ON QUASI-NORMAL \((f, g, u, v, \lambda)-STRUCTURES\)

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§ 0. Introduction.

Let \(M\) be a \(C^\infty\) differentiable manifold and assume that \(M\) admits a tensor field \(f\) of type \((1, 1)\), two vector fields \(U, V\), two 1-forms \(u, v\) and a function \(\lambda\) satisfying

\[
\begin{align*}
  f^2X &= -X + u(X)U + v(X)V, \\
  fU &= -\lambda V, & u(fX) &= +\lambda v(X), \\
  fV &= +\lambda U, & v(fX) &= -\lambda u(X), \\
  u(U) &= 1 - \lambda^2, & u(V) &= 0, \\
  v(U) &= 0, & v(V) &= 1 - \lambda^2
\end{align*}
\]

for any vector field \(X\). Such a manifold \(M\) is said to have an \((f, U, V, u, v, \lambda)\)-structure \([1], [2]\). A manifold \(M\) with \((f, U, V, u, v, \lambda)\)-structure is even-dimensional \([2]\).

An \((f, U, V, u, v, \lambda)\)-structure is said to be normal if the tensor field \(S\) of type \((1, 2)\) defined by

\[
S(X, Y) = N(X, Y) + (du)(X, Y)U + (dv)(X, Y)V
\]

vanishes, where \(N\) is the Nijenhuis tensor of \(f\) defined by

\[
N(X, Y) = [fX, fY] - f[X, fY] - f[Y, fX] + f^2[X, Y]
\]

for arbitrary vector fields \(X\) and \(Y\).

Assume that a differentiable manifold \(M\) with \((f, U, V, u, v, \lambda)\)-structure admits a Riemannian metric \(g\) such that

\[
g(fX, fY) = g(X, Y) - u(X)u(Y) - v(X)v(Y),
\]

\[
g(U, X) = u(X), \quad g(V, X) = v(X)
\]

for arbitrary vector fields \(X\) and \(Y\). We call an \((f, g, u, v, \lambda)\)-structure an \((f, U, V, u, v, \lambda)\)-structure with a Riemannian metric \(g\) satisfying (0.4) \([2]\).
The tensor field of type (0, 2) defined by
\[(0.5)\]
\[\omega(X, Y) = g(fX, Y)\]
for arbitrary vector fields \(X\) and \(Y\) is a 2-form [2].

Okumura and one of the present authors [2] proved

**Theorem A.** Let \(M\) be a complete manifold with normal \((f, g, u, v, \lambda)\)-structure satisfying
\[\phi\text{ being a differentiable function on } M.\]
If \(\lambda(1-\lambda^2)\) is an almost everywhere non-zero function and \(\text{dim } M > 2\), then \(M\) is isometric with an even-dimensional sphere.

We put
\[(0.6)\]
\[T(X, Y, Z) = g(S(X, Y), Z)\]
If
\[(0.7)\]
\[T(X, Y, Z) - (d\omega)(fX, Y, Z) - (d\omega)(fY, X, Z) = 0,\]
then we say that the \((f, g, u, v, \lambda)\)-structure is quasi-normal.

The main purpose of the present paper is first to prove that in a manifold with quasi-normal \((f, g, u, v, \lambda)\)-structure such that the function \(\lambda(1-\lambda^2)\) is almost everywhere non-zero, the conditions
\[\mathcal{L}_u g = -2\alpha g\quad\text{and}\quad dv = 2\omega\]
are equivalent, where \(\mathcal{L}_U\) denotes the operator of Lie differentiation with respect to the vector field \(U\) and \(\alpha\) is a function, and next to prove that in a manifold with quasi-normal \((f, g, u, v, \lambda)\)-structure such that the function \(\lambda(1-\lambda^2)\) is almost everywhere non-zero and
\[\mathcal{L}_u g = -2c\lambda g\quad\text{or}\quad dv = 2c\omega\]
is satisfied, \(c\) being a non-zero constant, we have
\[du = -2\phi g,\quad \mathcal{L}_v g = -2\phi g,\]
\(\phi\) being a function.

Combining Theorem A and this result, we see that a complete manifold with normal \((f, g, u, v, \lambda)\)-structure such that the function \(\lambda(1-\lambda^2)\) is almost everywhere non-zero, \(\text{dim } M > 2\) and \(\mathcal{L}_u g = -2c\lambda g\) or \(dv = 2c\omega\) is satisfied is isometric to an even-dimensional sphere.

This result is an improvement of Theorem A.

In § 1, we prove general formulas for an \((f, g, u, v, \lambda)\)-structure and in § 2, we specialize these formulas for a quasi-normal \((f, g, u, v, \lambda)\)-structure. In § 3, we
prove the equivalence of $\mathcal{L}_vg = -2\alpha g$ and $dv = 2\alpha \omega$ in a manifold with quasi-normal $(f, g, u, v, \lambda)$-structure.

In the last §4, we prove that for the normal $(f, g, u, v, \lambda)$-structure, the condition $\mathcal{L}_vg = -2\alpha g$ or $dv = 2\omega \omega$ implies $du = -2\phi_\omega$.

In the sequel, we assume that the function $\lambda(1-\lambda^2)$ is almost everywhere non-zero and we use the index notation.

§1. General formulas.

We consider a $C^\infty$ differentiable manifold $M$ with an $(f, g, u, v, \lambda)$-structure, that is, a Riemannian manifold with metric tensor $g$ which admits a tensor field $f$ of type $(1, 1)$, two 1-forms $u$ and $v$ (or two vector fields associated with them), and a function $\lambda$ satisfying

\[
\begin{cases}
  f^i_j f^h_j = -\partial^i_j + u^i_j u^h_j + v^i_j v^h_j, \\
  f^i_j g^j_k = g^i_k - u^i_j u^k_j - v^i_j v^k_j,
\end{cases}
\]

(1.1)

\[
f^i_i u^i = \lambda v^i \quad \text{or} \quad f^i_i v^i = -\lambda w^i,
\]

(1.2)

being skew-symmetric. Such an $M$ is even-, say, $2n$-dimensional.

We put

\[
S^h_j^i = f^i_j f^k_j - f^i_j f^k_j - (f^i_j f^k_j - f^k_j f^i_j) f^h_j + u^i_j u^h_j + v^i_j v^h_j,
\]

(1.3)

where

\[
u_{ji} = f^i_j u^i_j, \quad v_{ji} = f^i_j v^i_j,
\]

(1.4)

$F_j$ denoting the operator of covariant differentiation with respect to the Levi-Civita connection. If the tensor $S^h_j^i$ vanishes, the $(f, g, u, v, \lambda)$-structure is said to be normal.

Transvecting (1.3) with $u_{\lambda}$, and using (1.1), we find

\[
S^h_j^i u_{\lambda} = f^i_j [(F_j f^h_j u_{\lambda}) - f^h_j F_j u_{\lambda}] - f^i_j [(F_j f^h_j u_{\lambda}) - f^h_j F_j u_{\lambda}]
- \lambda [F_j (f^h_j v_{\lambda}) - f^h_j F_j v_{\lambda} - (f^h_j v_{\lambda}) + f^h_j F_j v_{\lambda}] + (1 - \lambda^2) u_{ji}
\]

\[
f^i_j [(F_j \lambda) v_{\lambda} + \lambda F_j v_{\lambda} - f^h_j F_j u_{\lambda}] - f^i_j [(F_j \lambda) v_{\lambda} + \lambda F_j v_{\lambda} - f^h_j F_j u_{\lambda}]
- \lambda [-(F_j \lambda) u_{\lambda} - f^h_j F_j u_{\lambda} - f^h_j F_j v_{\lambda} + (F_j \lambda) u_{\lambda} + \lambda F_j v_{\lambda} + f^h_j F_j v_{\lambda}] + (1 - \lambda^2) u_{ji}.
\]

that is,
\[ S_{jk}^b u_h = u_{jt} - f_j^i f_k^a u_{it} + \lambda (f_j^t v_{it} - f_j^t v_{ti}) \]

(1.5)

+ (f_j^t v_{it} - f_j^t v_{ti}) v_l \lambda + \lambda (v_j^i u_t - v_j^i u_t).

Similarly, we can prove

\[ S_{jk}^b v_h = v_{jt} - f_j^i f_k^a v_{it} - \lambda (f_j^t u_{it} - f_j^t u_{ti}) \]

(1.6)

\[- (f_j^t u_{it} - f_j^t u_{ti}) v_l \lambda + \lambda ((f_j^i v_l - (f_j^i v_l). \]

We now put

\[ f_{jth} = v_j f_{ih} + v_k f_{jh} + v_h f_{jh} \]

(1.7)

and consider the covariant components of \( S \):

\[ S_{jth} = f_j^i f_{kth} - f_t^i f_{kjh} + (v_j f_{ih} - v_j f_{jh}) f_h^i + u_{ji} u_{th} + v_{ji} v_{th}. \]

(1.8)

Then we have

\[ S_{jth} = f_j^i f_{kth} - v_j f_{ih} - v_h f_{jh} - f_j^i (v_j f_{ih} - v_h f_{jh} - f_k^i f_{jh}) \]

\[ + (v_j f_{ih} - v_k f_{jh}) f_h^i + u_{ji} u_{th} + v_{ji} v_{th} \]

\[ = f_j^i f_{kth} - f_j^i f_{kjh} - v_j (v_j f_{ih} - v_h f_{jh} + f_j^i f_{ih}) \]

\[ - f_j^i v_j f_{ih} + f_j^i v_h f_{jh} + u_{ji} u_{th} + v_{ji} v_{th} \]

\[ = f_j^i f_{kth} - f_j^i f_{kjh} - v_j (v_j f_{ih} - v_h f_{jh} + f_j^i f_{ih}) \]

\[ - f_j^i v_j f_{ih} + f_j^i v_h f_{jh} + u_{ji} u_{th} + v_{ji} v_{th}, \]

from which

\[ S_{jth} = (f_j^i f_{kth} - f_j^i f_{kjh}) \]

(1.9)

\[ = - (f_j^i v_j f_{ih} - f_j^i v_h f_{jh}) + u_{ji} (v_j f_{ih} - v_h f_{jh}) + v_{ji} (v_j f_{ih} - v_h f_{jh}). \]

Transvecting (1.9) with \( w^j \) and using (1.1), we find

\[ \omega^j [S_{jth} - (f_j^i f_{kth} - f_j^i f_{kjh})] \]

\[ = \lambda [v_j (f_j^i v^i) - f_j^i v_j v^i] + f_j^i [v_j (f_j^i w^i) - f_j^i v^i w^i] \]

\[ + \lambda (v_j v^i u_{ti} - u_t (v_j v^i u_{ti}) - v_i (v_j v^i u_{ti}) \]

\[ = \lambda [v_j \lambda u_{ti} + \lambda v_j v_t + f_j^i v^i v_t] \]

\[ + f_j^i \lambda (v_t - f_j^i v^i u_{ti}) \]

\[ + \lambda (v_j v^i u_{ti} - u_t (v_j v^i u_{ti}) - v_i (v_j v^i u_{ti}). \]
\[= -\lambda^2 u_{ih} + \mathcal{L}_{u} u_{ih} + 2\lambda f^t_i v_t - u.u^t \mathcal{L}_{u} u_{ih} - v_v u_{ih}\]
\[= -\lambda^2 u_{ih} + \mathcal{L}_{u} u_{ih} + \lambda f^t_i \left[ \mathcal{L}_{v} v_{ih} - v_v \right] - u.u^t \mathcal{L}_{u} u_{ih} - v_v u_{ih},\]

where \(\mathcal{L}_u\) and \(\mathcal{L}_v\) denote Lie differentiation with respect to \(u^h\) and \(v^h\) respectively, from which,

\[u'[S_{fth} - (f^t_i f_{sth} - f^t_i f_{sujh})] = \mathcal{L}_u u_{ih} - u.u^t \mathcal{L}_u u_{ih} + \lambda f^t_i \mathcal{L}_v v_{ih} - \lambda^2 u_{ih} - \left[ \lambda f^t_i + v_v \right] u_{ih}.
\]

Similarly, we have

\[v'[S_{fth} - (f^t_i f_{sth} - f^t_i f_{sujh})] = \mathcal{L}_v v_{ih} - v_v \mathcal{L}_v v_{ih} - \lambda f^t_i \mathcal{L}_u u_{ih} - \lambda^2 v_{ih} + (\lambda f^t_i - u_v) u_{ih}.
\]

§2. Formulas for quasi-normal \((f, g, u, v, \lambda)\)-structures.

If the condition

\[S_{fth} - (f^t_i f_{sth} - f^t_i f_{sujh}) = 0\]

is satisfied, then we say that the \((f, g, u, v, \lambda)\)-structure is quasi-normal.

If the structure is quasi-normal, we have, from (1.10) and (1.11),

\[\mathcal{L}_u u_{ih} - u.u^t \mathcal{L}_u u_{ih} + \lambda f^t_i \mathcal{L}_v v_{ih} = \lambda^2 u_{ih} + \left[ \lambda f^t_i + v_v \right] u_{ih},\]

and

\[\mathcal{L}_v v_{ih} - v_v \mathcal{L}_v v_{ih} - \lambda f^t_i \mathcal{L}_u u_{ih} = \lambda^2 v_{ih} - \left[ \lambda f^t_i - u_v \right] u_{ih},\]

respectively.

From (2.2), we find

\[\lambda^2 u_{ih} = \mathcal{L}_u u_{ih} - u.u^t \mathcal{L}_u u_{ih} + \lambda f^t_i \mathcal{L}_v v_{ih} - \lambda^2 u_{ih} - \left[ \lambda f^t_i + v_v \right] u_{ih}.
\]

From (2.3), we have

\[\lambda^2 v_{ih} = \mathcal{L}_v v_{ih} - v_v \mathcal{L}_v v_{ih} - \lambda f^t_i \mathcal{L}_u u_{ih} = \lambda^4 v_{ih} - \left[ \lambda f^t_i - u_v \right] (\lambda^2 u_{ih}).\]

Substituting (2.4) into this equation, we have

\[\lambda^3 \mathcal{L}_v v_{ih} - \lambda^2 v_v v_v \mathcal{L}_v u_{ih} - \lambda^3 f^t_i \mathcal{L}_u u_{ih}
\]
\[= \lambda^4 v_{ih} - \lambda f^t_i \left[ \mathcal{L}_u u_{ih} - u.u^t \mathcal{L}_u u_{ih} + \lambda f^t_i \mathcal{L}_v v_{ih} - \lambda^2 u_{ih} - \left[ \lambda f^t_i + v_v \right] u_{ih} \right]
\]
\[+ u_v v_v \left[ \mathcal{L}_u u_{ih} - u.u^t \mathcal{L}_u u_{ih} + \lambda f^t_i \mathcal{L}_v v_{ih} - \lambda^2 u_{ih} - \left[ \lambda f^t_i + v_v \right] u_{ih} \right],\]

\[\lambda^3 \mathcal{L}_v v_{ih} - \lambda^2 v_v v_v \mathcal{L}_v u_{ih} - \lambda^3 f^t_i \mathcal{L}_u u_{ih}.
\]
\[
\lambda^3(1 - \lambda^2)v_{th} + (\lambda^3v_{th} - \lambda^3v_{sh})v_{sh} = \{v_t\nu^4 + \lambda^3v_t\nu^4 - \lambda(1 - \lambda^2)f_i^4\}L_{u\theta_{th}}.
\]

Transvecting this equation with \( u^i \), we find
\[
\lambda^3(1 - \lambda^2)u^iv_{th} + (1 - \lambda^2)u^sv_{sh} = \{(1 - \lambda^2)v^4 + \lambda^3(1 - \lambda^3)v^4\}L_{u\theta_{th}},
\]
from which,
\[
(2.5) \quad \lambda^3(1 - \lambda^2)v_{th} + (\lambda^3v_{th} - \lambda^3v_{sh})v_{sh} = \{v_t\nu^4 + \lambda^3v_t\nu^4 - \lambda(1 - \lambda^2)f_i^4\}L_{u\theta_{th}}.
\]

Similarly, from (2.2) and (2.3), we obtain
\[
\lambda^3(1 - \lambda^3)u_{th} + (v_t\nu^4 - \lambda^2\nu_tu^4)u_{sh}
\]

or
\[
(2.10) \quad \lambda^3(1 - \lambda^3)v_{th} + (\lambda^3v_{th} - \lambda^3v_{sh})v_{sh} = v_{ij}(L_{u\theta_{th}})v^b,
\]

which shows that
\[
(2.7) \quad (L_{u\theta_{th}})u^iv^4 = 0
\]

and
\[
(2.8) \quad (L_{u\theta_{th}})v^4 = v_{ij}(L_{u\theta_{th}})v^b.
\]

Substituting (2.6) into (2.5), we obtain
\[
\lambda^3(1 - \lambda^2)v_{th} - \lambda^3v_{sh} = \{\lambda v_{th} - (1 - \lambda^2)f_i^4\}L_{u\theta_{th}}.
\]

Thus (2.9) can be written as
\[
(2.10) \quad \lambda^3(1 - \lambda^3)v_{th} = v_{ij}(L_{u\theta_{th}})v^b.
\]

or
\[
(2.11) \quad \lambda^3(1 - \lambda^3)v_{th} = v_{ij}(L_{u\theta_{th}})v^b.
\]
Transvecting this equation with $v^h$, we find
\[
\lambda(1-\lambda^2)v^h u_{th} + (1-\lambda^2)v^h u_{th} = [(1-\lambda^2)u^t + \lambda^2(1-\lambda^2)u^t] \mathcal{L}_v g_{th},
\]
from which,
\[
(2.12) \quad v^h u_{th} = u^t \mathcal{L}_v g_{th},
\]
which shows that
\[
(2.13) \quad (\mathcal{L}_v g_{ji}) u^i v^j = 0
\]
and
\[
(2.14) \quad (\mathcal{L}_v g_{ji}) u^i u^j = -u_{ji} u^i v^j.
\]

Substituting (2.12) into (2.11), we find
\[
(2.15) \quad \lambda(1-\lambda^2)u_{th} - \lambda u_t u^t u_{th} = [\lambda u_t v^t + (1-\lambda^2) f^t_i] \mathcal{L}_v g_{th}.
\]

Transvecting this equation with $u^h$, we find
\[
\lambda(1-\lambda^2)u_{th} u^h = [\lambda u_t v^t + (1-\lambda^2) f^t_i] (\mathcal{L}_v g_{th}) u^h,
\]
or, using (2.13),
\[
\lambda u_{th} u^h = f^t_i (\mathcal{L}_v g_{th}) u^h.
\]

Thus (2.15) can be written as
\[
\lambda(1-\lambda^2)u_{th} + u_h f^t_i (\mathcal{L}_v g_{th}) u^h = [\lambda u_t v^t + (1-\lambda^2) f^t_i] \mathcal{L}_v g_{th},
\]
or
\[
(2.16) \quad \lambda(1-\lambda^2)u_{th} = -u_h f^t_i (\mathcal{L}_v g_{th}) u^h + [\lambda u_t v^t + (1-\lambda^2) f^t_i] \mathcal{L}_v g_{th}.
\]

§3. Equivalence of $\mathcal{L}_u g_{ji} = -2\alpha \lambda g_{ji}$ and $v_{ji} = 2\alpha f_{ji}$ in a manifold with quasi-normal $(f, g, u, v, \lambda)$-structure.

In this section, we assume that the $(f, g, u, v, \lambda)$-structure is quasi-normal, that is,
\[
(3.1) \quad S_{jth} - (f^t_j f_{tih} - f^t_i f_{jth}) = 0.
\]

We moreover assume that
\[
(3.2) \quad \mathcal{L}_u g_{ji} = -2\alpha \lambda g_{ji},
\]
where $\alpha$ is a function, that is, the vector field $u^h$ defines an infinitesimal conformal transformation with dilatation factor $-\alpha \lambda$.

Then we have, from (2.10),
QUASI-NORMAL \((f, g, u, v, \lambda)\)-STRUCTURES

\[\lambda(1-\lambda)\nu_{th} = -2\alpha\lambda \nu_{f_1} g_{th} \nu^4 - 2\alpha\lambda \nu_{f_1} \nu^4(1-\lambda^2)f_{1,1}g_{th},\]

or

\[(3.3) \quad \nu_{th} = 2\alpha f_{th}.\]

Conversely, suppose that (3.3) is satisfied, \(\alpha\) being a function. Then from (1.10) we obtain

\[0 = \mathcal{L}_{u} g_{th} - u_t w^t \mathcal{L}_{u} g_{th} + \lambda f_{i}^t \mathcal{L}_{u} g_{th} - \lambda^2 u_{th} - 2\alpha(\lambda f_{i}^t + \nu_{I} \nu_{I}), f_{th},\]

\[0 = \mathcal{L}_{u} g_{th} - u_t w^t \mathcal{L}_{u} g_{th} + \lambda f_{i}^t \mathcal{L}_{u} g_{th} - \lambda^2 u_{th} - 2\alpha \lambda(-g_{th} + u_{th} + v_{I} v_{I}) + 2\alpha \lambda v_{I} v_{I}, \]

that is,

\[(3.4) \quad \lambda f_{i}^t \mathcal{L}_{u} g_{th} = -\mathcal{L}_{u} g_{th} - 2\alpha \lambda g_{th} + \lambda^2 u_{th} + u_{I}(w^t \mathcal{L} u_{th} + 2\alpha \lambda u_{th}).\]

We have also, from (1.11),

\[0 = \mathcal{L}_{v} g_{th} - v_{I} v^t \mathcal{L}_{v} g_{th} - \lambda f_{i}^t \mathcal{L}_{v} g_{th} - 2\alpha \lambda^2 f_{th} + (\lambda f_{i}^t - \nu_{I} \nu_{I}) u_{th},\]

that is,

\[(3.5) \quad 2\lambda f_{i}^t v_{I} u_{th} = \mathcal{L}_{v} g_{th} - 2\alpha \lambda^2 f_{th} - u_{I} v_{I} u_{th} - v_{I} v^t \mathcal{L}_{v} g_{th}.\]

Writing (3.5) as

\[2\lambda f_{i}^t v_{I} u_{th} = \mathcal{L}_{v} g_{th} - 2\alpha \lambda^2 f_{th} - u_{I} v_{I} u_{th} - v_{I} v^t \mathcal{L}_{v} g_{th},\]

and transvecting this with \(\lambda f_{i}^t\), we find

\[2\lambda^2 (-\delta^t_t + u_{I} w^t + v_{I} v^t) \mathcal{L}_{v} u_{th} = \lambda^2 f_{i}^t \mathcal{L}_{v} g_{th} - 2\alpha \lambda^2(-g_{th} + u_{I} u_{th} + v_{I} v_{I}) - \lambda^2 v_{I} v^t u_{th} + \lambda^2 u_{I} v^t \mathcal{L}_{v} g_{th}.\]

Substituting (3.4) into this equation, we find

\[2\lambda^2 [-\mathcal{L} u_{I} u_{I} + u_{I} w^t (\mathcal{L} u_{I} u_{I}) + v_{I} v^t (\mathcal{L} u_{I} u_{I})]\]

\[= -\mathcal{L} u_{I} g_{th} - 2\alpha \lambda g_{th} + \lambda^2 (\mathcal{L} u_{I} u_{I} - \mathcal{L} u_{I} u_{I}) + u_{I}(w^t \mathcal{L} u_{I} g_{th} + 2\alpha \lambda u_{th})\]

\[+ 2\alpha \lambda^2 (g_{th} - u_{I} u_{th} - v_{I} v_{I}) - \lambda^2 v_{I} v^t u_{th} + \lambda^2 u_{I} v^t \mathcal{L} u_{I} g_{th},\]

\[(1-\lambda^2) \mathcal{L} u_{I} g_{th}\]

\[= -2\alpha \lambda (1-\lambda^2) g_{th}\]

\[+ u_{I} [-2\lambda^2 \mathcal{L} u_{I} (\mathcal{L} u_{I} u_{I}) + u_{I} \mathcal{L} u_{I} g_{th} + 2\alpha \lambda (1-\lambda^2) u_{th} + \lambda^2 v^t \mathcal{L} u_{I} g_{th}]\]

\[+ v_{I} [-2\lambda^2 v^t (\mathcal{L} u_{I} u_{I}) - 2\alpha \lambda^2 v_{I} - \lambda^2 v^t (\mathcal{L} u_{I} u_{I} - \mathcal{L} u_{I} u_{I})],\]

or
\( (1 - \lambda^3) u^g_{ih} \)

\[ = -2\alpha\lambda(1 - \lambda^3)u^g_{ih} \]

\[ + u_i[-2\lambda^3u^h(F_hu_i) + u^i\mathcal{L}_u g_{ih} + 2\alpha\lambda(1 - \lambda^3)u_h + \lambda^2u^r\mathcal{L}_u g_{rh}] \]

\[ - \lambda^2v_i((\mathcal{L}_u g_{ih})\nu^r + 2\alpha\lambda v_h), \]

or, using (2. 6) and (3. 3),

\[ (1 - \lambda^3) u^g_{ih} \]

\[ = -2\alpha\lambda(1 - \lambda^3)u^g_{ih} \]

\[ + u_i[-2\lambda^3u^h(F_hu_i) + u^i\mathcal{L}_u g_{ih} + 2\alpha\lambda(1 - \lambda^3)u_h + \lambda^2u^r\mathcal{L}_u g_{rh}] . \]

Transvecting (3. 6) with \( u^h \) and using (2. 13), we find

\[ (1 - \lambda^3)(\mathcal{L}_u g_{ih})u^h \]

\[ = -2\alpha\lambda(1 - \lambda^3)u_i + u_i[(1 - \lambda^3)(\mathcal{L}_u g_{ih})u^i\nu^4 + 2\alpha\lambda(1 - \lambda^3)^3], \]

from which,

\[ (1 - \lambda^3)(\mathcal{L}_u g_{ih})u^i\nu^4 \]

\[ = -2\alpha\lambda(1 - \lambda^3)^3 + (1 - \lambda^3)^3(\mathcal{L}_u g_{ih})u^i\nu^4 + 2\alpha\lambda(1 - \lambda^3)^3, \]

that is,

\[ (\mathcal{L}_u g_{ih})u^i\nu^4 = -2\alpha\lambda(1 - \lambda^3). \]

Thus, from (3. 7), we find

\[ (1 - \lambda^3)(\mathcal{L}_u g_{ih})u^h \]

\[ = -2\alpha\lambda(1 - \lambda^3)u_i + u_i[-2\lambda^3u^h(F_hu_i) + 2\alpha\lambda(1 - \lambda^3)^3 + 2\lambda\lambda^3(\mathcal{L}_u g_{rh})], \]

that is,

\[ (\mathcal{L}_u g_{ih})u^h = -2\alpha\lambda u_i. \]

Thus (3. 6) becomes

\[ (1 - \lambda^3) u^g_{ih} \]

\[ = -2\alpha\lambda(1 - \lambda^3)u^g_{ih} \]

\[ + u_i[-2\lambda^3u^h(F_hu_i) - 2\alpha\lambda u_h + 2\alpha\lambda(1 - \lambda^3)u_h + \lambda^2u^r(\mathcal{L}_u g_{rh})], \]

that is,
\[(1-\lambda^2)\mathcal{L}_u g_{th} = -2\alpha \lambda (1-\lambda^2) g_{th}\]

(3.10)

from which, taking the skew-symmetric part,

\[u_t[2u^t(F_h u_s) + 2\alpha \lambda u_h - v^t(\mathcal{L}_v g_{th})] - u_h[2u^t(F_t u_s) + 2\alpha \lambda u_t - v^t(\mathcal{L}_v g_{th})] = 0.\]

Transvecting this equation with \(u^t\), we find

\[(1-\lambda^2)[2u^t(F_h u_s) + 2\alpha \lambda u_h - v^t(\mathcal{L}_v g_{th})] - u_h[(\mathcal{L}_u g_{th})u^t u^t + 2\alpha \lambda (1-\lambda^2) - (\mathcal{L}_v g_{th})u^t v^t] = 0,\]

from which, using (2.13) and (3.10),

\[2u^t(F_h u_s) + 2\alpha \lambda u_h - v^t(\mathcal{L}_v g_{th}) = 0.\]

Thus (3.10) becomes

\[\mathcal{L}_u g_{th} = -2\alpha \lambda g_{th}.\]

Thus we have proved

**THEOREM 3.1.** In a manifold with quasi-normal \((f, g, u, v, \lambda)\)-structure such that the function \(\lambda(1-\lambda^2)\) is almost everywhere non-zero, the conditions

\[\mathcal{L}_u g_{ji} = -2\alpha \lambda g_{ji} \quad \text{and} \quad v_{ji} = 2\alpha f_{ji}\]

are equivalent, \(\alpha\) being a function.

Now we assume that \(\alpha\) is a non-zero constant.

Since \(v_{ji} = 2\alpha f_{ji}\) implies

\[f_{ji} = 0,\]

we have, as a corollary to this theorem,

**COROLLARY 3.2.** A quasi-normal \((f, g, u, v, \lambda)\)-structure such that the function \(\lambda(1-\lambda^2)\) is almost everywhere non-zero and

\[\mathcal{L}_u g_{ji} = -2\alpha \lambda g_{ji},\]

\(\alpha\) being a non-zero constant, is normal.

§4. **Normal \((f, g, u, v, \lambda)\)-structures satisfying \(\mathcal{L}_u g_{ji} = -2\epsilon \lambda g_{ji}\) or \(v_{ji} = 2\epsilon f_{ji}\).**

In this section, we put the assumption that the \((f, g, u, v, \lambda)\)-structure we
consider is quasi-normal and satisfies $\mathcal{L}_u g_{ji} = -2c\lambda g_{ji}$ or $v_{ji} = 2cf_{ji}$, that is,

A. The $(f, g, u, v, \lambda)$-structure under consideration is normal and satisfies

$$\mathcal{L}_u g_{ji} = -2c\lambda g_{ji} \quad \text{or} \quad v_{ji} = 2cf_{ji},$$

$c$ being a non-zero constant.

Under the assumption A, (1.5) and (1.6) become

\begin{equation}
(4.1) \quad u_{ji} - f^{j'}_i f^{k'}_i u_{ik} + (f^{j'}_i v_i - f^{k'}_i v_j) = \lambda[(f^{j'}_i) u_i - (f^{k'}_i) u_j] = 0
\end{equation}

and

\begin{equation}
(4.2) \quad 2c\lambda(u_{ji} - v_{ji}) + \lambda(f^{j'}_i u_i - f^{k'}_i u_j)
\end{equation}

respectively. (4.2) can also be written as

\begin{equation}
2c\lambda(u_{ji} - v_{ji}) + 2\lambda(f^{j'}_i v_i - f^{k'}_i v_j + 2c\lambda f_{ji})
\end{equation}

(4.3)

$$+ (f^{j'}_i u_i - f^{k'}_i u_j) v_\lambda - \lambda((f^{j'}_i) v_i - (f^{k'}_i) v_j) = 0,$$

since

$$u_{ji} = 2f_{ji} u_i - \mathcal{L}_u g_{ji} = 2f_{ji} u_i + 2c\lambda g_{ji}.$$

Also, under the assumption A, (1.10) becomes

\begin{equation}
(4.5) \quad f^{\iota}_i \mathcal{L}_v g_{\iota h} = \lambda u_{ih}.
\end{equation}

Now we transvect (4.2) with $u^i v^i$ and find

$$2c\lambda(1 - \lambda^2) = \lambda(1 - \lambda^2) u^i v^i - \lambda(1 - \lambda^2) u^i v^i = 0,$$

that is,

\begin{equation}
(4.6) \quad u^i v^i \lambda = c(1 - \lambda^2).
\end{equation}

Equation (4.5) can be written as

$$f^{\iota}_i (v_{\iota h} + v_{\iota h}) = \lambda (v_{\iota h} - v_{\iota h})$$

$$f^{\iota}_i (2v_{\iota h} + v_{\iota h}) = \lambda (v_{\iota h} - 2v_{\iota h})$$

$$f^{\iota}_i (v_{\iota h} + c f_{\iota h}) = -\lambda (c \lambda g_{ih} + v_{\iota h})$$

that is,

\begin{equation}
(4.7) \quad \lambda v_{\iota h} u_i + f^{\iota}_i v_{\iota h} = -c(1 + \lambda^2) + c(u_{ih} + v_{\iota h}).
\end{equation}

Transvecting (4.3) with $v^i$, we find
-2c\lambda(1-\lambda^2)u_1+2\lambda[\lambda\lambda^2]+f_1(F,\phi)_u+2c\lambda^2u_1 \\
+\lambda\lambda_1u_1=F_\lambda-\lambda[(\psi^\lambda\lambda^2)\psi_\psi-(1-\lambda^2)F_\lambda]=0,

or, using (4.6),

-\psi\lambda(1-\lambda^2)u_1+2\lambda[^2\lambda\lambda^2]F_\lambda+\lambda\phi_\phi \lambda_1=F_\lambda \psi_\psi-(1-\lambda^2)F_\lambda]=0,

or, using (4.7),

-\psi\lambda(1-\lambda^2)u_1+2\lambda[^2\lambda\lambda^2]F_\lambda+\lambda\phi_\phi \lambda_1=F_\lambda \psi_\psi-(1-\lambda^2)F_\lambda]=0,

that is,

(4.8) \psi_\lambda=c\lambda_1+\phi_\phi_1,

where we have put

(4.9) \psi_\lambda=(1-\lambda^2)_\phi.

We have

(4.10) \psi_\lambda=c\lambda_1+\phi_\phi_1.

We have

(4.11) \psi_\lambda=c\lambda_1+\phi_\phi_1.

We have also

(4.12) \psi_\lambda=(1-\lambda^2)_\phi.
\[
\begin{align*}
&= -2c\lambda v_i + u^i F_i v_i \\
&= -2c\lambda v_i + u^i (2c f_i u + F_i v_i),
\end{align*}
\]
from which,

\[(4.12) \quad v^i F_i u_i = u^i F_i v_i.\]

On the other hand, transvecting (4.7) with \(u^k\), we find

\[
\begin{align*}
\lambda u^i F_i u_i - f_i^k (F_i u_i) v^k &= -c(1 + \lambda^2) u_i + c(1 - \lambda^2) u_i, \\
\lambda u^i F_i u_i - f_i^k (F_i u_i) v^k &= -2c \lambda^2 u_i,
\end{align*}
\]
from which, substituting (4.10),

\[
f_i^k (F_i u_i) v^k = \lambda^2 (c u_i + \phi v_i).
\]

Transvecting this equation with \(f_j^i\), we find

\[
(-\partial_j + u_j u^i + v_j v^i)(F_i u_i) v^i = \lambda^2 (c v_j - \phi u_j),
\]
or

\[
-(F_j u_i) v^i + u_j (u^i F_i u_i) v^i - v_j (v^i F_i u_i) v^i = \lambda^2 (c v_j - \phi u_j),
\]
or, using (4.10) and (4.11),

\[(4.13) \quad (F_i u_i) v^i = \lambda (\phi u_i - c v_i).\]

From (4.13), we find

\[
(\mathcal{L}_u g_{ij} - F_i u_i) v^i = \lambda (\phi u_i - c v_i),
\]
that is,

\[(4.14) \quad v_i F_i u_i = -\lambda (\phi u_i + c v_i).\]

From (4.13) and (4.14), we have

\[(4.15) \quad v_i u_i = -2\lambda \phi u_i.\]

Now differentiating (4.8) covariantly, we find

\[
F_j F_i \lambda = c F_j u_i + (F_j \phi) v_i + \phi F_j v_i,
\]
from which,

\[(4.16) \quad c u_{ji} + (F_j \phi) v_i - (F_i \phi) v_j + 2c \phi f_{ji} = 0.\]

Transvecting this equation with \(v^j\) and using (4.15), we find

\[
0 = -2c \lambda \phi u_i + (v^j F_j \phi) v_i - (1 - \lambda^2)(F_i \phi) + 2c \lambda \phi u_i,
\]
or
(4.17) \( (1 - \lambda^2) \langle F, \phi \rangle = (v \cdot F) \phi \) 
which shows that \( F \phi \) is proportional to \( v \), and consequently, (4.16) becomes 

(4.18) \[ u_{ji} = -2\phi f_{ji}. \]

From (4.18) and 

\[ F_j u_i + F_i u_j = -2c \lambda g_{ji}, \]
we have 

(4.19) \[ F_j u_i = -c \lambda g_{ji} - \phi f_{ji}. \]

Substituting (4.19) into (4.7), we find 
\[ f_j v_i = \lambda \phi f_{ji} + c(-g_{ih} + u_{ih} + v_{ih}). \]

Transvecting this with \( f_j \) and using (4.11) and (4.14), we obtain 

(4.20) \[ V_j v_i = -\lambda \phi g_{ji} + c f_{ji}, \]
and consequently 

\[ \mathcal{L}_v g_{ji} = -2\lambda \phi g_{ji}. \]

Thus we have proved

**Theorem 4.1.** In a manifold with quasi-normal \((f, g, u, v, \lambda)\)-structure such that the function \( \lambda(1 - \lambda^2) \) is almost everywhere non-zero and 

(4.23) \[ \mathcal{L}_u g_{ji} = -2c \lambda g_{ji} \quad \text{or} \quad v_{ji} = 2c f_{ji}, \]
is satisfied, \( c \) being a non-zero constant, we have 

\[ u_{ji} = -2\phi f_{ji}, \]
and

\[ \mathcal{L}_v g_{ji} = -2\lambda \phi g_{ji}, \]
\( \phi \) being a function.

If the condition of Theorem 4.1 is satisfied, the structure is normal, so applying Theorem 7.1 of [2], we have

**Theorem 4.2.** Let \( M \) be a complete manifold with normal \((f, g, u, v, \lambda)\)-structure satisfying 

\[ \mathcal{L}_u g_{ji} = -2c \lambda g_{ji} \quad \text{or} \quad v_{ji} = 2c f_{ji}, \]
c being a non-zero constant. If \( \lambda(1 - \lambda^2) \) is almost everywhere non-zero function and \( n > 1 \), then \( M \) is isometric with an even-dimensional sphere.
BIBLIOGRAPHY


Tokyo Institute of Technology and Kyungpook University.