INTEGRAL FORMULAS FOR SUBMANIFOLDS OF CODIMENSION 2 AND THEIR APPLICATIONS

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§ 1. Introduction

Various integral formulas for hypersurfaces of a Riemannian manifold have been found and applied to the study of closed hypersurfaces with constant mean curvature.

Integral formulas for submanifolds of codimension greater than 1 was first obtained by Okumura [6] for the case of submanifolds of codimension 2 of an odd dimensional sphere. He made use of the natural contact structure of the odd dimensional sphere. Integral formulas for general submanifolds of a Riemannian manifold have been obtained by Katsurada [1, 2, 3], Kōjyō [2], Nagai [3, 4], and Yano [9].

In a recent paper [7], Okumura obtained integral formulas for a submanifold of codimension 2, invariant under the curvature transformation, of a Riemannian manifold admitting an infinitesimal conformal transformation and used them to prove that, under certain conditions, the submanifold in question is totally umbilical.

In the present paper, we study a problem similar to that treated in [7]. In [7], the ambient Riemannian manifold was supposed to admit an infinitesimal conformal transformation, but in this paper, we assume instead that there exists a vector field along the submanifold whose covariant differential is proportional to the displacement. We do not assume that the submanifold is invariant under the curvature transformation but instead we put a condition on the integral of a quantity depending on the curvature.

We moreover study the case in which the ambient Riemannian manifold admits a scalar function \( v \) such that \( \nabla_j \nabla_i w = f(v) g_{ij} \) and prove that the submanifold satisfying certain conditions is isometric to a sphere by a method used in [8].

§ 2. Submanifolds of codimension 2.

We consider an \((n+2)\)-dimensional orientable Riemannian manifold \( M^{n+2} \) of differentiability class \( C^\infty \) covered by a system of coordinate neighborhoods \( \{ U; x^i \} \), where and in the sequel the indices \( h, i, j, \ldots \) run over the range \( \{ 1, 2, \ldots, n, n+1, n+2 \} \). We denote by \( g_{ij}, \{ h^i \}, \nabla_i \), and \( K_{kij}^\alpha \), the metric tensor, the Christoffel symbols formed...
with $g_{\beta\alpha}$, the operator of covariant differentiation with respect to $\{\beta^\alpha\}$, and the curvature tensor of $M^{n+2}$ respectively.

We consider an $n$-dimensional orientable submanifold $M^n$ differentiably imbedded in $M^{n+2}$ and denote by

$$x^b = x^b(u^e)$$

its parametric representation, where and in the sequel the indices $a, b, c, d, e$ run over the range $(1, 2, \cdots, n)$. If we put

$$B_b^h = \partial_b x^h, \quad (\partial_b = \partial/\partial u^b)$$

then $B_b^h$, for each fixed $b$, is a vector field tangent to $M^n$ and $B_b^h$ are linearly independent. A Riemannian metric

$$g_{cb} = g_{\beta\alpha}B_c^\beta B^\alpha_b$$

is induced on $M^n$. We denote by $\{\epsilon^a_b\}$, $F_c$ and $K_c^d_{\epsilon\alpha}$, the Christoffel symbols formed with $g_{\beta\alpha}$, the operator of covariant differentiation with respect to $\{\epsilon^a_b\}$ and the curvature tensor of $M^n$ respectively.

Now, the so-called van der Waerden-Bortolotti covariant derivative of $B_b^h$ is given by

$$F_c B_b^h = \partial_c B_b^h + \left\{ \begin{array}{c}
\{ h \cr i \end{array} \right\} B_i^j B_{b}^j - B_{a}^b \left\{ \begin{array}{c}
a \cr c \cr b \end{array} \right\}.$$ 

Since $F_c B_b^h$, as vectors of $M^{n+2}$, are normal to $M^n$, the vector field

$$H^h = \frac{1}{n} g^{cb} F_c B_b^h$$

is normal to the submanifold $M^n$ and is called the mean curvature vector of $M^n$.

We assume throughout the paper that the mean curvature vector never vanishes and take the first unit normal $C^h$ to $M^n$ in the direction of the mean curvature vector. We take the second unit normal $D^h$ in such a way that $B_1^h, B_2^h, \cdots, B_n^h, C^h$ and $D^h$ give the positive orientation of $M^{n+2}$.

Then the equations of Gauss and those of Weingarten are written as

$$F_c B_b^h = h_{cb} C^h + k_{cb} D^h$$

and

$$\begin{cases}
F_c C^h = -h_{cb} B_c^h + l_c D^h, \\
F_c D^h = -k_{cb} B_c^h - l_c C^h
\end{cases}$$

respectively, where $h_{cb}$ and $k_{cb}$ are the second fundamental tensors with respect to the normals $C^h$ and $D^h$ respectively and $l_c$ the third fundamental tensor, $h_{c}^a$ and $k_{c}^a$ being defined by
The normals $C^h$ and $D^h$ being chosen intrinsically, the quantities $h, k$ and $l$ are all intrinsic quantities of $M^n$.

Since $(1/n)g^{a b}V_e B_a^h$ is in the direction of $C^h$, we see from (2.6) that

$$(2.8) \quad g^{a b} k_{e b} = k^e = 0.$$

Now the equations of Gauss, those of Codazzi and those of Ricci are respectively written as

$$(2.9) \quad K_{k j i h} B^k B^j B^h = K_{k d e b} - h_d a h_{e b} + k_{d a} h_{e b} - k_{d b} k_{e d} + k_{e d} k_{d b},$$

$$(2.10) \quad \begin{cases} K_{k j i h} B^k B^j C^h = V_{d h e b} - V_{e h d b} - l_d k_{e b} + l_e k_{d b}, \\ K_{k j i h} B^k B^j D^h = V_{d e h b} - V_{e d h b} - l_d k_{e b} - l_e h_{d b}, \end{cases}$$

$$(2.11) \quad K_{k j i h} B^k B^j C^h D^h = V_{d e c} + h_{d a} k_c - h_{e a} k_c.$$

§ 3. Vector fields along the submanifold of codimension 2.

Take a normal vector field

$$(3.1) \quad V^h = \lambda C^h + \mu D^h.$$

Then, using equations of Weingarten, we have

$$V_e V^h = (-\lambda h_e^a - \mu k_e^a) B_a^h + (\partial_e \lambda - l_e \mu) C^h + (\partial_e \mu + l_e \lambda) D^h,$$

and consequently the connection induced in the normal bundle from the Riemannian connection of $M^{n+2}$ is given by

$$(3.2) \quad V_e' \lambda = \partial_e \lambda - l_e \mu, \quad V_e' \mu = \partial_e \mu + l_e \lambda.$$

Thus in order that a normal vector field $\lambda C^h + \mu D^h$ be parallel with respect to the connection induced in the normal bundle, it is necessary and sufficient that

$$(3.3) \quad \partial_e \lambda - l_e \mu = 0, \quad \partial_e \mu + l_e \lambda = 0.$$

These equations show that

$$\lambda^2 + \mu^2 = \text{constant},$$

that is, a normal vector field parallel with respect to the connection induced in the normal bundle is of constant length.

If $\lambda C^h (\neq 0)$ is parallel with respect to the connection induced in the normal bundle, then we have

$$\lambda = \text{const. and } l_e = 0,$$
and conversely. If $\mu D^h(\neq 0)$ is parallel, we have

$$\mu=\text{const. and } l_c=0,$$

and conversely. Thus, in order that the mean curvature vector $(1/n)g^{ab}\nabla_a B_b^h(\neq 0)$ be parallel with respect to the connection induced in the normal bundle, it is necessary and sufficient that

$$h_a^a=\text{const.} \neq 0, \quad l_c=0.$$

Take next a vector field $X^h$ defined along the submanifold $M^n$ and assume that the covariant differential of this vector field is always proportional to the displacement along the manifold. For such a vector field we have

$$(3.4) \quad \nabla_b X^h = f B_b^h,$$

$f$ being a scalar function of $M^n$.

If we put

$$(3.5) \quad X^h = z^a B_a^h + \alpha C^h + \beta D^h,$$

we have

$$\nabla_b X^h = (\nabla_b z^a - \alpha h_b^a - \beta k_b^a) B_a^h$$

$$+ (\partial \alpha - h_b^a + k_{ba} z^a) C^h$$

$$+ (\partial \beta + h_b^a + k_{ba} z^a) D^h.$$

Thus if we assume that the covariant differential of $X^h$ is proportional to the displacement along $M^n$, then we have

$$(3.6) \quad \nabla_b z^a = f \delta_b^a + \alpha h_b^a + \beta k_b^a$$

or

$$(3.7) \quad \nabla_b z^a = f g_{ba} + \alpha h_{ba} + \beta k_{ba}$$

and

$$(3.8) \begin{cases} 
\partial_b \alpha - h_b^a + k_{ba} z^a = 0, \\
\partial_b \beta + h_b^a + k_{ba} z^a = 0.
\end{cases}$$

§ 4. Integral formulas for a closed submanifold of codimension 2.

We consider an $(n+2)$-dimensional Riemannian manifold $M^{n+2}$ and a closed orientable submanifold $M^n$ of codimension 2 imbedded in it. We assume that there exists a vector field
\[ X^h = \alpha B^a + \beta C^a + \beta D^a \]

along \( M^n \) whose covariant differential along \( M^n \) is always proportional to the displacement:

\[ \nabla_c X^h = \alpha B_c^a. \]

Then we have

\[ \nabla_c z_b = f_{gb} + \alpha h_{gb} + \beta k_{gb}, \]

from which

\[ g^{cb} \nabla_c z_b = nf + \alpha a^a. \]

Thus, integrating over \( M^n \), we find

\[ \int_{M^n} (nf + \alpha a^a) dV = 0, \]

where \( dV \) denotes the volume element of \( M^n \).

We now compute \( \nabla_a (h_b^a z_b) \):

\[ \nabla_a (h_b^a z_b) = (\nabla_a h_b^a) z_b + h_b^a \nabla_z a \]
\[ = (\nabla_a h_b^a) z_b + h_b^a (f_{gb} + \alpha h_{ga} + \beta k_{gb}) \]
\[ = (\nabla_a h_b^a) z_b + f h_a^a + \alpha h_{ba} h_{ba} + \beta h_{ba} k_{ba}. \]

But, from the first of equations (2.10) of Codazzi, we have

\[ K_{kji} B_d^k B^j C_b = \nabla_a h_a^a - \nabla_a h_d^a + l_d^a k_a^a, \]

where

\[ B^{ik} = g^{cb} B_{c} B_{b}^i, \]

and consequently we have

\[ \nabla_a (h_b^a z_b) = -K_{kji} B_d^k z_d B^j C_b + z_d \nabla_a h_a^a + l_a k_d^a z_d \]
\[ + f h_a^a + \alpha h_{ba} h_{da} + \beta h_{ba} k_{ba}. \]

Thus, integrating over \( M^n \), we obtain

\[ \int_{M^n} K_{kji} B_d^k z_d B^j C_b dV \]
\[ = \int_{M^n} (z_d \nabla_a h_a^a + l_a k_d^a z_d + f h_a^a + \alpha h_{ba} h_{da} + \beta h_{ba} k_{ba}) dV. \]
§ 5. Closed submanifolds with mean curvature vector parallel with respect to the connection induced in the normal bundle.

We consider a closed orientable submanifold $M^n$ of codimension 2 of an $(n+2)$-dimensional Riemannian manifold $M^{n+2}$ and assume that $M^n$ admits a vector field $X^h$ whose covariant differential along $M^n$ is always proportional to the displacement:

\begin{equation}
\nabla_c X^h = fB^c_h
\end{equation}

and that the mean curvature vector $(1/n)g^{ab}\nabla_c B^a_b(\neq 0)$ is parallel with respect to the connection induced in the normal bundle:

\begin{equation}
h^a_c = \text{const.}, \ n_c = 0.
\end{equation}

Then we have first of all

\begin{equation}
\int_{M^n} (nf + ah^a_c)dV = 0.
\end{equation}

We next have from (4.5)

\begin{equation}
\int_{M^n} K_{kjih} B^k_a z^d B^{ij} C^h dV
\end{equation}

\begin{equation}
= \int_{M^n} (fh^a_c + ah^ba_hba + \beta h^ba_kba) dV.
\end{equation}

Now, forming (5.4)–(5.3) $\times (1/n)h^a_c$, we find

\begin{equation}
\int_{M^n} K_{kjih} B^k_a z^d B^{ij} C^h dV
\end{equation}

\begin{equation}
= \int_{M^n} \left[ \alpha \left( h^b_a h_ba - \frac{1}{n} h^b_a h^a_c \right) + \beta h^ba_kba \right] dV,
\end{equation}

or

\begin{equation}
\int_{M^n} K_{kjih} B^k_a z^d B^{ij} C^h dV
\end{equation}

\begin{equation}
= \int_{M^n} \left[ \alpha \left( h^b_a - \frac{1}{n} h^b_a g^ba \right) \left( h_ba - \frac{1}{n} h^a_d g^ba \right) + k^ba_kba \right] dV.
\end{equation}

We denote by $X'^h$ and $X''^h$ the tangential part and normal part of $X^h$ respectively.

Suppose that
\[
\int_{\mathcal{M}^n} K_{kjh} X'^k B'^h C^h dV \leq 0,
\]
\[
\alpha > 0,
\]
\[
h_{ba} k_{ba} \beta - k_{ba} k_{ba} \alpha \geq 0,
\]
that is, the vector
\[
Y^h = h_{ba} k_{ba} C^h + k_{ba} k_{ba} D^h
\]
vanishes or this vector and
\[
X'^h = \alpha C^h + \beta D^h
\]
have positive orientation in the normal bundle, or
\[
\int_{\mathcal{M}^n} K_{kjh} X'^k B'^h C^h dV \geq 0,
\]
\[
\alpha < 0,
\]
\[
h_{ba} k_{ba} \beta - k_{ba} k_{ba} \alpha \leq 0,
\]
that is, the vector \( Y^h \) vanishes or \( Y^h \) and \( X'^h \) have negative orientation in the normal bundle, then we have
\[
h_{cb} - \frac{1}{n} h_{d} g_{db} = 0, \quad k_{cb} = 0,
\]
that is, the submanifold under consideration is totally umbilical. Thus we have

**Theorem 5.1.** Let \( M^n \) be a closed orientable submanifold of codimension 2 of an \((n+2)\)-dimensional Riemannian manifold \( M^{n+2} \) and assume that \( M^n \) admits a vector field \( X^h \) whose covariant differential along \( M^n \) is always proportional to the displacement. If
\begin{enumerate}
  \item the mean curvature vector field \((\neq 0)\) is parallel with respect to the connection induced in the normal bundle,
  \item \[
  \int_{\mathcal{M}^n} K_{kjh} X'^k B'^h C^h dV \leq 0 \quad (\geq 0),
  \]
  \item \[
  \alpha > 0 \quad (< 0),
  \]
  \item \( Y^h = 0 \) or \( Y^h \) and \( X'^h \) have positive (negative) orientation in the normal bundle,
\end{enumerate}
then the submanifold is totally umbilical.

If the submanifold is invariant under the curvature transformation, then we have
\[
K_{kjh} X'^k B'^h C^h = 0
\]
and consequently the second condition of Theorem 5.1 is automatically satisfied.

If the ambient Riemannian manifold \( M^{n+2} \) admits a scalar function \( v \) such that
\begin{equation}
(5.6)
\nabla \mathcal{F} v = f(v) g_{ji},
\end{equation}
then we have

\[(5.7)\quad F_{a}v^{b} = f(v)B_{c}^{b}\]

along any submanifold, where we have put

\[v^{b} = v_{i}g^{ib}, \quad v_{i} = F_{i}v.\]

This equation shows that the vector field \(v^{b}\) defined along \(M^{n}\) has covariant differential always proportional to the displacement along \(M^{n}\).

Thus, under 4 conditions of Theorem 5.1, we have

\[(5.8)\quad h_{cb} = \lambda g_{cb}, \quad k_{cb} = 0, \quad l_{c} = 0,\]

\(\lambda\) being a constant different from zero, and consequently, (3.7) and (3.8),

\[(5.9)\quad F_{b}z_{a} = (f + \alpha \lambda)g_{ba}\]

and

\[(5.10)\quad \partial_{b}\alpha + \lambda z_{b} = 0.\]

But

\[z_{b} = B_{b}v_{i} = \partial_{b}v\]

and consequently, we have from (3.7) and (5.10),

\[\alpha + \lambda v = c \quad \text{(constant)}.\]

Thus, from (5.9),

\[(5.10')\quad F_{b}F_{a}v = (f(a\lambda - \lambda^{2}v)g_{ba}.\]

We examine two cases,

(1) \(f = kv, \quad k = \text{const.} \neq 0, \quad v \neq \text{const. along } M^{n}.\)

In this case, we have

\[(5.11)\quad F_{b}F_{a}v = [-\lambda^{2} - k)v + \lambda c]g_{ba}.\]

Here, \(\lambda^{2} - k \neq 0\), because if \(\lambda^{2} - k = 0\), then we have \(F_{b}F_{a}v = \lambda c g_{ba}\), from which \(g^{ba}F_{b}F_{a}v = n\lambda c\), which, the submanifold being closed, is impossible unless \(v = \text{constant} \) on \(M^{n}\).

Thus, \(\lambda^{2} - k\) being different from zero, we have from (5.11)

\[(5.12)\quad F_{b}F_{a}v - \frac{\lambda c}{\lambda^{2} - k}) = -\lambda^{2} - k\left(v - \frac{\lambda c}{\lambda^{2} - k}\right)g_{ba},\]

from which

\[g^{ba}F_{b}F_{a}v - \frac{\lambda c}{\lambda^{2} - k}) = -n(\lambda^{2} - k)\left(v - \frac{\lambda c}{\lambda^{2} - k}\right)\]
which shows that $\lambda^2 - k > 0$. Thus, by a famous theorem of Obata [5], the submanifold is isometric to a sphere.

(II) $f = k$, $k =$ constant, $v \neq \text{const.}$ along $M^n$.

In this case, we have

\[(5.13) \quad \mathcal{F}_b \mathcal{F}_a v = (-\lambda^2 v + k + c\lambda) g_{ba}.\]

Here $\lambda \neq 0$, because if $\lambda = 0$, then we have $v = \text{const.}$ along $M^n$. Thus we have

\[(5.14) \quad \mathcal{F}_b \mathcal{F}_a \left(v - \frac{k + c\lambda}{\lambda^2}\right) = -\lambda^2 \left(v - \frac{k + c\lambda}{\lambda^2}\right) g_{ba},\]

from which we conclude that the submanifold is isometric to a sphere. Thus we have

**Theorem 5.2.** Let $M^n$ be a closed orientable submanifold of codimension 2 of an $(n+2)$-dimensional orientable Riemannian manifold $M^{n+2}$ which admits a scalar function $v$ such that $\mathcal{F}_b \mathcal{F}_a v = f(v) g_{ba}$, where $f(v) = kv$, or $k$, $k$ being a constant, and $v \neq \text{const.}$ along $M^n$. Then under 4 conditions of Theorem 5.1 where $X^k = (\mathcal{F}_b \mathcal{F}_a) g^{ba}$, the submanifold is totally umbilical and is isometric to a sphere.

**Bibliography**


**Tokyo Institute of Technology, and Saitama University.**