EQUIMEASURABILITY OF FUNCTIONS AND DOUBLY STOCHASTIC OPERATORS

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1. As a continuous version of doubly stochastic matrices, a linear operator $T$ from the real Lebesgue space $L^1(0,1)$ into itself is called doubly stochastic (d.s., in short) if

(1.1) $T1 = 1$,
(1.2) $T^*1 = 1$,
and
(1.3) $T \geq 0$,

where $1$ denotes the function whose range is $\{1\}$, and (1.3) means that $Tf \geq 0$ whenever $f \geq 0$. (1.2) is equivalent to the requirement that $\int fTf d\mu = \int ffd\mu$ for all $f \in L^1$, where $\mu$ denotes the Lebesgue measure on $(0,1)$. As is easily seen, every d.s. operator is a contraction in both $L^1$ and $L^\infty$ norms ($\|T\|_1 \leq 1$, and $\|T\|_\infty \leq 1$). Furthermore, $Tf \leq f$ holds for all $f \in L^1$, where $\leq$ denotes the continuous version of the preorder of Hardy—Littlewood and Póly [2, 8].

In the sequel, we denote by $M$ the set of all Lebesgue measurable sets in $I = (0,1)$. $e \equiv e'$, $e, e' \in M$, means that the measure of the symmetric difference of $e, e'$ is zero, or equivalently, that $\chi_e$, the characteristic function of $e$, is identified with $\chi_{e'}$ as an element of $L^1$. Let $e_1, e_2 \in M$ with $\mu(e_1) = \mu(e_2)$. A mapping $\sigma$ from $e_1$ (exactly speaking, defined a.e. on $e_1$) into $e_2$ is called a measure preserving transformation (m.p. transformation, in short) from $e_1$ into $e_2$, if

(1.4) $\sigma^{-1}(e) \in M$ and $\mu(\sigma^{-1}(e)) = \mu(e \cap e_2)$ for all $e \in M$.

If $\sigma^{-1}$ is a m.p. transformation from $e_2$ into $e_1$ again, $\sigma$ is called invertible measure preserving from $e_1$ onto $e_2$. For each m.p. transformation $\sigma$ from $I$ into itself, the operator $T_\sigma$ defined by

(1.5) $T_\sigma f(t) = f(\sigma t)$ \hspace{1cm} ($t \in I$)

is a d.s. operator, and is called a d.s. operator induced by $\sigma$. In what follows, $\mathcal{D}$ stands for the set of all d.s. operators and $\Sigma(\mathcal{D})$ for the set of all m.p. (resp. invertible m.p.) transformations on $I$. Then $\mathcal{D}$ is a convex set and each $T_\sigma, \sigma \in \Sigma$ is, as is easily verified, multiplicative, that is, $T_\sigma(f \cdot g) = T_\sigma f \cdot T_\sigma g$ for all $f, g \in L^\infty$, and is

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1) Two such transformations will be identified if they differ on a set of measure zero.
on extreme point of $\mathcal{O}$ [7]. Also $T_* f \sim f$ holds, where $f \sim g$ means that $f$ and $g$ are equimeasurable.\(^2\) Since every $T \in \mathcal{O}$ acts as a contraction on $L^\alpha$, we can consider $\mathcal{O}$ as a subset of the operator space of $L^\alpha$. It is known \([8]\) that, according to a general compactness theorem of Kadison \([3]\), $\mathcal{O}$ is compact in the weak*-operator topology.

Let $\xi$ and $\eta$ be $n$-vectors $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ respectively. It is clear that

(1.6) if $\eta$ is a $n$-vector whose coordinates $y_i$ are obtained by a permutation of the coordinates of $\xi$, then there exists a $n$-square permutation matrix $P$ such that $\xi = \eta P$.

A continuous version of this statement would be the following:

(1.7) if $f \sim g$, $f, g \in L^1$, there exists an $\sigma \in \Sigma$ such that $T_\sigma f = g$.

Unfortunately, however, the statement (1.7) is not valid in general. It is only known \([1, 8]\) that if $f \sim g$, $f, g \in L^1$, there exists an $T \in \mathcal{O}$ such that $Tf = g$. More precisely, Ryff \([8]\) has shown that such a $T$ can be chosen from d.s. operators of the form $T_\sigma T_\sigma'$, $\sigma_1, \sigma_2 \in \Sigma$.

In \S 2, we shall present an alternative proof of this Ryff's theorem in a somewhat different form. Namely we shall show that if $f \sim g$, $f, g \in L^1$ there exists an $T \in \mathcal{O}$ such that $Tf = g$ which is a $w^\alpha$-cluster point of a sequence of members of $T_\sigma$, $\sigma \in \Sigma$.

In \S 3, some fundamental properties of d.s. operators will be studied. In \([6]\) Mirsky called a d.s. operator $T$ a permulator if $f \sim Tf$ holds for all $f \in L^1$. We shall show that each permulator $T$ is nothing but a d.s. operator induced by a m.p. transformation $\sigma$, i.e., $T = T_\sigma$ (Theorem 5). Also some characterizations for the d.s. operators induced by m.p. transformations will be given.

Finally, in \S 4, we shall give a necessary and sufficient condition for $f \sim g$, $f, g \in L^1$, under which we can find an $\sigma \in \Sigma$ such that $T_\sigma f = g$ holds.

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2. We shall give an alternative proof of the Ryff's theorem:

**Theorem 1.** If $f$ and $g$ are equimeasurable on $I = (0, 1)$, then $Tf = g$ holds for a d.s. operator $T$ which is a $w^\alpha$-cluster point of a sequence of members of $T_\sigma$, $\sigma \in \Sigma$.

To prove this theorem we use a lemma due to Lorentz \([4, p. 60]\).

**Lemma 1 (Lorentz).** Let $f$ and $g$ be equimeasurable. If $C$ is any set of real numbers for which $f^{-1}(C)$ is measurable, then so is $g^{-1}(C)$ and both sets have the same measure.

The following lemma is known. For the convenience of readers, we present here a proof based on the preceding lemma.

\(^2\) $f$ and $g$ are called equimeasurable if $d_f$, the distribution function of $f$, is equal to $d_g$.  

**Lemma 2.** If \( \mu(e_1) = \mu(e_2) \), \( e_1, e_2 \in \mathcal{M} \), then there exists an \( \sigma \in \Sigma_0 \) such that \( \sigma(e_1) \equiv e_2 \).

**Proof.** Let \( k_i(t) = \int I_{x_i} d\mu \), \( 0 < t < 1 \), \( i = 1, 2 \). The functions \( k_i \), \( i = 1, 2 \), are positive, continuous, and non-decreasing on \( I \). Also denote by \( f \) the function \( k_i \), \( i = 1, 2 \), are positive, continuous, and non-decreasing on \( I \). Then it is easy to see that \( f \) is measurable, and \( k_i^{-1}(\lambda) \) is a single point or a closed interval in \( I \) for any \( \lambda \in (0, \alpha) \), \( \alpha = \mu(e_1) = \mu(e_2) \). We put \( J_i \) the set of all \( \lambda \in (0, \alpha) \) such that \( k_i^{-1}(\lambda) \) is not a set of a single point. Then \( J_i \) is a countable set for each \( i \). Putting \( e_i = f^{-1} \cdot (0, \alpha) - J_i \cup J_2 \), \( i = 1, 2 \), we see that \( e_i \subset e_1 \) and \( e_1 \equiv e_2 \). If we define a mapping \( \sigma_i \) from \( e_1 \) onto \( e_2 \) by

\[
\sigma_i(s) = f^{-1}(f_i(s)), \quad s \in e_1, \tag{2.1}
\]

\( \sigma_i \) is a one to one mapping from \( e_1 \) onto \( e_2 \). Furthermore, \( \sigma_i \) is a m.p. transformation from \( e_1 \) onto \( e_2 \). Consequently, putting \( \sigma(s) = \sigma_i(s) \) if \( s \in e_i \), we see that \( \sigma \) is an invertible m.p. transformation on \( I \) for which \( \sigma(e_i) \equiv e_2 \).

From the proof above, it follows that if \( \{e_i \}_{i=1}^{n} \) and \( \{e_i' \}_{i=1}^{n} \) are two systems of mutually disjoint sets of \( \mathcal{M} \) with \( \mu(e_i) = \mu(e_i') \) for all \( 1 \leq i \leq n \), there exists an \( \sigma \in \Sigma_0 \) such that \( \sigma(e_i) \equiv e_i' \) for all \( 1 \leq i \leq n \). Now let \( S \) denote the set of all simple functions on \( I \). Then we have immediately

**Lemma 3.** If \( f \sim g \), \( f, g \in S \), then there exists an \( \sigma \in \Sigma_0 \) for which \( T_\sigma f = g \) holds.

**Proof of Theorem 1.** First we prove in the case that \( 0 \leq f, g \in L^1 \), and \( f \sim g \). For every \( n \in \mathbb{N} \) (\( \mathbb{N} \) stands for the set of all integers) let \( F_n = f^{-1}[n, \infty) \), \( G_{n, 0} = g^{-1}[n, \infty) \), \( F_{n, k} = f^{-1}[2^{-n}(k-1), 2^{-n}k) \), and \( G_{n, k} = g^{-1}[2^{-n}(k-1), 2^{-n}k) \), where \( k = 1, \ldots, 2^n \). Since \( f \sim g \) and both \( \{F_{n, k}\}_{k=0}^{2^n} \) and \( \{G_{n, k}\}_{k=0}^{2^n} \) are systems of mutually disjoint sets, Lemma 3 shows that for every \( n \in \mathbb{N} \) there exists an \( \sigma_n \in \Sigma_0 \) such that \( T_\sigma F_{n, k} = \lambda_{G_{n, k}} \) for all \( k = 0, \ldots, 2^n \). If we put

\[
f_n = \sum_{k=1}^{2^n} 2^{-n}(k-1) \chi_{F_{n, k}} + n \chi_{F_{n, 0}}, \quad g_n = \sum_{k=1}^{2^n} 2^{-n}(k-1) \chi_{G_{n, k}} + n \chi_{G_{n, 0}},
\]

\( T_\sigma f_n = g_n \), \( n \in \mathbb{N} \) holds. Moreover, since each \( F_{m, k}(G_{m, k}) \), \( 0 \leq k \leq 2^m \) is contained in an \( F_{n, k} \) (resp. \( G_{n, k} \)) if \( n \leq m \), we have

\[
T_{nm} f_n = g_n, \quad n \leq m. \tag{2.2}
\]

We write \( \mathcal{D}_i = \{T_{i, 0}, T_{i, 1}, \ldots\}^{-w} \), the closure of \( \{T_{i, 0}, T_{i, 1}, \ldots\} \) in the \( w^* \)-operator topology, for each \( i \). Since \( \mathcal{D} \), considered as a subset of the operator space of \( L^\infty \), is \( w^* \)-compact, there exists an \( T \in \mathcal{D} \) such that \( T \in \cap_{n=1}^{\infty} \mathcal{D}_i \). For each fixed \( m \in \mathbb{N} \), there is a subnet \( \{T_i\} \subset \{T_{nm}, T_{nm+1}, \ldots\} \) such that \( T = w^*\text{-lim}_n T_i \). Since \( T_{nm} f_m = g_m \) holds for every \( T_m \), by (2.2) and \( f_m \in L^\infty \), we have
for every \( \mu \in L^1 \). Hence \( T_{\mu} = g_{\mu} \) holds for every \( m \in N \). Finally, for every \( m \),
\[
||g - T_f||_1 \leq ||g - g_{\mu}||_1 + ||g_{\mu} - T_{f_m}||_1 + ||T_{f_m} - T_f||_1 \leq ||g - g_{\mu}||_1 + ||f_{m} - f||_1,
\]
which implies \( g = T_f \).

For a proof in the general case we have only to recall that if \( f \sim g \in L^1 \) we have \( f^+ \sim g^+ \), \( f^- \sim g^- \), and if we construct \( f^+_n \), \( g_n \), \( f^-_n \), \( g_n \in S \) in a similar way as above, we have \( f^+_n - f^-_n \leq g_n \in S \) and \( f^+_n - f^-_n \rightarrow f \), \( g_n \rightarrow g \) in \( L^1 \) norm.

3. In the sequel, we denote by \( R \) the set of all real numbers. For each \( f \in L^1 \) and each \( \lambda \in R \), we denote by \( e(f; \lambda) \) the \( \lambda \)-spectral set, that is, the set \( \{ t : f(t) \geq \lambda \} \subset \mathbb{R} \) and we denote by \( M_f \) the \( \sigma \)-algebra generated by these sets. \( f^{(\infty)} \) is the \( \alpha \)-truncation of \( f \):

\[
(3.1) \quad f^{(\infty)}(t) = \alpha(t) \quad \text{if} \quad f(t) > \alpha, \quad f^{(\infty)}(t) = f(t) \quad \text{if} \quad f(t) \leq \alpha.
\]

Each function \( f \in L^1 \) will be called smooth if \( \mu\{ t : f(t) = \lambda \} = 0 \) for all \( \lambda \in R \).

**Lemma 4.** Let \( Tf = g \), \( T \in \mathcal{B} \), and \( f, g \in L^1 \). Then the following statements are equivalent.

1. \( f \sim g \);
2. \( T(f^{(\infty)}) = g^{(\infty)} \) for all \( \alpha \in R \);
3. \( T \chi e^{\alpha}\lambda_n = \chi e^{\alpha} \lambda_n \) for all \( \lambda \in R \).

**Proof.** (1) implies (2): Since \( g = Tf \equiv T(f^{(\infty)}) \) and \( \alpha \equiv T(f^{(\infty)}) \), we have \( g^{(\infty)} \equiv T(f^{(\infty)}) \). Moreover (1) implies

\[
\int_0^1 g^{(\infty)}(t) \, d\mu = \int_0^1 f^{(\infty)}(t) \, d\mu = \int_0^1 T(f^{(\infty)})(t) \, d\mu.
\]

Hence we obtain (2).

(2) implies (3): For each \( f \in L^1 \) and each \( \lambda \in R \), let denote by \( \tilde{e}(f; \lambda) \) the set \( \{ t : f(t) \geq \lambda \} \). Then we have

\[
\mu\{ t : \eta \leq f(t) < \xi \} = \mu\{ t : f^{\xi}(t) - f^{\eta}(t) = \xi - \eta \} \chi e^{\xi}(\xi, \eta)
\]

for each \( f \in L^1 \) and each pair \( \xi, \eta \in R \) with \( \eta < \xi \). Hence we have

\[
(3.2) \quad \chi e^{\xi}(\xi, \eta) = \lim_{\eta \rightarrow \xi} \frac{f^{\eta}(t) - f^{\xi}(t)}{\xi - \eta} \quad \text{(in} \quad L^1 \text{norm)}
\]

for each \( f \in L^1 \) and each \( \xi \in R \). Therefore we get

\[
(3.3) \quad T \chi e^{\xi}(\xi, \eta) = \chi e^{\xi}(\xi, \eta), \quad \xi \in R,
\]
on account of (2). Then we can easily obtain (3) by the equality \( \tilde{e}(f; \lambda) \)
Finally, the implication $(3) \Rightarrow (1)$ is clear.

**Theorem 2.** Let $Tf=g$, $T \in \mathcal{D}$ and $f, g \in L^1$. Then $f \sim g$ if and only if $T^*g=f$.

**Proof.** Since $Tf=g$, $T \in \mathcal{D}$ implies $g \ll f$, it is easy to see that $Tf=g$ and $T^*g=f$ imply $f \sim g$. On the other hand, if $Tf=g$ and $f \sim g$, applying the statement $(3)$ in Lemma 4, we have

$$\int_0^1 T^*\chi_{e(f; \lambda)} d\mu = \int_0^1 \chi_{e(g; \lambda)} d\mu = \int_0^1 \chi_{e(g; \lambda)} T^*\chi_{e(f; \lambda)} d\mu = \int_0^1 T^*\chi_{e(g; \lambda)} \chi_{e(f; \lambda)} d\mu.$$

We also have

$$T^*\chi_{e(g; \lambda)} \geq T^*\chi_{e(g; \lambda)} \chi_{e(f; \lambda)}.$$

Therefore

$$(3.4) \quad T^*\chi_{e(g; \lambda)} = T^*\chi_{e(g; \lambda)} \chi_{e(f; \lambda)}$$

holds. $(3.4)$ means $T^*\chi_{e(g; \lambda)} \leq \chi_{e(f; \lambda)}$. Hence we obtain

$$T^*\chi_{e(g; \lambda)} = \chi_{e(f; \lambda)}.$$

From this we can show easily that $T^*g=f$ holds.

Ryff [8] proved the following:

**Theorem 3 (Ryff).** To each $f \in L^1$ there corresponds a $\sigma \in \Sigma$ such that $T_\sigma f^* = f$.

Now we prove the following theorem, which plays an essential role in the rest of the present paper.

**Theorem 4.** For every smooth function $f \in L^1$, there corresponds one and only one d.s. operator $T$ such that $Tf^* = f$. This operator $T$ is induced by some $\sigma \in \Sigma$. Moreover, $f^* = Sf$, $S \in \mathcal{D}$ implies $S = T^*$.

**Proof.** By virtue of Lemma 4, if $Tf^* = f$, and $T \in \mathcal{D}$, then we have $T\chi_{e(f; \lambda)} = \chi_{e(f; \lambda)}$, $\lambda \in \mathbb{R}$. And our assumption that $f$ be smooth implies $\mathcal{M}_{f^*} = \mathcal{M}$. Thus $T$ coincides with $T_\sigma$, where $\sigma \in \Sigma$ is obtained by Theorem 3.

Next, suppose $f^* = Sf$. Then we have $Sf^* = f$ by Theorem 2, we must have $S^* = T$, that is, $S = T^*$.

In Theorem 5 below, we shall give some simple characterizations of d.s. operators induced by m.p. transformations. Also, some of the statements are

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3) $T \in \mathcal{D}$ implies $T^* \in \mathcal{D}$, where $T^*$ is a unique extension of the adjoint of $T$ to an operator acting on $L^1$.

4) $f^*$ is the decreasing rearrangement of $f$. 

nearly clear if we use the result due to v. Neumann [6, p. 582, Satz 1]. For completeness and because the special case is much simpler than the general case, we intend to prove our Theorem 5 by mere use of the preceding arguments.\footnote{Also, Satz 2 of v. Neumann [7, p. 584] is easily proved by use of Lemma 1, in (0, 1) case.}

**Theorem 5.** Let $T$ be an d.s. operator. Then the following statements are equivalent.

1. $T$ is a permutator, that is, $f \sim Tf$ for all $f \in L^1$;
2. $T$ is truncation invariant, that is, $Tf^{(\alpha)} = (Tf)^{(\alpha)}$ for all $\alpha \in \mathbb{R}$ and all $f \in L^1$;
3. $T$ is multiplicative, that is, $T(f \cdot g) = Tf \cdot Tg$ for all $f, g \in L^\infty$;
4. $T$ is an isometry in $L^1$;
5. $T^*T = I$;
6. $T$ is induced by a $\sigma \in \Sigma$.

In particular, a d.s. operator $T$ is induced by a $\sigma \in \Sigma$ if and only if $TT^* = T^*T = I$.

**Proof.** First, the equivalence $(1) \iff (2)$ follows from Lemma 4.

Next, we have the implication $(1) \Rightarrow (6) \Rightarrow (3) \Rightarrow (5) \Rightarrow (1)$ as follows: Let $T$ be a permutator. Then, in particular, for smooth $f \in L^1$ we have $f \sim Tf$. Hence follows (6) from Theorem 4. The implication $(6) \Rightarrow (3)$ is obvious. If $T$ is multiplicative, $Tf = (Tf) \in L^1$. So, $Tf \sim Tf$ holds for each $E \in \mathcal{M}$. Therefore by virtue of Theorem 2, $T^*Tf = T^*f$ for all $E \in \mathcal{M}$, that is, $T^*T = I$. Finally, let $T^*T = I$. If there exists a function $f \in L^1$ such that $f \sim Tf$ does not hold, then we can find a numbers $s \in I$ for which

\[ \int_0^s (Tf)^* d\mu < \int_0^s f^* d\mu \quad (3.5) \]

holds on account of $Tf \prec f$. Thus we must have

\[ \int_0^s f^* d\mu = \int_0^s (T^*Tf)^* d\mu \leq \int_0^s (T^*Tf)^* d\mu < \int_0^s f^* d\mu \]

by (3.5), which is a contradiction.

The proof of the implication $(6) \Rightarrow (4) \Rightarrow (5)$ is also given as follows: The implication $(6) \Rightarrow (4)$ is obvious. To prove $(4) \Rightarrow (5)$, we recall an elementary formula that

\[ |a + b| + |a - b| = 2(|a| + |b|) \quad a, b \in \mathbb{R} \text{ if and only if } a \cdot b = 0. \quad (3.6) \]

Now let $T$ be an isometry. Then, for each $E \in \mathcal{M}$,

\[ \|TX_E + TX_Ec\|_1 + \|TX_E - TX_Ec\|_1 = 2(\|TX_E\|_1 + \|TX_Ec\|_1) \quad (3.7) \]
follows from
\[ ||X_E + X_Ec||_1 + ||X_E - X_Ec||_1 = 2(||X_E||_1 + ||X_Ec||_1).\]
Therefore we have \( TX_E + TX_Ec = 0 \) by (3.6) and (3.7). Moreover, \( T \in \mathcal{D} \) implies \( TX_E + TX_Ec = 1 \). It follows that \( X_E \sim T X_E \) for all \( E \in \mathcal{M} \). Finally by the same argument used for the proof of \( (3) \Rightarrow (5) \), we obtain the implication \( (4) \Rightarrow (5) \).

4. Two pairs of functions \((f, f')\) and \((g, g')\) on \( I \) are called simultaneously equimeasurable, if for each pair of \( \alpha, \beta \in R \), we have
\[ \mu(e(f; \alpha) \cap e(f'; \beta)) = \mu(e(g; \alpha) \cap e(g'; \beta)) \] 
We write
\[ (f, f') \sim (g, g') \]
if \((f, f')\) and \((g, g')\) are simultaneously equimeasurable.

Now we shall call an \( f \) to be strongly equimeasurable with \( g \), and write \( f \sim g \), if for each \( f' \in L^1 \), there corresponds some \( g' \) which satisfies \((f, f') \sim (g, g')\). It is clear that \( f \sim g \) implies \( f \sim g \).

**Theorem 6.** \( f \) is strongly equimeasurable with \( g \) if and only if \( T_x f = g \) holds for some m.p. transformation \( \sigma \).

**Proof.** If \( f \) is strongly equimeasurable with \( g \), by the definition, there is a function \( u \in L^1 \) which satisfies both \( x \sim u \) and
\[ (f, x) \sim (g, u). \]
Then, by Theorem 4, there is a unique \( \sigma \in \Sigma \) so that equality \( T_x x = u \), and for every \( \alpha \in \mathbb{R} \),
\[ (4.3) \int_{[\beta, \beta']}(x_{e(f, \alpha)} d\mu = \int_{[\beta, \beta']} x_{e(g, \alpha)} d\mu \quad (\beta, \beta' \in I) \]
holds. It is easy to see that (4.3) implies
\[ (4.4) \int_{E} x_{e(f, \alpha)} d\mu = \int_{E} x_{e(g, \alpha)} d\mu = \int_{E} x_{e(f, \alpha)} T_{x_{e(f, \alpha)} d\mu,} \]
for each \( E \in \mathcal{M} \).

Substituting \( E = e(f; \alpha) \) in (4.4), we have, on account of \( f \sim g \),
\[ (4.5) \int_{0}^{1} x_{e(g, \alpha)} d\mu = \int_{0}^{1} x_{e(f, \alpha)} d\mu = \int_{0}^{1} x_{e(g, \alpha)} T_{x_{e(f, \alpha)} d\mu.} \]
(4.5) means \( x_{e(g, \alpha)} = T_{x_{e(f, \alpha)}} \). Thus we have \( x_{e(g, \alpha)} = T_{x_{e(f, \alpha)}} \), that is, \( T_x f = g \), since

6) \( x \) denote the function \( x(t) = t \).
\( \alpha \in R \) is arbitrary.

The converse implication is clear; we have only to set \( g' = T_{x} f' \) for each \( f' \in L^1 \).

**THEOREM 7.** For each \( f \in L^1 \), the following conditions are equivalent.

1. \( M_f = M \);
2. for each \( g \) with \( f \sim g \), there corresponds a unique \( T \in \mathcal{G} \) such that \( T f = g \);
3. for each \( g \) with \( f \sim g \), there corresponds a unique \( u \in L^1 \) such that \( \langle f ; x \rangle \sim \langle g ; u \rangle \).

**Proof.** The implication (1) \( \Rightarrow \) (2) is an immediate consequence of the statement (1) in Lemma 4.

(2) implies (1): In general, if \( f \sim g \in L^1 \) is not smooth, we can easily construct two d.s. operators \( T_i, i = 1, 2 \) such that \( T_{x} f = g \), \( i = 1, 2 \), since the Lebesgue measure on \( I \) is non-atomic. Thus, under the condition (2), \( f \) is smooth.

Then, by the condition \( f \sim g \), and by use of Theorem 6, we have

\[(4.6) \quad T_{x} f = g, \quad \text{for a unique } \sigma \in \Sigma.\]

On the other hand, we have

\[(4.7) \quad T_{x} f = f, \quad \text{for a unique } \sigma \in \Sigma, \text{ by Theorem 4.}\]

Then (4.6) and (4.7) imply

\[(4.8) \quad T_{x_{1}} T_{x_{2}} f = g.\]

Therefore we must have

\[(4.9) \quad T_{x_{1}} = T_{x_{1}} T_{x_{2}} T_{x_{2}}^{*}.\]

And (4.9) holds if and only if \( T_{x_{2}} T_{x_{2}}^{*} = I \), by (5) of Theorem 5; this is equivalent to \( \sigma \in \Sigma_{0} \), by Theorem 5 again.

If \( \sigma \in \Sigma_{0} \), then there exists, for each \( E \in \mathcal{M} \), an \( F \in \mathcal{M} \) such that \( \chi_{E} = T_{x_{2}} \chi_{F} \). Since \( F \) is smooth, \( F \) must belong to \( \mathcal{M}_{f} \). Consequently, (4.7) implies \( M_{f} = \mathcal{M} \).

Finally, the implication (3) \( \Leftrightarrow \) (2) is implicit in the proof of Theorem 6.

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