intersections with them are on a compact subset $K_m$ of them in $\Omega$ respectively. In fact, if a noncompact component of $A_m$ and $B_m$ terminates at a component $\alpha$ which is a point, its intersection with $K_m$ is the complement of the interior of an $E_k$ with respect to it. If it terminates at a continuum $\alpha$, its intersection is the conformally equivalent arc of the arc in Lemma 1, by applying the lemma to the conformally equivalent unit disc of the component of $\alpha$ containing $\Omega$.

Thus, by diagonal process, we may assume that the two sequences of the above intersections are convergent to the points $a_m$ and $b_m$ on $A_m$ and $B_m$ respectively. Since $||\rho_\beta||^2 < \infty$, there exist two sequences of discs $|z - a_m| \leq r_m$ and $|z - b_m| \leq r_m$ contained in $\Omega$ and such that

$$\int_{|z - a_m| = r_m} \rho_s |dz| + \int_{|z - b_m| = r_m} \rho_s |dz| < \frac{\eta}{2^{m+1}} \quad (m = 1, 2, \ldots).$$

We denote by $c^{(m)}_n$ and $c^{(-m)}_n$ the subarcs of $c_n$ between its last intersections with $A_m$ and $A_{m+1}$ and first intersections with $B_m$ and $B_{m+1}$ respectively. Let $c^{(0)}_n$ denote its subarc between its last and first intersections with $A_1$ and $B_1$. Setting

$$\lim_{n \to \infty} \int_{c^{(m)}_n} \rho_s |dz| = l_m, \quad m = 0, \pm 1, \ldots,$$

we have easily

$$\sum_{m = -\infty}^{\infty} l_m \leq l'.$$

Hence we can take arcs $c^{(m)}_n$ such that

$$\int_{c^{(m)}_n} \rho_s |dz| < l_m + \frac{\eta}{4^{m+2}}$$

and intersecting both the circles around $a_m$ and $b_m$ for sufficiently large $n$. Connecting these arcs by the circular arcs passing through $\Omega - \mathcal{G}_n$, we obtain a curve $\hat{c}$ such that

$$\int_{\hat{c}} \rho_s |dz| < l' + \eta.$$  

The curve $\hat{c}$ belongs to $\Gamma_{\eta}$. It is sufficient to show that $\hat{c}$ has no cluster points in $\Omega$. Indeed, if any, there exists a relatively compact subregion containing the closure of a neighborhood of a cluster point. Then it intersects both the boundaries of these sets infinitely often and since $\rho_s$ has a positive lower bound in any compact subregion by virtue of $\rho_s'$, we have $\int \rho_s |dz| = \infty$, which contradicts (5). Since $\eta$ is arbitrary in (5), we get a contradiction to the admissibility of $\rho_s$, which implies (2).
Then by (2) \((1+\eta)\rho_n\) is admissible for \(\Gamma_n\) for a sufficiently large \(n\) and we have

\[
\lim_{n \to \infty} \text{mod } \Gamma_n \leq \|\rho_n\|^2.
\]

Moreover \(\|\rho_n\|^2 \leq \|\rho_n\|^2 + \varepsilon\) holds by definition and since the \(\varepsilon\) is arbitrary, we have \(\text{mod } \Gamma_0 \geq \lim \text{mod } \Gamma_n\)

and

\[
(6) \quad \text{mod } \Gamma_0 = \lim_{n \to \infty} \text{mod } \Gamma_n.
\]

10. To show that \(\lambda(\Gamma_n) \to \lambda(\Gamma_0)\), we have at first

\[
\lim_{n \to \infty} \text{mod } \Gamma_n \leq \text{mod } \Gamma_0,
\]

since \(P(\Gamma_n) \subset P(\Gamma_0)\). Next let \(A_n\) be the subfamily of \(\Gamma_n\) consisting of curves which never intersect \(D_v\). Then we have \(\Gamma_n = \Gamma_{nv} \cup A_n\) and from Lemma 2

\[
\text{mod } \Gamma_n \leq \text{mod } \Gamma_{nv} + \text{mod } A_n.
\]

There exists a metric \(\mu \in P(\Gamma_0)\) such that \(\|\mu\|^2 < \infty\) by the assumption \(\lambda(\Gamma_0) > 0\). Then the metric

\[
\mu_n = \begin{cases} 
\mu & \text{on } \Omega - D_v, \\
0 & \text{elsewhere}
\end{cases}
\]

is admissible for \(A_n\), \(n=1, 2, \cdots\) and such that \(\|\mu_n\|^2 \to 0\) as \(v \to \infty\). Hence there exists an \(N\) such that \(\text{mod } A_n < \varepsilon\) for \(v \geq N\). Letting \(n \to \infty\), we have from (6)

\[
\lim_{n \to \infty} \text{mod } \Gamma_n \leq \text{mod } \Gamma_0 + \varepsilon,
\]

for \(v \geq N\), and by letting \(v \to \infty\), from Lemma 3

\[
\lim_{n \to \infty} \text{mod } \Gamma_n \leq \text{mod } \Gamma_0 + \varepsilon.
\]

Since \(\varepsilon\) is arbitrary, we get the assertion.

As for the convergence of \(\rho_n\), we have from (1)

\[
\|\rho_m - \rho_n\|^2 \leq \|\rho_m\|^2 - \|\rho_n\|^2
\]

for \(n \geq m\), which implies the strong convergence of \(\rho_n\) to a metric \(\rho_0\). Since \(\bigcup P(\Gamma_n) \subset P(\Gamma_0)\), \(\rho_0 \in P^*(\Gamma_0)\), whence \(\rho_0\) is extremal for \(\Gamma_0\).


11. After the continuity of extremal distance is established, we can give a conformal representation of \(\Omega\) onto a slit rectangle by a function related to the extremal metric for \(\Gamma_0\) of special \(A\) and \(B\),
Let $A$ and $B$ be two disjoint boundary parts with $\bar{A} \cap \bar{B} = \emptyset$ which are on a common boundary component $\alpha$ of $\Omega$. Suppose that $\Omega$ can be divided into two subregions by a Jordan curve in such a way that $A$ and $B$ are regarded as the boundary parts of different parts. Each $\bar{A}$ consists of a finite number of components $\partial \bar{A}$ whose relative boundary is a single Jordan curve. Then we can select a sequence $\{\partial \bar{A}_n\}_{n=1}^\infty$ which makes a boundary part. Such a boundary part is said to be elementary. Then $A$ and $B$ are decomposed into elementary boundary parts.

Let $\Omega_n$ be an exhaustion of $\Omega$ towards $A$ and $B$ and let $A_n$ and $B_n$ be the relative boundaries of $\partial \bar{A}$ and $\partial \bar{B}$ respectively. Let $a^\prime_n$ denote the boundary component of $\Omega_n$ containing them. The component of $a^\prime_n$ containing $\Omega_n$ is mapped onto the unit disc by Riemann’s mapping theorem. Then the images of $\partial \bar{A}_n$ and $\partial \bar{B}_n$ are open arcs, denoted by $\partial \bar{A}_n$ and $\partial \bar{B}_n$, on the unit circle, where $\partial \bar{A}_n$ and $\partial \bar{B}_n$ are the relative boundaries of components $\partial \bar{A}_n$ and $\partial \bar{B}_n$ respectively. When the circle is positively oriented with respect to the disc, we may assume that $\partial \bar{A}_n$ and $\partial \bar{B}_n$ are arranged in such a way that $\partial \bar{A}_1, \partial \bar{A}_2, \ldots, \partial \bar{B}_1, \partial \bar{B}_2, \ldots$ lie in this orientation.

Next we can construct a defining sequence $\{\partial \bar{A}_n\}$ of $a^\prime_n$, each of whose relative boundaries intersects every $\partial \bar{A}_n$ and $\partial \bar{B}_n$ just twice. We denote by $\partial \bar{A}_n^\prime$ the non-compact subregion of $\partial \bar{A}_n$ bounded by the subarcs of $\partial \bar{A}_n$ and $\partial \bar{B}_n$ and the relative boundary of $\partial \bar{A}_n$. The $\{\partial \bar{A}_n^\prime\}$ defines an elementary boundary part $\partial \bar{A}_n$, which is called the boundary part determined by these two end arcs. We denote by $\partial \bar{B}_n, \partial \bar{C}_n$ and $\partial \bar{D}_n$ the boundary part between $\partial \bar{B}_n$ and $\partial \bar{B}_{n+1}$, the last element of $\partial \bar{A}_n$ and the first of $\partial \bar{B}_n$, and the last of $\partial \bar{B}_n$ and the first of $\partial \bar{A}_n$ similarly defined respectively.

Then we can define an exhaustion $\{\Omega_m\}$ of $\Omega$ towards the sets $\partial \bar{A}_1, \partial \bar{B}_1, \partial \bar{C}_1, \partial \bar{D}_1$ and the other boundary components than $\alpha$. We designate the relative boundaries of the $m$-th members of the defining sequences of the first four as $\partial \bar{A}_m, \partial \bar{B}_m, \partial \bar{C}_m, \partial \bar{D}_m$. It should be remarked that these sets may be void except for $\partial \bar{C}_n$ and $\partial \bar{D}_n$.

12. Since the $\Omega_m$ is regularly imbedded, by a standard method [1], we can construct a unique function $u_m$ harmonic in $\Omega_m$ and such that

i) $u_m = 0$ on the intersection of $\partial \bar{A}_m$ with $\partial \bar{C}_m$, denoted by $\partial \bar{A}_m^\prime$,

ii) $u_m = 1$ on the intersection of $\partial \bar{B}_m$ with $\partial \bar{D}_m$, denoted by $\partial \bar{B}_m^\prime$, and

iii) $\partial u_m = 0$ along the relative boundary of $\Omega_m$ in $\Omega$.

Set $\psi_m = u_m + iv_m$, where $v_m$ is the conjugate harmonic function of $u_m$ which is single-valued from iii), normalized so that $v_m = 0$ on $\partial \bar{C}_m$. We have $\psi_m = h_m$ on $\partial \bar{D}_m^\prime$. The function $\psi_m$ maps $\Omega_m$ onto an incised horizontal slit rectangle in such a way that

i) $\psi_m(\alpha_m)$ is the boundary of a horizontally incised rectangle, whose closure is such that $0 \leq \text{Re} \psi_m \leq 1$ and $0 \leq \text{Im} \psi_m \leq h_m$, where $\alpha_m$ is the boundary component containing $\partial \bar{A}_m^\prime$,

ii) $\psi_m(\partial \bar{A}_m)$ and $\psi_m(\partial \bar{B}_m)$ are arcs on the imaginary axis and the line $\text{Re} \psi_m = 1$ which are non-overlapping and whose sums over $j$ and $k$ are equal to $h_m$ respectively,

iii) $\psi_m(\partial \bar{A}_m^\prime)$ and $\psi_m(\partial \bar{B}_m^\prime)$ are horizontal incisions emanating from the
boundary points of adjacent vertical arcs,

iv) \( \psi_{nm}(C_{n}^{(m)}) \) and \( \psi_{nm}(D_{n}^{(m)}) \) are horizontal sides on the real axis and the line \( \text{Im} \ \psi_{nm}=h_{nm} \) respectively,

v) the images of the boundary components other than \( \alpha_{nm} \) under \( \phi_{nm} \) are a finite number of horizontal slits, and

vi) the metric \( \rho_{nm}=|\psi'_{nm}| \) is extremal for the family \( \Gamma_{nm} \) of curves joining two points of \( A_{nm}^{\infty} \)'s and \( B_{nm}^{\infty} \)'s and \( \text{mod} \ \Gamma_{nm}=h_{nm} \).

The properties from i) to v) are obvious since \( \Omega_{nm} \) is a regularly imbedded subregion of \( \Omega \). We remark that the \( \phi_{nm} \) can be directly constructed by duplication [13].

The last property vi) is verified as follows: We have ||\( \rho \)=h_{nm} || \( \rho \in P(\Gamma_{nm}) \) by Schwarz's inequality and since \( \rho_{nm}=|\phi'_{nm}| \) belongs to \( P(\Gamma_{nm}) \) with ||\( \rho_{nm} \)||=h_{nm}, it is thus extremal.

13. The sequence \( \Gamma_{nm} \) is increasing and we set \( \Gamma_{n}=\bigcup_{m} \Gamma_{nm} \). Note that a curve of \( \Gamma_{n} \) joins two points of \( A_{n} \) and \( B_{n} \) within \( \Omega \). We have by Lemma 3

\[
\text{mod} \ \Gamma_{n}=\lim_{m \to \infty} \text{mod} \ \Gamma_{nm},
\]

and the metrics \( \rho_{nm} \) tend to the extremal metric \( \rho_{n} \) strongly, if \( \lambda(\Gamma_{n})>0 \). We can deduce, from this convergence, that there exists a univalent function \( \psi_{n} \) such that \( \rho_{n}=|\phi'_{n}| \) and ||\( \phi_{nm}'-\phi_{n}' \)||_{n} \to 0 \) [15]. Clearly \( \text{Re} \ \psi_{n}=0 \) on \( A_{n} \) and \( \text{Re} \ \phi_{n}=1 \) on \( B_{n} \), since a half neighborhood of every point of them is regularly imbedded.

Finally letting \( n \to \infty \), we have \( \lambda(\Gamma_{n}) \to \lambda(\Gamma_{0}) \) from Theorem 1 and the metrics \( \rho_{n} \) tend to its extremal metric \( \rho_{0} \) strongly where \( \Gamma_{0} \) is the family of curves joining \( A \) and \( B \). Suppose further that \( \lambda(\Gamma_{0})<\infty \), that is \( h_{0}=\text{mod} \ \Gamma_{0}>0 \). Then \( \phi_{n} \) tends to a univalent function \( \phi_{0} \) in such a way that ||\( \phi'_{n}-\phi'_{0} \)||_{n} \to 0 \) as \( n \to \infty \). Then we state

**Theorem 2.** Suppose \( \lambda(\Gamma_{1})>0 \) and \( \lambda(\Gamma_{0})<\infty \). The function \( \phi_{0}(z) \), above constructed, is univalent and possesses the following properties:

i) \( I(\phi_{0}(\xi)) \) is the boundary of horizontally incised rectangle whose closure is such that \( 0 \leq \text{Re} \ \phi_{0} \leq 1 \) and \( 0 \leq \text{Im} \ \phi_{0} \leq h_{0} \),

ii) \( I(\phi_{0}(A)) \) (resp. \( \phi_{0}(B) \)) contains the vertical side \([0,ih_{0}]\) (resp. \([1,1+ih_{0}]\)) and is disjoint from the open opposite side including its possible incisions,

iii) \( I(\phi_{0}(\xi)), \xi \in A \) (resp. \( \xi \in B \)), is either a closed arc (possibly a point) on the vertical side \([0,ih_{0}] \) (resp. \([1,1+ih_{0}] \)) with possible horizontal incisions emanating from it or a segment (possibly a point) on an incision, where \( \xi \) denotes an elementary boundary part,

iv) \( I(\phi_{0}(\partial \Omega-\xi)) \) is a minimal set of horizontal slits,

v) the area of \( I(\phi_{0}(\partial \Omega)) \) vanishes, and

vi) the metric \( \rho_{0}=|\phi'_{0}| \) is extremal for \( \Gamma_{0} \) and \( \text{mod} \ \Gamma_{0}=h_{0} \).

14. **Proof.** We have already proved vi). To prove i), iv) and v), we shall make use of Reich's method [8].
Let \( c \) be a curve of \( \Gamma_0 \) with its parametrization \( z = z(t), \ 0 < t < 1 \), which intersects all the members of a defining sequence of \( A \) as \( t \to 0 \). Then we show
\[
\lim_{t \to 1} \text{Re } \phi_0(z(t)) - \lim_{t \to 0} \text{Re } \phi_0(z(t)) = 1
\]
for almost all curves \( c \in \Gamma_0 \). In fact, the function \( \psi_n \) in No. 13 is such that \( \text{Re } \psi_n = 0 \) and \( \text{Re } \phi_n = 1 \) on \( A_n \) and \( B_n \) respectively. Set
\[
\mu_n = \begin{cases} 
|\text{grad Re}(\psi_0 - \phi_n)| & \text{in } \Omega_n, \\
0 & \text{elsewhere},
\end{cases}
\]
then we have by a lemma given in [6] or directly as in [13]
\[
\left| \int_{c \in \sigma_n} d \text{Re } \phi_0 - d \text{Re } \phi_n \right| \to 0
\]
for almost all \( c \in \Gamma_0 \) as \( n \to \infty \), since \( ||\mu_n||^2 \to 0 \), which implies (7).

Let \( A \) denote the exceptional family of curves along which the equality (7) does not hold. Set \( w = \phi_0, \ A = \phi_0(\Omega) \) and denote by \( \Gamma_0' \) and \( A' \) the families of the image curves of the members of \( \Gamma_0 \) and \( A \) under \( \phi_0 \), respectively. Then \( \text{mod } \Gamma_0' = \text{mod } (\Gamma_0' - A') = h_0 \), since \( A' \) is also exceptional. Next consider two rectangles \( R_1: 0 < \text{Re } w < s, \ 0 < \text{Im } w < h_0 \) and \( R_2: 1 - s < \text{Re } w < 1, \ 0 < \text{Im } w < h_0 \) and the family, denoted by \( \Gamma'' \), of curves joining two points of vertical sides within \( A = \Omega \cup R_1 \cup R_2 \). Then we have \( \Gamma'' \ni \Gamma_0' - A' \) and \( \text{mod } \Gamma'' \geq h_0 \). On the other hand the metric \( \rho = 1 \) is admissible, which thus extremal and \( \text{mod } \Gamma'' = h_0 \). Then by a criterion of minimality given in [13] (cf. [15]) the inner boundary components of \( A \), form a minimal set of horizontal slits. Thus we have proved the properties other than ii) and iii).

In order to prove ii), suppose at first that, for example, \( I(\phi_0(A)) \) does not contain the side \([0, ih_0]\). Then there exists an open segment \( J \) contained in the vertical side \([0, ih_0]\) and disjoint from \( I(\phi_0(A)) \). As before, the module of the subfamily \( \Gamma'' \) of \( \Gamma_0' \) consisting of the curves satisfying (7) is equal to \( h_0 \). Consider another module problem for the family \( \Gamma''' \) of curves joining the vertical side \([0, ih_0]\) less \( J \) and the opposite vertical side in the rectangle: \( 0 < \text{Re } w < 1, \ 0 < \text{Im } w < h_0 \). Clearly \( \Gamma'' \ni \Gamma'' \) and \( \text{mod } \Gamma'' \geq \text{mod } \Gamma'' \). On the other hand, from the result in No. 12 and the unicity of the extremal metric, we get \( \text{mod } \Gamma'' < \text{mod } \Gamma'' \), which is a contradiction.

Next suppose that \( I(\phi_0(A)) \) contains an interior point of the side \([1, 1+ih_0]\) or a point of an incision emanating from it. Let \( \omega \) be that point with \( 0 < \text{Im } \omega < h_0 \) and let \( J_1 \) and \( J_2 \) be two closed subsets of \( I(\phi_0(B)) \) divided by the line \( \text{Im } w = \text{Im } \omega \). Then one of \( J_k \) is contained in a single \( I(\phi_0(\xi_0)), \xi_0 \in A \). If it were not, we would construct a Jordan curve with ends in \( \phi_0(B) \) dividing a boundary part of \( \phi_0(A) \) containing \( \omega \) from its boundary parts on \([0, ih_0]\), which contradicts the assumption in No. 11.

Let \( J' \) denote the projection of the \( J_k \) into the line \( \text{Re } w = 1 \). Then there exists
a disc \(|w-w_0|<r\), contained in \(\Delta\) whose projection to the line \(\text{Re } w=1\) is contained in the segment \(J'\). By the same reason as above, a curve \(c^*\) joining the disc and the \(\phi_0(\xi_0)\) enjoys

\[
\text{Re } w \to 0
\]

as \(w\) tends to \(\phi_0(\xi_0)\) along almost all \(c^*\). Let \(A^*\) be the exceptional family for the property (8). Then from Strebel's inequality given in [15] for the disc and the segment we get

\[
\text{mod } A^* \geq \int_{\text{Im } w_0}^{\text{Im } w_0 + r} \frac{dy}{k(y)} > 0
\]

where \(l(y)\) is the length of the horizontal segment with \(\text{Im } w = y\) \(|y-\text{Im } w_0|<r\) between them, which is a contradiction to the property (8). Thus the property ii) has been completed. The property iii) immediately follows from the connectedness of \(\phi_0(\xi)\).

15. Remarks. The realizations \(I(\phi_0(\xi))\) and \(I(\phi_0(\xi'))\) with \(\xi, \xi' \subset A \cup B\) may have their intersection containing a continuum (not a point), if they are distinct. Note that these two elementary boundary parts are disjoint by definition. There exists an elementary part \(\xi\) disjoint from \(A\) and \(B\) such that \(I(\phi_0(A)) \cap I(\phi_0(\xi))\) contains a segment on the vertical side \([0, i\theta_0]\). The latter example was given in [15] and examples of the former type can be constructed by making use of a characterization theorem in the next No. 16 which will be omitted.

16. Extremal property of \(\psi_0\). Let \(\mathcal{F}\) be the family of univalent function \(f(z)\) in \(\Omega\) such that \(0<\text{Re } f<1\), \(\inf \text{Im } f=0\), and \(\text{Re } f(z)\rightarrow 0\) and \(\text{Re } f(z)\rightarrow 1\) as \(z\) tends to \(A\) and \(B\) along almost all curves joining \(A\) and \(B\) respectively. Setting \(H(f) = \sup_{e \in \mathcal{E}} \text{Im } f\), we have similarly as in [15]

**Theorem 3.** Suppose that the extremal distance between \(A\) and \(B\) in \(\Omega\) is positive and finite. Then the function \(\phi_0\), if it exists, is the unique function which minimizes the quantity \(H(f)\) in \(\mathcal{F}\).

**Proof.** We have from (1) for \(\phi_0\) and \(f\)

\[
||f'|| - ||\psi'||^2 \leq ||f'||^2 - ||\psi'||^2 \leq H(f) - h_0,
\]

with \(\rho = |f'|\) and \(\rho_0 = |\psi'|\). It is obvious, from Theorem 2 and its proof, that \(\phi_0\) belongs to \(\mathcal{F}\).

An example in which the assumption of the theorem does not guarantee the construction of \(\phi_0\) was given in [15].

The image is called a minimal horizontal slit rectangle with respect to \(A\) and \(B\). We cite its characterization without proof (cf. [15]).

**Corollary 1.** Let \(\Omega\) be a region whose outer boundary is the periphery of the
rectangle \(0<\Re z < 1, 0<\Im z < h\) with possible horizontal incisions emanating from its vertical sides and let \(A\) and \(B\) be two disjoint boundary parts on the outer boundary component \(a\) whose realizations contain the sides \([0, ih]\) and \([1, 1+i h]\) respectively. Suppose that the extremal distance of the relative boundaries of the first members of their defining sequences is positive. Then any two of the following three conditions imply the minimality of \(\Omega\) with respect to \(A\) and \(B\):

i) \(\partial \Omega - \alpha\) is a minimal set of horizontal slits,

ii) \(\Re z \to 0\) and \(\Re z \to 1\) as \(z\) tends to \(A\) and \(B\) along almost all curves joining \(A\) and \(B\) respectively, and

iii) the extremal distance of \(A\) and \(B\) is equal to \(1/h\).

Conversely a minimal slit rectangle with respect to \(A\) and \(B\) possesses all the above properties.

§ 5. Boundary elements.

17. Introduction. In this section we shall generalize the notion of prime ends for an arbitrary region, which introduces a well-defined boundary correspondence. Several approaches to the theory of prime ends in terms of extremal length and its extension were given by Schlesinger [9], Gál [4] and Ohtsuka [7]. Here we shall define a boundary element which corresponds to either a point or an incision in a radial slit disc mapping with a finite radius.

The author expresses his sincere thanks to Professor K. Oikawa with whom he had many valuable discussions.

18. Definitions. Let \(\Omega\) be a region which is not the Riemann sphere. Let \(K\) be a compact disc \(|z-a| \leq r\) contained in \(\Omega\). Consider an elementary boundary part \(\xi = \{\Delta_n\}_{n=1}^{\infty}\) satisfying \(\Delta_n \supset \Delta_{n+1}\). The relative boundary of every \(\Delta_n\) is either closed or open. In either case \(V_n = \Delta_n - \Delta_{n+1}\) is a region and we denote by \(\hat{\Gamma}_n\) the family of curves running within \(V_n\) and intersecting all the curves joining the relative boundaries of \(\Delta_n\) and \(\Delta_{n+1}\). Such a family is called a dividing curve family. An elementary boundary part \(\xi\) with the following properties is called a boundary element:

I) \(\lambda(\hat{\Gamma}_n) < \infty\) i.e. \(\text{mod} \hat{\Gamma}_n > 0\), and

II) the extremal distance between \(K\) and \(\xi\) is infinite.

The condition II) is equivalent to the following

II*) \(\lim \lambda(\Gamma'_n) = \infty\), where \(\Gamma'_n\) is the family of curves joining \(K\) and \(\Delta_n\), from Theorem 1.

19. Radial slit disc mapping. We review the properties of radial slit disc mappings. Let \(\alpha\) be a boundary component of \(\Omega\). Suppose that the extremal distance of \(K\) and \(\alpha\), denoted by \(d(K, \alpha)\) for simplicity, is finite. Then the radial slit disc mapping \(g_{\alpha}\) with a finite radius \(R=R(\alpha)\) normalized by \(g_{\alpha}(a)=0\) and \(g'_{\alpha}(a)=1\) is uniquely constructed, which has the following properties:

i) \(I(g_{\alpha}(\alpha))\) is a circle \(|w|=R\) with possible radial incisions emanating from it,

ii) \(I(g_{\alpha}(\partial \Omega - \alpha))\) is a minimal set of radial slits,

iii) the area of \(I(g_{\alpha}(\partial \Omega))\) vanishes, and
iv) \(|g_\ast(z)| \to R\) as \(z\) tends to \(\alpha\) along almost all curves joining \(K\) and \(\alpha\).

The properties except for iv) are essentially due to Strebel \([11]\) (cf. \([8]\), \([10]\)) and iv) was shown in \([13]\) and \([6]\). We called \(g_\ast(\Omega)\) a minimal radial slit disc.

If \(d(K, \alpha)=\infty\), the \(\alpha\) is a boundary element by definition and a radial slit disc mapping with radius infinite can be constructed \([11, 12]\), but the uniqueness is not yet established. We show

**Lemma 4.** Let \(\Omega\) be a minimal radial slit disc with a finite radius \(R\) in the \(w\)-plane. The points \(Re^\theta\) from which no incisions emanate are accessible in \(\Omega\). They form a \(G_\delta\) set on the circle \(|w|=R\) with angular measure \(2\pi\). Moreover each \(Re^\theta\) is accessible in the intersection of \(\Omega\) with a sector \(|\arg w-\theta|<\varepsilon\) for an arbitrary positive \(\varepsilon\).

**Proof.** The set of incisions whose lengths are not less than \(1/\eta\) is compact, whence so is its projection into the circle \(|w|=R\), denoted by \(\Theta_\eta\). Since the angular measure of \(\Theta_\eta\) vanishes from the property iii), so does the union \(\bigcup_\eta \Theta_\eta\). Hence the set in the lemma is a \(G_\delta\) set of angular measure \(2\pi\). Next let \(Re^\theta\) be a point in the lemma. Consider an exhaustion \(\{\Omega_n\}\) of \(\Omega\) towards \(\alpha\) and set \(V_n=\Omega_n-\Omega_{n+1}\). Let \(A_n\) denote the set of boundary components of \(V_n\) other than its relative boundary. Each boundary component \(\beta \in A_n\) is enclosed by a Jordan curve along which the oscillation of \(\arg w\) is less than \(\varepsilon\), since \(I(\beta)\) is a radial segment. We denote by \(U_n\) its interior. \(I(A_n)\) is covered by a finite number of \(U_n\)'s, since it is compact. Then the segments on the radius \([0, Re^\theta]\) interior of all the \(U_n\)'s are at most countable and replacing them by a union of subarcs of the boundary of \(U_n\), we get a desired curve.

20. **Boundary correspondence.** If a boundary component \(\alpha\) is a boundary element, then it contains no other boundary element. To see it we prove

**Theorem 4.** If a boundary element \(\xi\) exists on a boundary component \(\alpha\) with \(\xi \neq \alpha\), then the extremal distance between \(K\) and \(\alpha\) is finite.

**Proof.** Set \(\xi=\{J_n\}\). We may suppose that the relative boundary of every \(J_n\) is an open Jordan curve, since \(\alpha \neq \xi\). The end arcs of the relative boundary of \(V_n=J_n-J_{n+1}\) determine two boundary parts \(\xi_n\) and \(\xi'_n\). Take an auxiliary disc \(K': |z-b| \leq \delta\) contained in the region \(V_n\). We show that the extremal distances \(d(K', \xi_n)\) and \(d(K', \xi'_n)\) in \(\Omega\) are both finite. In fact, if one of them, for example, \(d(K', \xi_n)=\infty\), the subfamily \(\hat{P}_{K'}\) of curves of \(\hat{P}_n\) passing through \(K'\) is exceptional, since every curve of \(\hat{P}_{\xi'}\) contains a curve of the former module problem as a subset. The finiteness of the extremal distance is independent of such a reference set \([13]\). Enlarging \(K'\) so that it may remain simply connected in the region \(V_n\) and approach to both the relative boundary of \(J_n\) and \(J_{n+1}\), we see that \(\Gamma_n\) is the union of such \(\hat{P}_{\xi'}\). We have \(\lambda(\hat{P}_n)=\infty\) from Lemma 3 which contradicts the condition i) of the definition of a boundary element. So the extremal distances \(d(K', \xi_n)\) and \(d(K', \xi'_n)\) or equivalently \(d(K, \xi_n)\) and \(d(K, \xi'_n)\) are finite. Since they are on \(\alpha\), we have \(d(K, \alpha)<\infty\).
We now state

**Theorem 5.** If the extremal distance between $K$ and a boundary component $\alpha$ is finite and if a boundary element $\xi$ is on the $\alpha$, then the image $I(g_\alpha(\xi))$ is either a point on the circle $|w|=R$ or a possible incision including its end point on the circle, where $g_\alpha$ is the radial slit disc mapping with radius $R$.

**Proof.** $g_\alpha(K)$ contains a disc $|w|\leq r'$, denoted by $K'$. $I(g_\alpha(\xi))$ is a continuum and so is its projection into the circle $|w|=R$. If it is an arc $Re^{i\theta_1}Re^{i\theta_2}$ (not a point), we get as in No. 13 by Strebel's inequality [10] a contradiction

$$d(K', \xi) = \left( \int_{\theta_1}^{\theta_2} \frac{d\theta}{I(\theta)} \right)^{-1} < \infty,$$

where $I(\theta)$ is the logarithmic length of the radial segment between $r'e^{i\theta}$ and the set $I(g_\alpha(\xi))$ for $\theta_1 \leq \theta \leq \theta_2$.

If no incision emanates from the projection of $I(g_\alpha(\xi))$, it is really a point. We show that if an incision $\tau$ emanates from the projection, $I(g_\alpha(\xi))$ coincides with $\tau$. We parametrize the relative boundary of $g_\alpha(\Delta_n)$, denoted by $\gamma_n$, by $w_n(t)$ ($0 < t < 1$) in such a way that $w_n(t)$ runs positively with respect to $g_\alpha(\Delta_n)$ as it moves from 0 to 1, where $\{\Delta_n\}$ denotes a defining sequence of $\xi$. We can deduce that the intersections of the projections of the cluster sets $C(w_n, 0)$ and $C(w_{n+1}, 0)$ and the cluster sets at $t=1$ are both void. Suppose, for example, that the intersection of the projections of the cluster sets at $t=0$ contains a point $Re^{i\theta}$. If no incision emanates from the $Re^{i\theta}$, we can construct, by Lemma 4, a defining sequence of an elementary boundary part $\xi^* = \{\Delta^*_n\}$ such that $I(\xi^*) = Re^{i\theta}$ and that the relative boundary of $\Delta^*_n$ intersects both the $\Delta_n$ and $\Delta_{n+1}$. Then every curve of $\hat{\Gamma}_n$ clusters at the point $Re^{i\theta}$ since it must pass through all the $\Delta^*_n$ and the metric

$$\mu_n = \frac{\varepsilon}{|w-\cdot Re^{i\theta}|} \frac{|\log|w-\cdot Re^{i\theta}||}{|}$$

is admissible for $\hat{\Gamma}_n$ [15]. We get $||\mu_n||^2 \to 0$ as $\varepsilon \to 0$, which contradicts the condition I.

Next, if an incision emanates from the point $Re^{i\theta}$, we take again, by Lemma 4, a defining sequence $\{\Delta^*_n\}$ such that its realization coincides with the incision and that the relative boundary of $\Delta^*_n$ intersects both the $\Delta_n$ and $\Delta_{n+1}$. Then every curve of $\hat{\Gamma}_n$ clusters either at the point $Re^{i\theta}$ or a point on the incision and in the disc $|w|<R$. Let $K''$ be a compact disc in the region $g_\alpha(\Delta_n-\bar{\Delta}_{n+1})$. Then both the families of curves of the former and the latter type are exceptional. Enlarging $K''$ as before, we get the same contradiction.

Therefore the projection of the cluster set $C(w_n, 0)$ is left to the incision $\tau$ and that of $C(w_{n+1}, 1)$ is right to it with respect to the outer normal. Then the set $Cl(g_\alpha(\Delta_n))$ contains $\tau$ which implies the assertion.

21. Next we show, using the above conformal representation,
Theorem 6. If two boundary elements ξ and ξ' are distinct, then they are disjoint.

Proof. Set ξ = \{Δ_n\} and ξ' = \{Δ'_n\}. If they are on distinct boundary components separately, there is nothing to prove. Suppose that they are on a boundary component α. Then from Theorem 4 we have \(d(K, α) < \infty\) and there exists the radial slit disc mapping \(g_α\) with finite radius \(R\) normalized at a point \(Ω\). Contrary to the assertion, we have Δ_n ⊆ Δ'_n for all \(n\) and \(I(g_α(ξ)) = I(g_α(ξ'))\). To see their equivalence, we denote by Θ_n (resp. Θ'_n) the circular arc between the left and right ends of the projections of the cluster sets of the relative boundary of Δ_n (resp. Δ'_n) at \(t=0\) and \(t=1\), respectively, where the relative boundaries are parametrized as before. Θ_{n+1} is contained in the open arc between the right and left ends of the projections of the cluster sets from \(Θ_n\) as is seen in the proof of the preceding theorem. Let \(n_0\) be fixed. Then Θ'_n is contained in the open arc of Θ_\(n_0\) for a sufficiently large \(n \geq n_0\), since Θ_n and Θ'_n are shrinking to a point as \(n \to \infty\). Therefore \(g_α(Δ_n)\) is contained in \(g_α(Δ'_n)\) for a sufficiently large \(m\) and, hence, we get their equivalence.

We have immediately

Corollary 2. If the extremal distance between a boundary component α and a compact disc K is finite, there exists a one to one correspondence between the boundary elements and the points \(Re^φ\) under the radial slit mapping \(g_α\) with finite radius \(R\).

22. A compactification. We denote by \(E\) the set of all boundary elements. We set \(Ω = Ω \setminus \partial\) and define a topology for it by giving a base for the topology. Let ξ be a boundary element with a defining sequence \(\{Δ_n\}\). We denote by \(J_n\) the union of \(Δ_n\) and all the boundary elements a member of each of whose defining sequences is contained in \(Δ_n\). The base is given by the union of the original open sets in \(Ω\) and such a set \(J_n\) for every \(ξ ∈ Ω\). Using the Stoïlow compactification and conformal representations onto radial slit discs, we show

Theorem 7. The topological space \(Ω\) is a compact Hausdorff space.

Proof. We first see the compactness. Let \(O\) be a covering of \(Ω\) and let \(O' \cap J_n\) be the class of its elements containing a boundary element on α, where α is a boundary component. Then there exist a finite number of \(O'_j\)'s such that \(∪ O'_j\) contains all the boundary elements on α and a member of a defining sequence of α. Indeed, it is obvious if α is a boundary element. Otherwise, let \(g_α\) be a radial slit disc mapping with radius \(R\). Since every O containing ξ contains a \(J_n\) of its defining sequence \(\{Δ_n\}\), the \(g_α(O)\) contains the intersection of \(g_α(Ω)\) with a disc \(|w - Re^φ| < δ\) for a sufficiently small δ, where \(Re^φ\) is the projection of \(g_α(ξ)\) into the circle \(|w| = R\). The compactness of the circle implies the assertion. Using the Stoïlow compactification [1], we cover all the boundary components by a finite number of such unions. The complementary set of their union is compact on the Riemann sphere, whence \(Ω\) is compact.
The separation of distinct $\xi$ and $\xi'$ of $E$ follows from Theorem 6.

23. Remarks. Every conformal mapping of $\Omega$ is topologically extended to the $\hat{\Omega}$ and especially the radial slit disc mapping $g_a$ with a finite radius maps distinct boundary elements on $\alpha$ onto those with disjoint realizations.

A boundary component is identified with a connected component of $E$.

An elementary boundary part with extremal distance infinite from a compact disc in $\Omega$ is contained in a unique boundary element. It is easily verified by making use of radial slit disc mappings. But there exists a boundary part with a finite extremal distance which contains no boundary elements.

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