DISTRIBUTION AND CRITICAL CURVES
IN A RIEMANNIAN MANIFOLD

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Let $\mathcal{D}$ be a $C^\infty$ distribution in a $C^\infty$ Riemannian manifold $M$. In the present paper a curve of $M$ where every tangent vector lies in $\mathcal{D}$ is called a $\mathcal{D}$-curve. Let $P$ and $Q$ be two points of $M$ such that there exist $\mathcal{D}$-curves joining $P$ and $Q$. We call a $\mathcal{D}$-curve $C$ a critical $\mathcal{D}$-curve with the fixed end points $P, Q$ if the length $l$ of $C$ takes a critical value in the set of $\mathcal{D}$-curves joining $P$ and $Q$. The purpose of the present paper is to find differential equations of critical $\mathcal{D}$-curves when $n-m=\dim \mathcal{D}$ satisfies $n<2(n-m)$, where $n=\dim M$, and to study properties of such critical $\mathcal{D}$-curves in some special cases.

§ 1. The differential equations of a critical $\mathcal{D}$-curve.

Let $M$ be an $n$-dimensional Riemannian manifold and $\mathcal{D}$ (or $\mathcal{D}^{n-m}$) an $(n-m)$-dimensional distribution given locally by $n-m$ linearly independent $C^\infty$ vector fields $X_\lambda (\lambda = m+1, \ldots, n)$.\(^1\) Their components with respect to a local coordinate system will be denoted by $X^h$. The distribution $\mathcal{D}$ will also be represented by $m$ linearly independent covector fields $\varphi (\alpha = 1, \ldots, m)$ whose components $\varphi^a_\lambda$ satisfy

$$\varphi^a_\lambda X^\lambda = 0.$$

A $\mathcal{D}$-curve $C$ is by definition a curve $x^h = x^h(t)$ such that

$$\frac{\varphi^a_\lambda}{\varphi^a_\lambda} \frac{dx^\lambda}{dt} = 0$$

holds throughout the curve.

We assume that $2m$ covectors

$$\varphi_1, \ldots, \varphi_m, \psi_1, \ldots, \psi_m$$

Received November 5, 1970.

1) We let the indices $h, i, j, \ldots$ run over the range $\{1, \ldots, n\}$, $\alpha, \beta, \gamma, \ldots$ over the range $\{1, \ldots, m\}$ and $\kappa, \lambda, \mu, \ldots$ over the range $\{m+1, \ldots, n\}$. The summation convention is used for all such indices.
are linearly independent at every point of \( C \), where \( \phi_i \) are defined by

\[
\phi_i = \left( \partial_j \phi_i - \partial_i \phi_j \right) \frac{dx^j}{dt}.
\]

Let \( P \) and \( Q \) be the end points of \( C \) and the parameter \( t \) be such that \( t=0 \) and \( t=1 \) correspond respectively to \( P \) and \( Q \). Then the length \( l \) of \( C \) is given by the integral

\[
J(C) = \int_C ds = \int_0^1 \sqrt{g_{ij} \left( \frac{dx^i}{dt} \frac{dx^j}{dt} \right)} \ dt.
\]

Let us consider an infinitesimal deformation of the curve \( C \) with the points \( P \) and \( Q \) fixed assuming that any curve obtained is also a \( \mathcal{D} \)-curve. Then the vector of deformation \( \xi^h(t) \) must satisfy

\[
\xi^h \frac{dx^i}{dt} \partial_j \phi_i + \phi_i \frac{d\xi^h}{dt} = 0.
\]

As the points \( P \) and \( Q \) are fixed, \( \xi^h \) must also satisfy

\[
\xi^h(0) = \xi^h(1) = 0.
\]

Then it is a consequence of an ordinary argument in the calculus of variations that \( C \) is a critical \( \mathcal{D} \)-curve if and only if

\[
\int_0^1 \left[ \frac{d^2 x^\alpha}{ds^2} + \sum_{i,j} \frac{dx^i}{ds} \frac{dx^j}{ds} g_{ij} \xi^h(s) ds \right] dt = 0
\]

is satisfied by every set of functions \( \xi^h(t) \) satisfying (1.5) and (1.6). Notice that the arc length \( s \) is used in (1.7) as the parameter and that \( l \) is the length of \( C \).

Now let \( f(t) (\alpha = 1, \ldots, m) \) be a set of arbitrary \( C^\infty \) functions. Then we find that

\[
\int_0^1 \left[ \left( \frac{d}{dt} f \right) \phi_i + f(t) (\partial_j \phi_i - \partial_i \phi_j) \frac{dx^j}{dt} \right] \xi^h(t) dt = 0
\]

is equivalent to (1.5). (1.8) is also equivalent to

\[
\int_0^1 \left[ \left( \frac{d}{ds} f \right) \phi_i + f(s) (\partial_j \phi_i - \partial_i \phi_j) \frac{dx^j}{ds} \right] \xi^h(s) ds = 0.
\]

and again to

\[
\int_0^1 \left[ \left( \frac{d}{ds} f \right) \phi_i + f(s) (\partial_j \phi_i - \partial_i \phi_j) \frac{dx^j}{ds} \right] \xi^s(s) ds = 0.
\]

If we put
we can write (1.9) in the form

\[
\psi_i = \frac{dx^i}{ds} (\partial_i \psi_i - \partial_j \psi_j),
\]

We prove in §2 the following lemma.

**Lemma 1.1.** In an \(n\)-dimensional Euclidean space let there be given \(2m+1\) \(C^\infty\) vector functions \(A_i(t), \varphi_i(t), \psi_i(t) (a=1, \ldots, m)\) where \(2m\) vectors \(\varphi_i(t), \ldots, \varphi_i(t), \psi_i(t), \ldots, \psi_i(t)\) are linearly independent at each value of \(t, 0 \leq t \leq a\). If, for every functions \(\xi(t)\) which satisfy

\[
\xi(0) = \xi(a) = 0
\]

and

\[
\int_0^a \left[ \left( \frac{d}{dt} f \right) \varphi_i(t) + f(t) \psi_i(t) \right] \xi(t) dt = 0
\]

for every choice of \(C^\infty\) functions \(f(t)\), we have

\[
\int_0^a A_i(t) \xi(t) dt = 0,
\]

then there exist functions \(\chi(t), \ldots, \chi(t)\) such that

\[
A_i(t) = \left( \frac{d}{dt} \chi \right) \varphi_i(t) + \chi(t) \psi_i(t).
\]

**Remark.** It is easily found that (1.13) is a consequence of (1.12) and (1.14).

Applying Lemma 1.1 to the case of \(D\)-curves, we easily obtain the following lemma.

**Lemma 1.2.** Let \(M\) be an \(n\)-dimensional Riemannian manifold equipped with an \((n-m)\)-dimensional distribution \(D\) determined locally by \(m\) covector fields \(\varphi_i\). Let \(C\) be a \(D\)-curve \(x^b = x^b(s), 0 \leq s \leq 1\), such that \(2m\) covectors

\[
\varphi_i, \frac{dx^i}{ds} (\partial_j \varphi_i - \partial_j \varphi_j) (a=1, \ldots, m)
\]

are linearly independent at each point of \(C\). A necessary and sufficient condition
for the curve $C$ to be a critical $\partial$-curve with fixed end points is that there exist functions $\chi(s)$ satisfying the equations

\begin{equation}
\frac{d^2x^h}{ds^2} + \left[ \frac{\lambda}{j} \right] \frac{dx^i}{ds} \frac{dx^a}{ds} = \left[ \left( \frac{d}{ds} \phi \right)^a \phi + \chi \frac{dx^i}{ds} (\varphi_{j\phi} - \varphi_{i\phi}) \right] \phi^{ab}.
\end{equation}

Differentiating the equations

\begin{equation}
\frac{\phi^a}{dx^i} \frac{dx^i}{ds} = 0
\end{equation}

covariantly along the curve $C$, we get

\begin{equation}
(\varphi_{j\phi}) \frac{dx^i}{ds} \frac{dx^a}{ds} + \frac{\phi^a}{ds} + \left[ \frac{\lambda}{j} \right] \frac{dx^i}{ds} \frac{dx^a}{ds} = 0.
\end{equation}

Then applying (1.15) we obtain

\begin{equation}
\phi^a \frac{dx^i}{ds} \frac{dx^a}{ds} + \phi^a (\varphi_{j\phi} - \varphi_{i\phi}) \frac{dx^i}{ds} \frac{dx^a}{ds} = 0.
\end{equation}

Let us consider a system of differential equations composed of (1.15) and (1.17) in the unknown functions $x^h(s)$ and $\chi(s)$. As far as only these equations are considered, $s$ may not be the arc length and the curve $x^h = x^h(s)$ may not be a $\partial$-curve. But, if the initial condition is chosen in such a way that

\begin{equation}
g_{ab} \frac{dx^a}{ds} \frac{dx^b}{ds} = 1, \quad \frac{\phi^a}{dx^i} \frac{dx^a}{ds} = 0
\end{equation}

hold at $s=0$, then we can easily see that $s$ is the arc length of the curve $x^h = x^h(s)$ and (1.6) is satisfied by the curve.

Thus we obtain the

**Theorem 1.3.** Let $M$ and $\partial$ be the same as those assumed in Lemma 1.2. A necessary and sufficient condition for a $\partial$-curve $C$, for which the same is also assumed as in Lemma 1.2 and parametrized by the arc length $s$, to be a critical $\partial$-curve with the fixed end points is that the functions $x^h(s)$ satisfy with some functions $\chi(s)$ the differential equations (1.15), (1.16) and (1.17). If a solution $x^h = x^h(s), \chi = \chi(s)$ of the system of differential equations composed of (1.15) and (1.17) satisfies the initial condition

\begin{equation}
\left( g_{ab} \frac{dx^a}{ds} \frac{dx^b}{ds} \right)_0 = 1, \quad \left( \frac{\phi^a}{dx^i} \frac{dx^a}{ds} \right)_0 = 0
\end{equation}

and the $2m$ covectors

\begin{equation}
\frac{\phi^a}{dx^i} (\varphi_{j\phi} - \varphi_{i\phi})
\end{equation}
are linearly independent at each point $x^h(s) \ (0 \leq s \leq l)$, then the curve $x^h = x^h(s)$ is a critical $\mathcal{D}$-curve with the fixed end points $x^h(0), x^h(l)$ and $s$ is the arc length.

§ 2. Proof of Lemma 1.1.

Let $\tau$ be any number such that $0 < \tau < a$ and put
\begin{equation}
\xi^h(t) = a^h \delta(t - \tau)
\end{equation}
where $a^h$ is a constant vector and $\delta$ is the Dirac function. Then (1.12) becomes
\begin{equation}
\frac{d}{dt} f(\tau) \phi^a(\tau) a^i + f(\tau) \psi^a(\tau) a^i = 0.
\end{equation}
As we can take arbitrary $C^\infty$ functions as $f(t)$, we get
\begin{equation}
\phi^a(\tau) a^i = 0, \quad \psi^a(\tau) a^i = 0
\end{equation}
from (2.2).

On the other hand we have
\begin{equation}
A_i(\tau) a^i = 0
\end{equation}
from (1.13). Since any vector $a^h$ satisfying (2.3) must satisfy (2.4) by assumption, there exist $2m$ numbers $\rho(\tau), \sigma(\tau)$ such that
\begin{equation}
A_i(\tau) = \rho^a(\tau) \phi^a(\tau) + \sigma^a(\tau) \psi^a(\tau).
\end{equation}
Thus we obtain
\begin{equation}
A_i(t) = \rho^a(\tau) \phi^a(t) + \sigma^a(\tau) \psi^a(t)
\end{equation}
where $\rho(t)$ and $\sigma(t)$ are $C^\infty$ functions, for $\phi^a(t)$ and $\psi^a(t)$ are linearly independent.

We now proceed to find a relation between $\rho(t)$ and $\sigma(t)$.

From (1.13) and (2.5) we get
\begin{equation}
\int_0^a \left[ \rho(t) \phi^a(t) \xi^a(t) + \sigma(t) \psi^a(t) \xi^a(t) \right] dt = 0.
\end{equation}

Let $\lambda$ be an arbitrary number, $0 < \lambda < a$, and $\epsilon > 0$ a sufficiently small number such that $[\lambda - \epsilon, \lambda + \epsilon] \subset (0, a)$ and such that a determinant of order $2m$ composed of some components of the $2m$ covectors $\phi^a, \psi^a$ does not vanish at any point of $[\lambda - \epsilon, \lambda + \epsilon]$. Then we can consider for example
In this case, if we take $C^\infty$ functions $\tilde{h}(t)$ such that

\begin{align*}
\tilde{h}(t) &= \cdots = \tilde{h}(t) = 0, \\
\tilde{h}(t) &= 0, \quad 0 \leq t \leq \lambda - \varepsilon, \quad \lambda + \varepsilon \leq t \leq a, \\
\tilde{h}(t) &> 0, \quad \lambda - \varepsilon < t < \lambda + \varepsilon
\end{align*}

and determine $\xi^h(t)$ by

\begin{align*}
\xi^{\tilde{h}}(t) &= \cdots = \xi^{\tilde{h}}(t) = 0, \\
\xi^a(t) &= \cdots = \xi^a(t) = 0, \quad 0 \leq t \leq \lambda - \varepsilon, \quad \lambda + \varepsilon \leq t \leq a, \\
\frac{\tilde{h}}{\varphi_1(t)} \xi^a(t) &= \tilde{h}(t), \\
\frac{\tilde{h}}{\varphi_2(t)} \xi^a(t) &= \frac{d}{dt} \tilde{h}(t)
\end{align*}

then $\xi^h(t)$ satisfy $\xi^h(0) = \xi^h(a) = 0$ and (1.12). On the other hand we get from (2.6)

\begin{align*}
\int_0^a \left[ \rho(t) \xi^a(t) + \rho(t) \frac{d}{dt} \tilde{h}(t) \right] dt = 0,
\end{align*}

and consequently,

\begin{align*}
\int_0^a \left[ \rho(t) - \frac{d}{dt} \sigma(t) \right] \tilde{h}(t) dt = 0.
\end{align*}

As we can take the positive valued function $\frac{1}{\tilde{h}}(t)$ arbitrarily, and, as we can take the number $\lambda (0 < \lambda < a)$ arbitrarily, we have

\begin{align*}
\rho(t) = \frac{d}{dt} \sigma(t).
\end{align*}

Similarly we have

\begin{align*}
\rho(t) = \frac{d}{dt} \sigma(t).
\end{align*}

Hence we get (1.14) and the lemma is proved.
§ 3. Some examples.

In § 3 some examples are given. Another example which is concerned with the normal contact metric structure of $S^{n-1}$ is studied in § 4.

1° A distribution which is orthogonal to a Killing vector field of constant magnitude.

Let $X$ be a Killing vector field in an odd dimensional Riemannian manifold such that

$$g_{ij}X^iX^j=1$$

and such that the rank of the matrix $(F, JX)$ is $n-1$. $X$ satisfies

$$(F, X_i-X_j)X^i=2X^iF_{ij}X_j=0$$

and, since the rank of $(F, X_i)$ is $n-1$, $Y^jF_{ij}X_j$ does not vanish if $Y^iX_i=0$ and $Y \neq 0$. Hence the covectors $X_i$ and $Y^i(F, X_i-X_j)X_j$ are linearly independent. Consider the $(n-1)$-dimensional distribution $\mathcal{D}$ determined by the covector field $X$. Then from the above argument, for any $\mathcal{D}$-curve $C$: $x^h=x^h(s)$, the covectors

$$X_i, \quad \frac{dx^j}{ds} (F, X_i-X_j)$$

are linearly independent on $C$.

The differential equations of the a critical $\mathcal{D}$-curve are

$$\frac{d^2x^h}{ds^2} X^i + \left| \begin{array}{c} h \\ j \\ i \end{array} \right| \frac{dx^j}{ds} \frac{dx^i}{ds} = \left( \frac{d}{ds} \chi \right) X^h + 2\chi \frac{dx^j}{ds} F_{ij} X^h,$$

but it is easily seen from (1.17) that $\chi$ is a constant. Hence we have

$$\frac{d^2x^h}{ds^2} + \left| \begin{array}{c} h \\ j \\ i \end{array} \right| \frac{dx^j}{ds} \frac{dx^i}{ds} = c \frac{dx^j}{ds} F_{ij} X^h.$$

2° A distribution in the Euclidean 3-space.

Let $\mathcal{D}$ be a distribution orthogonal to a Killing vector field defined by

$$\varphi_1 = -y, \quad \varphi_2 = x, \quad \varphi_3 = 1.$$

Then we have

$$\frac{d^2x}{ds^2} = \frac{dx}{ds} (-y) - 2\chi \frac{dy}{ds},$$

$$\frac{d^2y}{ds^2} = \frac{dx}{ds} (x + 2\chi \frac{dx}{ds},$$

$$\frac{d^2z}{ds^2} = \frac{dx}{ds}.$$
for (1.15),

\[-y \frac{dx}{ds} + x \frac{dy}{ds} + \frac{dz}{ds} = 0\]

for (1.16) and

\[(x^2+y^2+1) \frac{d\chi}{ds} + 2\left(x \frac{dx}{ds} + y \frac{dy}{ds}\right) \chi = 0\]

for (1.17). Then we get

\[\chi = \frac{c}{x^2+y^2+1}\]

and \(\chi\) is not a constant in general, although there exist some critical \(\mathcal{D}\)-curves where \(\chi\) is constant.

Suppose

\[a\phi_i + b \frac{dx^j}{ds} (\partial_i \phi_k - \partial_k \phi_i) = 0\]

for some \(a\) and \(b\). Then we have

\[-ay - 2b \frac{dy}{ds} = 0, \quad ax + 2b \frac{dx}{ds} = 0, \quad a = 0,\]

and consequently

\[b = 0 \quad \text{or} \quad \frac{dx}{ds} = \frac{dy}{ds} = 0.\]

But the latter contradicts

\[\frac{dz}{ds} = \frac{y}{ds} \frac{dx}{ds} - x \frac{dy}{ds} \cdot \left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2 = 1.\]

Thus we see that

\[\phi_i \frac{dx^j}{ds} (\partial_j \phi_k - \partial_k \phi_j)\]

are linearly independent for all \(\mathcal{D}\)-curves.

3° A distribution in a contact metric manifold.

A contact metric manifold \(M\) is a Riemannian manifold of odd dimension endowed with a vector field \(\phi^a\) satisfying the following conditions,

(i) \(\phi^a \phi_a = 1\) where \(\phi_i = \partial_i \phi_a\),

(ii) \((V_{\phi_i} - \phi_i) \phi^i = 0\),
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(iii) \[ \frac{1}{4} (F_{ij} \varphi^i - F_{kj} \varphi^j)(F_{jkl} \varphi^k - F_{pl} \varphi^l) = -\partial_j \varphi + \varphi \varphi^h. \]

Let \( \mathcal{D} \) be a distribution which is orthogonal to the vector field \( \varphi^h \). Let \( x^h = x^h(s) \) be a \( \mathcal{D} \)-curve.

Suppose

\[ a \varphi^h + b \frac{dx^k}{ds} (F_{klm} \varphi^l - F_{mnp} \varphi^p) = 0. \]

Transvecting \( \varphi_h \) we get

\[ a = 0. \]

Transvecting with \( F_{ik} \varphi^i - F_{kj} \varphi^j \) we get

\[ b \left( -\frac{dx^k}{ds} + \frac{dx^l}{ds} \varphi_{kl} \right) = 0. \]

But, as we have

\[ \varphi^f \frac{dx^f}{ds} = 0 \]

for a \( \mathcal{D} \)-curve, we get \( b = 0 \). Hence

\[ \varphi_i, \frac{dx^i}{ds} (F_{ij} \varphi^j - F_{jk} \varphi^k) \]

are linearly independent for all \( \mathcal{D} \)-curves.

§ 4. A \((2n-2)\)-dimensional distribution on \( S^{2n-1} \) and the critical \( \mathcal{D} \)-curves of this distribution.

In their study of normal contact metric structure Sasaki and Hatakeyama [1] showed that \( S^{2n-1} \) is an example of normal contact metric manifolds. A normal contact metric structure of \( S^{2n-1} \) induces a \((2n-2)\)-dimensional distribution \( \mathcal{D} \) and it is the purpose of § 4 to study critical \( \mathcal{D} \)-curves of this distribution. On the other hand Yano and Ishihara [3] showed that \( S^{2n-1} \) is a fibred space with invariant Riemannian metric with a base space \( M^* \) which is a \((2n-2)\)-dimensional Kähler manifold of constant holomorphic sectional curvature.\(^2\) A \( \mathcal{D} \)-curve is a horizontal curve with respect to this fibre structure and a critical \( \mathcal{D} \)-curve \( C \) has a projection curve \( C^* \) on \( M^* \). We shall study some properties of \( C^* \).

1° When we regard \( S^{2n-1} \) as a hypersphere

\(^2\) See also Steenrod [2] where it is shown on page 108 that \( S^{2n-1} \) is a 1-sphere bundle over the projective space of \( n \) homogeneous complex variables.
in a $2n$-dimensional Euclidean space $E^{2n}$ where a rectangular coordinate system $(x^1, \ldots, x^{2n})$ is fixed, $x^1, \ldots, x^{2n}$ can be considered as local coordinates of $S^{2n-1}$ in domains $x^{2n} > 0$ and $x^{2n} < 0$.

There exists on $E^{2n}$ a complex structure induced canonically from the given rectangular coordinate system, and this complex structure and the metric of $E^{2n}$ induce on $S^{2n-1}$ a normal contact metric structure. The contravariant vector field $\varphi$ of this structure has components

\begin{align*}
\varphi^1 &= -x^3, \quad \varphi^3 = x^1, \quad \varphi^8 = -x^4, \quad \varphi^4 = x^3, \\
\varphi^8 &= -x^2, \quad \varphi^{2n-1} = -x^{2n}
\end{align*}

in the local coordinates $(x^\kappa)$. We consider again the distribution $\mathcal{D}$ which is orthogonal to the vector field $\varphi$.

As the metric tensor of $S^{2n-1}$ has components

\begin{equation}
\gamma_{\mu\nu} = \delta_{\mu\nu} + \frac{x^\mu x^\nu}{(x^{2n})^2}
\end{equation}

in the local coordinates $(x^\kappa)$, the components $\varphi_\mu$ of the covector field of the distribution $\mathcal{D}$ are

\begin{equation}
\varphi_\mu = \varphi^\mu + \frac{x^\mu x^1}{(x^{2n})^2} x^\nu,
\end{equation}

hence we have

\begin{equation}
\varphi_\mu \varphi^\mu = 1.
\end{equation}

Let $\{\gamma_{\kappa\lambda}\}$ be the Christoffel constructed from $\gamma_{\mu\nu}$ and let $\nabla_\mu$ be the operator of covariant differentiation with respect to the Riemannian metric of $S^{2n-1}$. If indices $a, b, c$ are used in the range $\{1, \ldots, 2n-2\}$, the components

\begin{equation}
\varphi_{a\mu} = \nabla_\mu \varphi_a - \nabla_a \varphi_\mu - \partial_a \varphi_\mu
\end{equation}

have the following values,

\begin{equation}
\varphi_{\kappa\kappa} = 0 \text{ except } \varphi_{12} = \varphi_{22} = \cdots = \varphi_{2n-1,2n-1} = -\varphi_{21} = -\varphi_{43} = \cdots = -\varphi_{2n-2,2n-1} = 2.
\end{equation}

3) In §4 indices $\kappa, \lambda, \mu, \cdots$ run over the range $\{1, \ldots, 2n-1\}$. Summation convention is used in the usual way and also in the following way, $A^1B^1 + \cdots + A^{2n-1}B^{2n-1}$.

4) The summation convention of the following form is also used,

\begin{equation}
A^aB^a = A^1B^1 + \cdots + A^{2n-2}B^{2n-2}.
\end{equation}
The rank of $\langle \varphi, \mu \lambda \rangle$ is $2n-2$.

As we have

\begin{equation}
\kappa \begin{bmatrix} \mu \\ \lambda \end{bmatrix} = \delta_{\mu \lambda} x^r + \frac{x^r x^s x^t}{(x^s)^2} = g_{\mu \lambda} x^r,
\end{equation}

the differential equation of a critical $\varphi$-curve is

\begin{equation}
\frac{d^2 x^r}{ds^2} + x^r = C\varphi^r \frac{dx^s}{ds}.
\end{equation}

The study of critical $\varphi$-curves is facilitated by the use of local coordinates $y^1, \ldots, y^{2n-1}$ such that

\begin{align}
x^1 &= y^1 \cos z + y^2 \sin z, \\
x^2 &= -y^1 \sin z + y^2 \cos z,
\end{align}

\begin{equation}
x^{2n-3} = y^{2n-3} \cos z + y^{2n-2} \sin z, \\
x^{2n-2} &= -y^{2n-3} \sin z + y^{2n-2} \cos z, \\
x^{2n-1} &= r \sin z, \\
x^{2n} &= r \cos z,
\end{equation}

where $z = y^{2n-1}$ and

\begin{equation}
r^2 = 1 - (x^1)^2 - \cdots - (x^{2n-3})^2 = 1 - (y^1)^2 - \cdots - (y^{2n-3})^2.
\end{equation}

Notice that these coordinates are used only in the range

\[ r > 0, \quad -\frac{\pi}{2} < z < \frac{\pi}{2}. \]

Let us define $f_{cb}$ by

\begin{equation}
f_{cb} = 0 \text{ except } f_{12} = f_{34} = \cdots = f_{2n-3,2n-2} = -f_{21} = -f_{43} = \cdots = -f_{2n-2,2n-1} = 1.
\end{equation}

Then the components $h_{\mu \lambda}$ of the metric tensor of $S^{2n-1}$ in local coordinates $(y^r)$ are

\begin{equation}
h_{cb} = \delta_{cb} + \frac{y^c y^b}{r^2},
\end{equation}

\begin{equation}
h_{c.2n-1} = f_{c1} y^1, \quad h_{2n-1.2n-1} = 1.
\end{equation}
If we define $h^{\nu\iota}$ by
\[ h_\nu h^{\iota} = \delta_\nu^\iota, \]
we have
\[ h^{\kappa\iota} = \delta_{\kappa\iota} - y^\iota y^\kappa + \frac{1}{r^2} f_{0\kappa} y^i f_{a\kappa} y^a, \]
(4.11)
\[ h^{b,2n-1} = -\frac{1}{r^2} f_{0\kappa} y^i, \quad h^{\kappa,2n-1} = \frac{1}{r^2}. \]

When we use the coordinate system $(y^i)$, the corresponding contravariant components of the vector $\varphi$ will be denoted by $\psi^i$, hence
\[ \psi^i = \frac{\partial y^i}{\partial x^1} \varphi^1. \]

Then we have
(4.12) \[ \psi^a = 0, \quad \psi^{2n-1} = -1. \]

We have for the corresponding covariant components
(4.13) \[ \psi_a = -f_{0\kappa} y^i, \quad \psi^{2n-1} = -1 \]

2° Remember that $\psi^i$ are the components of a Killing vector of unit length to which the distribution $\mathcal{D}$ is orthogonal. (4.12) shows that the $y^{2n-1}$-curves (curves on which $y^a$ are constant) are fibres of the fibred space $S^{2n-1}$. This fibred space which has been studied by Yano and Ishihara [3], has a base space $M^*$ of dimension $2n-2$ and, if we use the local coordinates $(y^i)$, namely $(y^a, y^{2n-1})$, in $S^{2n-1}$, the projection $\pi$: $S^{2n-1} \to M^*$ is given by $\pi$: $(y^a, y^{2n-1}) \to (y^a)$.

Let us introduce a metric into $M^*$ by the standard of Yano and Ishihara. If the metric tensor of $M^*$ is written $h^*_{cb}$ in the coordinate system $(y^i)$, $h^*_{cb}$ are obtained from
\[ h_{cb} dy^c dy^b = h^*_{cb} dy^c dy^b \]
by putting $\psi_a dy^i = 0$. The explicit formula is
(4.14) \[ h^*_{cb} = \delta_{cb} + \frac{y^c y^b}{r^2} f_{0c} y^i f_{a0} y^a. \]

The inverse $(h^{ba})$ of the matrix $(h^*_{cb})$ has the elements
(4.15) \[ h^{ba} = \delta_{ba} - y^b y^a + \frac{1}{r^2} f_{bc} y^i f_{a0} y^0. \]

The Christoffel $(\gamma^a_b)_c$ is
\[
\begin{bmatrix}
a \\
\varepsilon \\
b 
\end{bmatrix}^* = \left(\frac{y^c y^b}{r^2}\right)y^a
\]

(4.16)

\[+ f_{a c} y^i f_{a b} + f_{b c} y^i f_{c a} - 2 f_{a c} y^i f_{b c} y^a \]

\[- (f_{a c} y^i y^b + f_{b c} y^i y^f) \frac{1}{r^2} f_{a i} y^a.\]

On the other hand, if we define \( \psi_\mu^e \) by

\[
\psi_\mu^e = \left(\frac{\partial \psi_\mu}{\partial y^e} - \frac{\partial \psi_\mu}{\partial y^f}\right)h_{i e},
\]

we can write the differential equations of a critical \( \theta \)-curve in the form

(4.17)

\[
\frac{d^2 y^e}{ds^2} + \left[\kappa_\mu \lambda \right] \frac{dy^a}{ds} \frac{dy^i}{ds} = C_{\psi_\mu^e} \frac{dy^a}{ds}.
\]

Calculating the Christoffel \( \{_{a b}^c \} \) of \( h_{\mu i} \), we get from (4.17)

\[
y''^a = \left[-y'^c y'^e - \frac{(y'^c y'^e)^2}{r^2} + 2 \rho^2 - 2C\rho\right]y^a
\]

(4.18)

\[+ \frac{2y'^c y'^e}{r^2} (\rho - C)f_{a i} y^i + 2(\rho - C)f_{a i} y^i y^t\]

where

\[
y''^a = \frac{dy^a}{ds}
\]

and \( \rho \) is defined by

(4.19)

\[
\rho = f_{a i} y^i y^a.
\]

We can regard (4.18) as a curve \( C^* \) in \( M^* \), the projection of a critical \( \theta \)-curve \( C \). In order to find some properties of \( C^* \) we use (4.16) and write (4.18) in the form

\[
y''^a + \left[\begin{bmatrix}
a \\
\varepsilon \\
b 
\end{bmatrix}^* \right] y'^c y'^b
\]

(4.20)

\[= -2C\left(\rho y^a + \frac{1}{r^2} y'^c y'^e f_{a i} y^i + f_{a i} y^i y^t\right).
\]

Differentiating (4.20) covariantly along the curve \( C^* \) we get after some straightforward calculation
This shows that $C^*$ is a Riemannian circle of curvature $2|C|$. A Riemannian circle is by definition a curve in a Riemannian space whose development in a tangent space is a circle. Its global properties are quite various according to the enveloping manifold. Thus, for example, we cannot even guess the period of $C^*$.

But, as for the function $r(s)$ only, we can find its period. As $r$ is given by $y^a y^a = 1 - r^2$, we have

\[ y^c y^c = -rr', \]

(4.22)

\[ y^c y^c + y^c y'' = -r'r'' - rr''. \]

We also get from $h_{cb} y^c y^c y'' = 1$ and (4.14)

(4.23)

\[ y^c y^c + r'r'' = 1 + \rho^2. \]

On the other hand, if we substitute (4.18) into $y^c y''$, the second equation of (4.22) gives

\[ rr'' = -r^2(1 - \rho^2 + 2C\rho) = -r^2[1 + C^2 - (\rho - C)^2]. \]

As we assume $r > 0$, we get

(4.24)

\[ r'' = -r[1 + C^2 - (\rho - C)^2]. \]

We also obtain from (4.18), (4.19) and (4.22)

\[ \rho' = -\frac{2(\rho - C)r'}{r}. \]

Hence we have

(4.25)

\[ \rho - C = \frac{k}{r^2} \]

where $k$ is a constant. Substituting this into (4.24) we get

\[ r'' = -(1 + C^2)r + \frac{k^2}{r^3}. \]

The general solution of this differential equation is
where

\[ k^2 = (1 + C^2)(C_1^2 - C_2^2). \]

Thus we find that \( r(s) \) has period \( \pi / \sqrt{1 + C^2} \) or \( r(s) \) is reduced to a constant. The only exceptional cases will occur if \( k = 0 \). Then we have \( \rho = C \). Such cases will be studied in the appendix.

3° It was shown by Yano and Ishihara [3] that the base space \( M^* \) is a Kähler manifold of constant holomorphic sectional curvature.

Let us turn to the Euclidean space \( E^{2n} \) equipped with a fixed rectangular coordinate system \((x^1, \ldots, x^{2n})\) and introduce a complex coordinate system

\[ Z^1 = x^1 + ix^2, \ldots, Z^{n-1} = x^{2n-3} + ix^{2n-2}, \]

\[ Z^n = x^{2n-1} + ix^n. \]

Then we have a complex space \( \mathbb{C}^n \). In \( \mathbb{C}^n - \{0\} \) we can regard \((Z^0, Z^1, \ldots, Z^{n-1})\) as a system of homogeneous complex coordinates of the complex projective space \( P^{n-1}(\mathbb{C}) \). If we assume \( Z^0 \neq 0 \), we can introduce an inhomogeneous complex coordinate system by

\[ z^1 = \frac{Z^1}{Z^0}, \ldots, z^{n-1} = \frac{Z^{n-1}}{Z^0}, \]

and, if we introduce real local coordinates \( w^1, \ldots, w^{2n-2} \) in \( P^{n-1}(\mathbb{C}) \) by

\[ z^1 = w^1 + iw^2, \ldots, z^{n-1} = w^{2n-3} + iw^{2n-2}, \]

then we obtain

\[ w^1 = \frac{x^1 x^{2n-1} + x^2 x^{2n}}{(x^{2n-1})^2 + (x^{2n})^2}, \]

\[ w^2 = \frac{x^2 x^{2n-1} - x^1 x^{2n}}{(x^{2n-1})^2 + (x^{2n})^2}, \]

\[ \ldots \ldots \ldots \ldots \ldots \]

\[ w^{2n-3} = \frac{x^3 x^{2n-1} - x^{2n-3} x^{2n}}{(x^{2n-1})^2 + (x^{2n})^2}, \]

\[ w^{2n-2} = \frac{x^3 x^{2n-1} + x^{2n-3} x^{2n}}{(x^{2n-1})^2 + (x^{2n})^2}. \]

If the ordinary Kähler metric of \( P^{n-1}(\mathbb{C}) \) is multiplied by a suitable constant, the corresponding metric tensor has following components \( g_{\alpha \beta}^F \) in real coordinates.
\( w^1, \ldots, w^{2n-1} \),

\[
(4.30) \quad g^a_{\alpha\beta} = \frac{\delta_{\alpha\beta}}{1 + w^a w^b} - \frac{w^a w^b + f_{\alpha\gamma} w^\gamma f_{\beta\delta} w^\delta}{(1 + w^a w^b)^2}
\]

which will be easily proved by direct calculation.

The relation between \( w^1, \ldots, w^{2n-1} \) and \( y^1, \ldots, y^{2n-1} \) is obtained from (4.7) and (4.29) to be

\[
(4.31) \quad w^a = \frac{1}{r} f_{a\gamma} y^\gamma, \quad w^a w^b = \frac{1}{r^2} - 1.
\]

Hence we can write (4.30) in the form

\[
(4.32) \quad g^a_{\alpha\beta} = r^2 (\delta_{\alpha\beta} - y^\alpha y^\beta - f_{\alpha\gamma} y^\gamma f_{\beta\delta} y^\delta).
\]

That the metric tensor whose components are \( g^a_{\alpha\beta} \) in local coordinates \((w^a)\) is identical with the metric tensor whose components are \( h^a_{\alpha\beta} \) in local coordinates \((y^a)\) is immediately shown since we have

\[
g^a_{\alpha\beta} y^\alpha w^\beta = h^a_{\alpha\beta} y^\alpha y^\beta
\]

because of (4.14), (4.31) and (4.32).

As \( f_{\alpha\beta} \) satisfies

\[
P^a_{\gamma} f_{\alpha\beta} = \begin{bmatrix} e & \ast \\ c & b \end{bmatrix} f_{\alpha\beta} + \begin{bmatrix} e & \ast \\ c & a \end{bmatrix} f_{\alpha\beta} = 0
\]
on account of (4.16), \((h^a_{\alpha\beta}, f_{\alpha\beta})\) is a Kähler structure of \( P^{n-1}(C) \).

4° Let

\[
(4.33) \quad \alpha^a Z^a + \alpha^1 Z^1 + \cdots + \alpha^{n-1} Z^{n-1} = 0
\]

be the equation of a hyperplane of \( P^{n-1}(C) \). If we use only real numbers, we can write (4.33) in the form

\[
(4.34) \quad A^a y^a = Kr, \quad A^a f_{a\gamma} y^\gamma = Lr
\]

where \( r \) is given by (4.8). Hence, to a complex hyperplane of \( P^{n-1}(C) \) corresponds a subspace \( M' \) of codimension 2 in \( M^* \). The subspace \( M' \) determined by (4.34) will be denoted by \( M'(A^a, K, L) \).

If we define functions \( X(s) \) and \( Y(s) \) by

\[
X(s) = A^a y^a(s) - Kr(s),
\]

\[
(4.35) \quad Y(s) = A^a f_{a\gamma} y^\gamma(s) - Lr(s)
\]

along a curve \( C^* \), these satisfy
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\[ X'' = \left[-(1+C^2) + \frac{k^2}{r^4}\right] X - \frac{2kr'}{r^3} Y + \frac{2k}{r^2} Y', \]

\[ Y'' = \left[-(1+C^2) + \frac{k^2}{r^4}\right] Y + \frac{2kr'}{r^3} Y - \frac{2k}{r^2} X', \]

for we get

\[ y'' = \left[-(1+C^2) + \frac{k^2}{r^4}\right] y' - \frac{2kr'}{r^3} f_{at} y' + \frac{2k}{r^2} f_{at} y'^r, \]

from (4.18), (4.23) and (4.25). Hence we get \( X(s) = Y(s) = 0 \) if \( X(s) \) and \( Y(s) \) satisfy \( X(0) = Y(0) = X'(0) = Y'(0) = 0 \).

This proves the following lemma.

**Lemma 4.1.** Let \( C^* \) be a curve of \( M^* \) which is the projection of a critical \( \mathcal{Q} \)-curve \( C \) in \( S^{2n-1} \). If, in the corresponding curve in \( P^{n-1}(C) \), which will also be denoted by \( C^* \), a point \( P \) and the tangent of \( C^* \) at \( P \) lie in a complex hyperplane, then \( C^* \) lies completely in this complex hyperplane.

From (4.20) we observe that a curve \( C^* \) where \( C=0 \) is a geodesic of \( M^* \) and that any geodesic of \( M^* \) is a curve \( C^* \). Hence \( M'(A^a, K, L) \) is a totally geodesic subspace. Notice that \( M'(f_{at} A^i, -L, K) \) is the same subspace as \( M'(A^a, K, L) \).

A subspace \( M'(A^a, K, L) \) tangent to a given curve \( C^* \) at the point \( s=0 \) is obtained if we take \( A^a, K, L \) satisfying

\[ A^a y^a(0) - Kr(0) = 0, \quad A^a y'^a(0) - Kr'(0) = 0, \]

(4.37)

\[ A^a f_{at} y^a(0) - Lr(0) = 0, \quad A^a f_{at} y'^a(0) - Lr'(0) = 0. \]

If we define \( M \) by

\[ M = \begin{pmatrix}
 y^1 & y^2 & \ldots & y^{2n-3} & y^{2n-2} & r & 0 \\
 y'^1 & y'^2 & \ldots & y'^{2n-3} & y'^{2n-2} & r' & 0 \\
 y^2 & -y^1 & \ldots & y^{2n-3} & -y^{2n-3} & 0 & r \\
 y'^2 & -y'^1 & \ldots & y'^{2n-3} & -y'^{2n-3} & 0 & r'
\end{pmatrix}, \]

the rank of \( M \) is 4, since we have

\[ MM^\tau = \begin{pmatrix}
 1 & 0 & 0 & -\rho \\
 0 & 1 + \rho^2 & \rho & 0 \\
 0 & \rho & 1 & 0 \\
 -\rho & 0 & 0 & 1 + \rho^2
\end{pmatrix}, \quad \det(MM^\tau) = 1 \]
because of (4.22) and (4.23). Hence we have $2n-4$ linearly independent solutions of (4.37). We also observe that, if $(A^a, K, L)$ is a solution of (4.37), $(f_{at} A^t, -L, K)$ is also a solution.

Suppose that $(A^a, K, L)$ $(\xi=1, \ldots, 2p)$ are $2p$ linearly independent solutions of (4.37) where

$$A^a = f_{at} A^t \quad (u=1, \ldots, p).$$

If $(A^a, K, L)$ is a solution of (4.37) such that

$$A^a = k A^a + \cdots + k A^a, \quad (1) \quad (2p) \quad (2p)$$

then we find immediately that

$$K = k K + \cdots + k K, \quad (1) \quad (2p) \quad (2p)$$

$$L = k L + \cdots + k L, \quad (1) \quad (2p) \quad (2p)$$

hence $(A^a, K, L)$ is a linear combination of $(A^a, K, L), \ldots, (A^a, K, L)$. Then we also find that the $2p$ $(2n-2)$-tuples $A^a, \ldots, A^a$ are linearly independent, for a solution $(A^a, K, L)$ must satisfy $K=L=0$ if $A^a=0$.

From the above result we can deduce that there exists a set of $2n-4$ linearly independent solutions $(A^a, K, L)$ $(\xi=1, \ldots, 2n-4)$ of (4.37) where

$$A^a = f_{at} A^t, \quad (\xi=1, \ldots, 2n-4)$$

and such that the $(2n-2)$-tuples $A^a, \ldots, A^a$ are linearly independent.

We can interpret this result geometrically as follows.

**Lemma 4.2.** For any curve $C^*$ there exists in $M^*$ a totally geodesic subspace of dimension 2 which contains $C^*$ and is determined by a system of equations

$$A^a y^a = K r \quad (\xi=1, \ldots, 2n-4)$$

where

$$A^a = f_{at} A^t \quad (u=1, \ldots, n-2).$$

This subspace is common to all curves $C^*$ passing a point $P$ and having a common tangent vector at the point $P$.

The contents of §4 can be resumed in the following theorem.

**Theorem 4.3.** According to Sasaki and Hatakeyama an $S^{2n-1}$ in $E^{2n}$ can be treated as a normal contact metric manifold. According to Yano and Ishihara $S^{2n-1}$
can also be treated as a fibred space with invariant Riemannian metric. The base space \( M^* \) is a Kähler space of constant holomorphic sectional curvature. By virtue of these structures an \( S^{2n-1} \) becomes a space equipped with a distribution \( \mathcal{D} \) of dimension \( 2n-2 \) where the \( \mathcal{D} \)-curves are horizontal curves of the fibred space. If \( C \) is a critical \( \mathcal{D} \)-curve, the projection \( C^* \) on \( M^* \) of \( C \) has following properties.

(I) \( C^* \) is a Riemannian circle of \( M^* \).

(II) Let \( \{M'\} \) be the set of \((2n-4)\)-dimensional totally geodesic subspaces of \( M^* \) such that each subspace \( M' \) is a complex hypersurface if \( M^* \) is regarded as a complex projective space. Then any curve \( C^* \) passing a point \( P \) of a subspace \( M' \) and tangent at \( P \) to this \( M' \) is contained completely in this subspace \( M' \).

(III) For any curve \( C^* \) there exists in \( M^* \) a totally geodesic subspace of dimension 2 which contains \( C^* \) and is obtained as an intersection of \( n-2 \) elements of \( \{M'\} \). This subspace is common to all curves \( C^* \) passing a common point \( P \) and having a common tangent at \( P \).

**Appendix. The exceptional cases.**

In this appendix we study critical \( \mathcal{D} \)-curves \( C \) of \( S^{2n-1} \) where \( k=0, \rho=C \).

In this case the differential equation of \( r \) is reduced to the form

\[
A.1 \quad r'' = -(1+C^p) \hspace{1pt} r
\]

and the general solution \( r=r_0 \cos(\sqrt{1+C^p} (s-s_0)) \) does not obey the restriction \( r>0 \). Hence, for the study of global properties of such exceptional critical \( \mathcal{D} \)-curves \( C \), we use rectangular coordinates \( x^1, \ldots, x^{2n} \) of \( E^{2n} \).

Since the equations (4.18) of the projection curve \( C^* \) are written in local coordinates \( y^1, \ldots, y^{2n-3} \), we must use (4.7) and (4.8) to return to the coordinates \( x^1, \ldots, x^{2n} \). \( C \) is obtained by the process of lifting in which we use

\[
A.2 \quad \phi_ay''+\phi_{2n-1}z'=0,
\]

which becomes

\[
A.3 \quad z' = -C
\]

because of (4.13) and \( \rho=C \).

Differentiating (4.7) and using (A.3) we obtain

\[
\begin{align*}
\dot{x}^{a} &= \dot{y}^{a} \cos z + y^{a} \sin z \hspace{1pt} C \hspace{1pt} x^{a}, \\
\dot{x}^{2} &= -y^{a} \sin z + y^{a} \cos z \hspace{1pt} C \hspace{1pt} x^{a}, \\
\cdots & \\
\dot{x}^{a} &= \dot{y} \sin z \hspace{1pt} C \hspace{1pt} x^{a}, \\
\dot{x}^{2n} &= \dot{y} \cos z \hspace{1pt} C \hspace{1pt} x^{2n-1},
\end{align*}
\]

and
\[
x''^1 = y''^1 \cos z + y''^2 \sin z + 2C(y'^1 \sin z - y'^2 \cos z) - C^2 x^1,
\]
\[
x''^2 = -y''^1 \sin z + y''^2 \cos z + 2C(y'^1 \cos z + y'^2 \sin z) - C^2 x^2,
\]
(A. 5)
\[
x''^m = r'' \sin z - 2Cr' \cos z - C^2 x^m - 1,
\]
\[
x''^{2n} = r'' \cos z + 2Cr' \sin z - C^2 x^{2n}.
\]

Since \( r \) and \( y'' \) satisfy
\[
 r'' = -(1 + C^2)r, \quad y'' = -(1 + C^2)y''
\]
along \( C \), we obtain
\[
x''^1 + 2C x''^2 + x^1 = 0,
\]
\[
x''^2 - 2C x''^1 + x^3 = 0,
\]
(A. 6)
\[
x''^{2n-1} + 2C x''^{2n} + x^{2n-1} = 0,
\]
\[
x''^{2n} - 2C x''^{2n-1} + x^{2n} = 0.
\]

\( C \) satisfies moreover
\[
(x^1)^2 + \ldots + (x^{2n})^2 = 1,
\]
\[
(x''^1)^2 + \ldots + (x''^{2n})^2 = 1,
\]
(A. 7)
\[
x^1 x''^1 + \ldots + x^{2n} x''^{2n} = 0,
\]
\[
x^1 x''^2 - x^2 x''^1 + \ldots + x^{2n-1} x''^{2n} - x^{2n} x''^{2n-1} = 0.
\]

The fourth equation of (A. 7) is obtained from (A. 4) and \( p = C \).
If \( F_{ji} \) is defined by
\[
F_{ji} = 0 \text{ except } F_{12} = F_{34} = \ldots = F_{2n-1,2n} = -F_{11} = -F_{43} = \ldots = -F_{2n,2n-1} = 1,
\]
then we can write the fourth equation of (A. 7) in the form
(A. 8)
\[
F_{ji} x''^i x^j = 0.
\]

Now we can write (A. 6) in the form
\[
x''^h + 2C F_{hi} x''^i + x^h = 0.
\]
If in \( E^{2n} \) the vector \( x^h \) is denoted by \( X \) and the vector \( F_{hi} x^i \) by \( FX \), (A. 6) is written

Differentiating repeatedly and eliminating $FX, FX'$ we get

\[(A. 10) \quad X^{(6)} + 2(2C^2 + 1)X'' + X = 0.\]

Let us put

\[(A. 11) \quad \alpha = \sqrt{1 + C^2 + |C|}, \quad \beta = \sqrt{1 + C^2 - |C|}.\]

Assuming $C \neq 0$, we have $\alpha > \beta > 0$. $-\alpha^2$ and $-\beta^2$ are the roots of $\lambda^2 + 2(2C^2 + 1)\lambda + 1 = 0$. Hence

\[(A. 12) \quad X = A_1 \cos \alpha s + A_2 \sin \alpha s + B_1 \cos \beta s + B_2 \sin \beta s\]

is the general solution of (A. 10).

Substituting (A. 12) into (A. 7) we can deduce

\[(A_1, A_1) = (A_2, A_2) = \frac{1}{2} - \frac{|C|}{2 \sqrt{1 + C^2}}, \]

\[(B_1, B_1) = (B_2, B_2) = \frac{1}{2} + \frac{|C|}{2 \sqrt{1 + C^2}}\]

and that $A_1, A_2, B_1, B_2$ are mutually orthogonal.

Substituting (A. 12) into (A. 9) we can deduce

\[FA_1 = -\frac{|C|}{C} A_2, \quad FB_1 = \frac{|C|}{C} B_2.\]

Thus we have

\[(A. 13) \quad X = A \cos \alpha s + \varepsilon F A \sin \alpha s + B \cos \beta s + \varepsilon F B \sin \beta s\]

where $\varepsilon = \pm 1$ and

\[(A, A) = \frac{1 - \beta^2}{\alpha^2 - \beta^2}, \quad (B, B) = \frac{\alpha^2 - 1}{\alpha^2 - \beta^2}, \quad (A, B) = 0.\]

If $C = 0$ we have the simplest case,

\[(A. 14) \quad X = A \cos s + B \sin s\]

where $(A, A) = (B, B) = 1, (A, B) = 0$.

Thus we have the following result.

The equations of the exceptional critical \(\mathcal{C}\)-curves are (A. 13) or (A. 14) according as $C \neq 0$ or $C = 0$. 

REFERENCES


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