UNIVALENCY OF ANALYTIC MAPPINGS OF A RIEMANN SURFACE INTO ITSELF

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1. In the present paper we shall study a Riemann surface whose every non-constant analytic mapping into itself is univalent.

Let $S$ be the class of Riemann surfaces whose every non-constant analytic mapping into itself is univalent, and let $K$ be the class of Riemann surfaces whose every non-constant analytic mapping into itself is univalent and onto. It is easy to see that $\phi^K \subseteq S \subseteq O_{AB} \cap H$ where $H$ is the class of Riemann surfaces whose universal covering are conformally equivalent to the unit disk. Heins [5] showed $O_{AB} \cap H \subseteq S$ and $K_A \subseteq K$ where $K_A$ denotes the class of Riemann surfaces with a finite positive genus or with a finite number of planar boundary elements belonging to $O_{AB} \cap H$. Kubota [8] introduced a class of Riemann surfaces and showed that the class is a subclass of $K$. In §2 we construct an example of Riemann surface of class $O_{AB} \cap H$ on which there exists a non-univalent analytic mapping into itself. Namely we show $S \supseteq O_{AB} \cap H$. In §3 we introduce a class $K_{HD}$ of Riemann surfaces and show $K_{HD} \subseteq K_H$, where $K_H$ denotes the class of Riemann surfaces introduced by Kubota. Heins [5] showed that if $W$ is of class $K_A$ and of finite genus, then the number of non-constant analytic mappings of $W$ into itself is finite. In §4 we show the same result with respect to a Riemann surface of class $K_{HD}$.

2. We construct an example of a Riemann surface $W$ of class $O_{AB} \cap H$ on which there exists a non-univalent analytic mapping into itself. It will be given as a covering surface of the $z$-plane. We introduce $E$, $F$ and $D$ as follows:

$$E = \{0 < |z| < \infty\} - \bigcup_{n=-\infty}^{\infty} [4^n, 2 \cdot 4^n],$$

$$F = E - \{ |z+1| \leq 1 \} - [-6, -4],$$

$$D = \{ |z+5| < 2 \} - [-6, -4],$$

where $[a, b] = \{ z | a \leq \text{Re} z \leq b, \text{Im} z = 0 \}$. We joint copies of $E$ and $F$ along their common slits identifying the upper edges of the slits of $E$ with the corresponding lower edges of the slits of $F$ and vice versa. The edges of the remained free slit of $F$ are identified with the opposite edges of the corresponding slit of a copy of $D$. Thereby a Riemann surface $W$ is constructed as a covering surface $(W, \pi)$ of the $z$-plane (cf. Ahlfors-Sario [1], pp. 119-120).

Received September 26, 1970.
Let $G$ be the covering of $\{|z+4|^4\}$ lying in the joining of $F$ and $D$. Then, by using the same arguments in Myrberg's paper [9], we see that $W-G$ is of class $O_{AB}$. Hence $W$ is of class $O_{AB} \cap H$. Let $\varphi$ be a mapping of $W$ into itself which satisfies $\pi \circ \varphi \circ \pi^{-1}(z) = 4z$ and carries the points of $E$, $F$ and $D$ onto the points of $E$, $F$ and $F$ respectively. Then $\varphi$ is analytic and non-univalent.

3. In this section we introduce the class $K_{HD}$ of Riemann surfaces such that $K_{HD} \subset K$. We show first the following lemma.

**Lemma 1.** Let $W$ be a Riemann surface whose fundamental group is non-abelian, and let $\varphi$ be an analytic mapping of $W$ into itself whose valence function $\nu_{\varphi}$ is a constant $n_\varphi (\leq \infty)$ except a set of zero area. If there exists a non-constant harmonic function $u$ with finite Dirichlet integral which satisfies

$$(1) \quad u \circ \varphi = cu,$$

where $c$ is a real constant, then $c$ is equal to $\pm 1$ and $\varphi$ has a finite period $p$ (i.e. the $p$-th iterate $\varphi_p$ of $\varphi$ is the identity mapping of $W$ onto itself).

**Remark 1.** If the fundamental group of $W$ is abelian then there is an example such that $\varphi$ has no period: $W=\{r<|z|<l\}$ ($r>0$), $\varphi(z)=e^{2\pi i \theta} z$ ($\theta$ is an irrational real number), $u=\log |z|$, $c=1$.

**Remark 2.** If $\varphi$ does not satisfy the condition on the valence function, then it is easy to construct an example such that $\varphi$ is not univalent.

**Remark 3.** If $u$ is a harmonic function with infinite Dirichlet integral, then there is an example such that the valence function is a constant $n(\geq 2)$ except one point: $W=\{0<|z|<l\}$ ($r>0$), $\varphi(z)=e^{2\pi i \theta} z$ ($\theta$ is an irrational real number), $u=\log |z|$, $c=n$.

**Remark 4.** If $u(\equiv \text{const})$ is a bounded harmonic function with finite Dirichlet integral, then we are able to replace the condition on $\varphi$ in lemma 1 by a weaker condition that $W$ is covered by the image $\varphi(W)$ of $\varphi$ except a set of zero area. In fact, we may assume without loss of generality that $\sup_u u$ is positive. For the 2nd iterate $\varphi_2$ of $\varphi$ we have

$$\sup_{\varphi_2(W)} u = \sup_{W} (u \circ \varphi_2) = \sup_{W} (c^2u) = c^2 \sup_{W} u,$$

Hence $c^2 \leq 1$. Therefore we have

$$D_{\varphi_2(W)}(u) = D_{\varphi_2(W)}(u) = D_{\varphi_2(W)}(cu) = c^2 D_{\varphi_2(W)}(u) \leq D_{\varphi_2(W)}(u),$$

where

$$D_{\varphi_2(W)}(u) = \int_{W} \nu_{\varphi_2} \cdot du^*.$$
On the other hand, by the above condition we have

$$D_\psi(W(u)) \geq D_W(u).$$

Hence the valence function $\nu_\psi$ is equal to 1 except a set of zero area.

**Proof of lemma 1.** We use the following result due to Heins [5]:

Let $W$ denote a non-compact Riemann surface whose fundamental group is non-abelian, and let $\varphi$ denote an analytic mapping of $W$ into itself. If $\varphi$ neither

i) possesses a fixed point $\zeta$, nor

ii) has a finite period $p$, then

iii) for every given compact subsets $K_1$, $K_2$ of $W$ there exists a natural number $N$ such that $\varphi^N(K_1) \subset W - K_2$.

We show first $n_\varphi = c^2 < \infty$. This follows from the following formulae.

$$D_W(u \circ \varphi) = D_\varphi(W(u)) = n_\varphi D_W(u),$$

$$D_W(cu) = c^2 D_W(u).$$

We show next that iii) leads to a contradiction. Let $(W_n)_{n=1}^\infty$ be a canonical exhaustion of $W$. Since $D_W(u)$ is finite, for any given positive number $\varepsilon$ there is a natural number $n$ such that $D_W(W_n(u)) < \varepsilon$. Setting $K_1 = K_2 = \overline{W_n}$, we find a natural number $N = N(n)$ such that $\varphi^N(W_n) \subset W - \overline{W_n}$. Hence we have

$$D_{W_n}(u \circ \varphi^N) = D_{\varphi^N(W_n)}(u) \leq n_\varphi^N D_{W-n}(u).$$

By formula (1) we have

$$D_{W_n}(c^N u) = c^{2N} D_{W_n}(u) = n_\varphi^N D_{W_n}(u),$$

and hence

$$D_W(u) = D_{W_n}(u) + D_{W-n}(u)$$

$$\leq 2D_{W-n}(u) < 2\varepsilon.$$ 

Therefore $u$ must reduce to a constant. This is a contradiction.

Finally we show that i) implies ii). Let $(|z|<1), \pi)$ be the universal covering surface of $W$ such that $\pi$ is analytic and satisfies $\pi(0) = \zeta$. We consider $\pi^{-1}$ in the neighborhood of $\zeta$ satisfying $\pi^{-1}(\zeta) = 0$ and set $f = \pi^{-1} \circ \varphi \circ \pi$ around 0. We continue analytically the function element of $f$ onto $|z|<1$. Then $f$ satisfies $f(0) = 0$, $|f(z)| < 1$ and $\varphi_k \circ \pi = \pi \circ f_k$ ($k = 1, 2, \cdots$). Setting $v = u \circ \pi$, we have $v \circ f = cv$. Let $h$ be an analytic function on $|z|<1$ having $v$ as its real part and set $q = h - h(0)$. Then we have

$$g \circ f = cq$$

and $g(0) = 0$. If $f$ and $g$ have the expansions around the origin
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\[ f(z) = az^2 + a_1z^{j+1} + \cdots, \quad a \neq 0, \quad j \geq 1, \]
\[ g(z) = bz^k + b_1z^{k+1} + \cdots, \quad b \neq 0, \quad k \geq 1, \]

then from (2) we have \( j = 1 \), \( a^k = c \) and \(|a^k| = |c| = \sqrt{n_2} \geq 1\). Using Schwarz's lemma we have \( a^k = c^k = 1 \), \( f(z) = az \) and \( \varphi_{2k} \circ \pi = \pi \circ f_{2k} = \pi \). Hence we have \( \varphi_{2k} = \iota \). Therefore \( \varphi \) has a finite period \( p \). It follows immediately that \( u = u \circ \varphi_p = c^p u \), and hence we have \( c = \pm 1 \).

We consider next a problem whether there exists a harmonic function \( u(= \text{const}) \) satisfying (1) for a given analytic mapping \( \varphi \) of \( W \) into itself. This is an eigenvalue problem in the following sense. For every harmonic function \( u \) on \( W \) the composition \( u \circ \varphi \) is also harmonic on \( W \). We denote by \( H(W) \) the class of harmonic functions on \( W \) and set \( \varphi^*(u) = u \circ \varphi \). Then \( \varphi^* \) is a linear operator of \( H(W) \) into itself and (1) is represented using \( \varphi^* \) as follows:

\[ \varphi^*(u) = cu \]

where \( c \) is an eigenvalue of \( \varphi^* \) and \( u \) is its eigenelement. From this point of view we consider an eigenvalue problem of the restriction \( \varphi^*|X \) of \( \varphi^* \) to \( X \), where \( X \) is a linear subspace of \( H(W) \) such that \( \varphi^*(X) \subseteq X \). If \( X \) is a finite dimensional lattice-ordered linear space (vector lattice) with respect to the natural order, then \( X \) has a base consisting of \( X \)-minimal functions (cf. Constantinescu-Cornea [3]). From this fact we obtain a matricial representation of \( \varphi^*|X \).

**Lemma 2.** Let \( \varphi \) be an analytic mapping of a Riemann surface \( W \) into itself such that \( W \) is covered by \( \varphi(W) \) except a set of zero area, and let \( X \subseteq H(W) \) be a finite dimensional lattice-ordered linear space satisfying \( \varphi^*(X) \subseteq X \). Choose a base \( u_1, u_2, \ldots, u_n \) of \( X \) consisting of \( X \)-minimal functions and set

\[
\begin{pmatrix}
\varphi^*(u_1) \\
\varphi^*(u_2) \\
\vdots \\
\varphi^*(u_n)
\end{pmatrix} = \Phi
\begin{pmatrix}
u_1 \\
u_2 \\
\vdots \\
u_n
\end{pmatrix}
\]

where \( \Phi \) is a square matrix of degree \( n \). Then \( \Phi \) is regular and equal to \( (c_i \cdot \delta_{i(j)}) \), where \( c_i \) (\( i=1,2,\ldots,n \)) are positive constants, \( \delta_{ij} \) is Kronecker's symbol and \( \sigma \) is a permutation of degree \( n \). Consequently, if we denote by \( s \) the order of \( \sigma \), then \( \Phi^s \) is a diagonal matrix and all its diagonal elements are positive.

**Proof.** The regularity of \( \Phi \) follows from the fact that \( \varphi^* \) is injective and \( X \) is of finite dimension. If we set

\[
\begin{pmatrix}
u_1 \\
u_2 \\
\vdots \\
u_n
\end{pmatrix} = \Phi^{-1}
\begin{pmatrix}
u_1 \\
u_2 \\
\vdots \\
u_n
\end{pmatrix},
\]

then from (2) we have \( j = 1 \), \( a^k = c \) and \(|a^k| = |c| = \sqrt{n_2} \geq 1\). Using Schwarz's lemma we have \( a^k = c^k = 1 \), \( f(z) = az \) and \( \varphi_{2k} \circ \pi = \pi \circ f_{2k} = \pi \). Hence we have \( \varphi_{2k} = \iota \). Therefore \( \varphi \) has a finite period \( p \). It follows immediately that \( u = u \circ \varphi_p = c^p u \), and hence we have \( c = \pm 1 \).

We consider next a problem whether there exists a harmonic function \( u(= \text{const}) \) satisfying (1) for a given analytic mapping \( \varphi \) of \( W \) into itself. This is an eigenvalue problem in the following sense. For every harmonic function \( u \) on \( W \) the composition \( u \circ \varphi \) is also harmonic on \( W \). We denote by \( H(W) \) the class of harmonic functions on \( W \) and set \( \varphi^*(u) = u \circ \varphi \). Then \( \varphi^* \) is a linear operator of \( H(W) \) into itself and (1) is represented using \( \varphi^* \) as follows:

\[ \varphi^*(u) = cu \]

where \( c \) is an eigenvalue of \( \varphi^* \) and \( u \) is its eigenelement. From this point of view we consider an eigenvalue problem of the restriction \( \varphi^*|X \) of \( \varphi^* \) to \( X \), where \( X \) is a linear subspace of \( H(W) \) such that \( \varphi^*(X) \subseteq X \). If \( X \) is a finite dimensional lattice-ordered linear space (vector lattice) with respect to the natural order, then \( X \) has a base consisting of \( X \)-minimal functions (cf. Constantinescu-Cornea [3]). From this fact we obtain a matricial representation of \( \varphi^*|X \).
then we have

\[
\begin{pmatrix}
\varphi^*(v_1) \\
\varphi^*(v_2) \\
\vdots \\
\varphi^*(v_n)
\end{pmatrix}
= \Phi^{-1}
\begin{pmatrix}
\varphi^*(u_1) \\
\varphi^*(u_2) \\
\vdots \\
\varphi^*(u_n)
\end{pmatrix}
= \Phi^{-1} \Phi
\begin{pmatrix}
u_1 \\
u_2 \\
\vdots \\
u_n
\end{pmatrix}
= 
\begin{pmatrix}
u_1 \\
u_2 \\
\vdots \\
u_n
\end{pmatrix},
\]

and hence \( v_i \circ \varphi = u_i \) \((i=1,2,\ldots,n)\). Since \( W \) is covered by \( \varphi(W) \) except a set of zero area, the functions \( v_i \) are positive. For any \( v \in X \) such that \( v > 0 \), \( v \leq v_i \) it follows that \( v \circ \varphi \in X, v \circ \varphi > 0 \) and \( v \circ \varphi \leq v_i \circ \varphi = u_i \). Hence \( v \circ \varphi = c(v_i \circ \varphi) = (cv_i) \circ \varphi \). This implies that \( v_i \) are also \( X \)-minimal functions. Hence there exists a permutation \( \tau \) of degree \( n \) satisfying \( v_i = k_i u_{\tau(i)} \) \((i=1,2,\ldots,n)\) with positive constants \( k_i \). Setting \( \sigma = \tau^{-1} \) and \( c_i = 1/k_{\tau^{-1}(i)} \), we have the desired result.

From lemma 1 and 2 we have the following lemma.

**Lemma 3.** Let \( W \) be a Riemann surface whose fundamental group is non-abelian, and let \( \varphi \) be a non-constant analytic mapping of \( W \) into itself whose valence function is finite and constant except a set of zero area. If there exists a lattice-ordered linear space \( X \subset H(W) \) which satisfies (i) \( \varphi^*(X) \subset X \) and that (ii) \( X \cap HD(W) \) is of finite dimension and contains at least one non-constant function, then \( \varphi \) has a finite period.

**Proof.** Since \( HD = HD(W) \) is a lattice-ordered linear space, \( X \cap HD \) is a finite dimensional lattice-ordered linear space. By the condition on the valence function we have \( \varphi^*(HD) \subset HD \) and hence \( \varphi^*(X \cap HD) \subset X \cap HD \). We apply now lemma 2 to \( X \cap HD \). Then there exists a natural number \( s \) such that every \( X \cap HD \)-minimal function is an eigenelement of \( \varphi^s | X \cap HD \). We apply further lemma 1 to \( X \cap HD \)-minimal functions. Then the matrix \( \Phi^s \) is equal to the unit one and \( \varphi \) has a finite period.

We introduce now the class \( K_{HD} \).

**Definition.** We denote by \( K_{HD} \) the class of Riemann surfaces \( W \) which satisfy the following conditions:

i) Every non-constant analytic mapping of \( W \) into itself is a Dirichlet mapping and of type \( Bl \), i.e. the valence function is finite and constant except a set of capacity zero.

ii) Let \( M_Y \) be the linear space generated by all \( Y (Y=HP, HB, HD) \)-minimal functions. The space \( M_Y \cap HD \) is of finite dimension and contains at least one non-constant function.

The class \( K_{HD} \) is not empty. In fact, the class \( O_{HB}^n - O_{HD} \) is a subclass of \( K_{HD} \). If \( W \) is of class \( O_{HB}^n - O_{HD} \), then we have \( M_{HB} = HB \supset HD \). This implies that the condition ii) is fulfilled for \( Y = HB \). Since \( W \) is of class \( O_{HB}^n \), each non-constant analytic mapping \( \varphi \) of \( W \) into itself is of type \( Bl \) and satisfies \( \varphi^*(HD) \subset HD \).

Using the same argument in the proof of lemma 3 and remark 4 \( \varphi \) is univalent,
and hence the condition i) is satisfied.

The class $K_{HB}$ which is introduced by Kubota [8] is a proper subclass of $K_{HD}$. If $W$ is of class $K_{HB}$ then the condition i) is fulfilled (cf. Kubota [8]). In the following we use the notation in [8]. Let $B_{t}$ be a set of positive measure. Then the harmonic measure $\omega_{t}=\lim_{y\to 0}\omega_{y}^{(t)}$ of $B_{t}$ is non-constant and its Dirichlet integral is finite since by the definition of $K_{HB}$ there exists another set $B_{t}'$ of positive measure. We assume that $B_{t}$ consists of $HB$-indivisible sets and a set of measure zero. Then $\omega_{t}$ belongs to $M_{HB}$ and hence the condition ii) is satisfied. To see $K_{HB} \nsubseteq K_{HD}$, we consider a Riemann surface $W$ which is of class $O_{HB}-O_{HD}$ and has one ideal boundary component (cf. Constantinescu-Cornea [2], pp. 230–231). Then from the above argument $W$ is of class $K_{HD}$, but by the definition of $K_{HB}$, $W$ is not of class $K_{HB}$.

**Theorem 1.** The class $K_{HD}$ is a subclass of $K$.

**Proof.** Suppose that $W$ is of class $K_{HD}$. Then $M_{T}$ is a lattice-ordered linear space and satisfies $\varphi^{*}(M_{T}) \subseteq M_{T}$ for every non-constant analytic mapping $\varphi$ of $W$ into itself (cf. Constantinescu-Cornea [3], pp. 123–124). Applying lemma 3, we have that $W$ is of class $K$.

4. In this section we show the following theorem.

**Theorem 2.** If $W$ is of class $K_{HD}$, then the number of non-constant analytic mappings of $W$ into itself is finite.

**Proof.** Let $\{\varphi_{t}^{(k)}\}_{k=1}^{\infty}$ be a sequence of non-constant analytic mappings of $W$ into itself. From theorem 1 we know that each $\varphi_{t}^{(k)}$ is univalent and onto. We apply lemma 2 to $M_{T} \cap HD$ and denote by $\sigma_{k}$ the permutation of $\varphi_{t}^{(k)}$. Then there exists a permutation $\sigma_{k}$ and a subsequence $\{\varphi_{t}^{(k)}\}$ of $\{\varphi_{t}^{(k)}\}$ such that $\sigma_{k} = \sigma_{k}$ ($l=1,2,\ldots$). For the sake of simplicity we write $\{\varphi_{t}^{(k)}\}$ for $\{\varphi_{l}^{(k)}\}$.

From lemma 1 all the matrices $\varphi_{t}^{(k)}$ of $\varphi_{t}^{(k)} = \varphi_{l}^{(k)} \circ \varphi_{l}^{(k)}$, where $\varphi_{l}^{(k)}$ is the inverse mapping of $\varphi_{l}^{(k)}$, are equal to the unit one. Hence there exists at least one non-constant harmonic function $u$ with finite Dirichlet integral such that $u \circ \varphi_{t}^{(k)} = u$ ($k=1,2,\ldots$). If $\{\varphi_{t}^{(k)}\}_{k=1}^{\infty}$ is a sequence of mutually distinct mappings, then for every two compact sets $K_{1}, K_{2}$ there exists a natural number $N$ such that $\varphi_{t}^{(k)}(K_{1}) \subseteq W-K_{2}$ (cf. Heins [4], Komatu-Mori [6] and Kubota [7]). Let $\{W_{n}\}_{n=1}^{\infty}$ be a canonical exhaustion of $W$. Since $D_{w}(u)$ is finite, for any given positive number there exists a natural number $n$ such that $D_{w-W_{n}}(u) < \varepsilon$. Setting $K_{1}=K_{2}=W_{n}$, we find a natural number $N=N(n)$ such that $\varphi_{t}^{(N)}(W_{n}) \subseteq W-W_{n}$. Hence we have

$$D_{w-n}(u) = D_{w-n}(u \circ \varphi_{t}^{(N)}) = D_{\varphi_{t}^{(N)}(w-n)}(u) \leq D_{w-W_{n}}(u),$$

and

$$D_{w}(u) = D_{w-n}(u) + D_{w-W_{n}}(u) \leq 2D_{w-W_{n}}(u) < 2\varepsilon.$$

Therefore $u$ must reduce to a constant. This is a contradiction.
References


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