ON THE GROWTH RATE OF COMPOSITIONS OF ENTIRE FUNCTIONS

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1. Let \( f(z) \) be an entire function, \( M(r, f) \) its maximum modulus on \(|z| = r\) and \( T(r, f) \) its Nevanlinna characteristic function. Recently Gross and Yang [4] proved the following:

Suppose that \( f(z), g(z) \) are entire functions such that

\[
T(ar, g) = o(T(r, f)) \quad \text{as} \quad r \to \infty
\]

for some constant \( a > 1 \). Then for any non-constant entire function \( h(z) \),

\[
T(r, h \circ g) = o(T(r, h \circ f)) \quad \text{as} \quad r \to \infty
\]

In this paper we shall consider the asymptotic behavior of the ratio

\[
\log \frac{M(r, ar \circ g)}{M(r, h \circ f)}
\]

replacing \( T(r, \cdot) \) by \( \log M(r, \cdot) \) in the above condition (1.1).

Our results are the following:

**Theorem 1.** Let \( g(z) \) and \( f(z) \) be entire functions such that

\[
\lim_{r \to \infty} \frac{\log M(ar, g)}{\log M(r, f)} = 0
\]

for some constant \( a > 1 \). Then for any non-constant entire function \( h(z) \),

\[
\lim_{r \to \infty} \frac{\log M(r, h \circ g)}{\log M(r, h \circ f)} = 0.
\]

**Theorem 2.** Let \( g(z) \) and \( f(z) \) be entire functions such that

\[
\lim_{r \to \infty} \frac{\log M(r, g)}{\log M(r, f)} = 0.
\]

Then for any non-constant entire function \( h(z) \),

\[
\lim_{r \to \infty} \frac{\log M(r, h \circ g)}{\log M(r, h \circ f)} = 0.
\]

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Theorem 3. Let $g(z)$ and $f(z)$ be entire functions satisfying (1.3). Suppose that $f(z)$ is of finite order. Then for any non-constant entire function $h(z)$,

$$\lim_{r \to \infty} \frac{\log M(r, h \circ f)}{\log M(r, h \circ g)} = 0.$$ 

The next theorem deals with the possibility still left open in Theorem 2.

Theorem 4. There exist entire function $g(z)$ and $f(z)$ such that

$$\lim_{r \to \infty} \frac{\log M(r, g)}{\log M(r, f)} = 0 \quad \text{and} \quad \lim_{r \to \infty} \frac{\log M(r, \exp \circ g)}{\log M(r, \exp \circ f)} = \infty.$$ 

2. Lemmas. We start from the following lemmas which will be used in the proof of our Theorems.

Lemma 1 ([3, 5, 6, 7]). Let $h(z)$ and $f(z)$ be entire with $f(0) = 0$. Let $\rho$ satisfy $0 < \rho < 1$ and let $c(\rho) = (1 - \rho)^2/4\rho$. Then for $r \geq 0$,

$$M(r, h \circ f) \leq M(c(\rho)M(\rho, f), h).$$

Lemma 2 ([1, 3]). Let $h(z)$ and $f(z)$ be entire. Then

$$M(r, h \circ f) \geq M(1 + o(1))M(r, f), h) \quad \text{as} \quad r \to \infty,$$

outside a set of $r$ of finite logarithmic measure which depends, as does $o(1)$, on $f(z)$.

Lemma 3. For any transcendental entire function $f(z)$, there exists an entire function $g(z)$ such that

$$\lim_{r \to \infty} \frac{\log M(r, g)}{\log M(r, f)} = 0 \quad \text{and} \quad \lim_{r \to \infty} \frac{\log M(r, \exp \circ g)}{\log M(r, \exp \circ f)} = \infty.$$ 

Proof. Let $f(z)$ be a transcendental entire function. Then Hadamard's three-circle theorem asserts that $\log M(r, f)$ is a convex, increasing function of $\log r$. Hence, by the well-known property of logarithmically convex function,

$$\log M(r, f) = \log M(r_0, f) + \int_{r_0}^{r} \frac{\phi(t)}{t} \, dt \quad (r \geq r_0),$$

where $r_0 > 0$ and $\phi(t)$ is a non-negative, non-decreasing function of $t$. Since $f(z)$ is transcendental, we have

$$\lim_{r \to \infty} \frac{1}{\log r} \int_{r_0}^{r} \frac{\phi(t)}{t} \, dt = \infty \quad \text{and} \quad \lim_{t \to \infty} \phi(t) = \infty.$$

We put

$$\phi(r) = \sup_{r_0 < r < \infty} \frac{\phi(t)}{\log \log t} \quad (r_0 > e) \quad \text{when} \quad \lim_{t \to \infty} \frac{\phi(t)}{\log t} = \infty.$$
and

\[ \phi(r) = \sup_{r_0 \leq t \leq r} \frac{\phi(t)}{\log \phi(t)} \quad (\psi(r_0) > 1) \quad \text{when} \quad \lim_{r \to \infty} \frac{\phi(t)}{\log t} < \infty \]

Then it follows from (2.3), (2.4) and (2.5) that \( \phi(r) \) is non-decreasing and

\[ \lim_{r \to \infty} \phi(r) = \infty \quad \text{and} \quad \lim_{r \to \infty} \frac{\phi(r)}{\phi(r)} = 0. \]

Put

\[ \Phi(r) = \int_{r_0}^{r} \frac{\phi(t)}{t} \, dt \quad (r \geq r_0). \]

Then we obtain

\[ \lim_{r \to \infty} \frac{\phi(r)}{\log M(r, f)} = 0 \]

and

\[ \lim_{r \to \infty} \frac{\phi(r)^2}{\log M(r, f)} = \infty. \]

In fact, it follows from (2.3) and (2.6) that

\[ \lim_{r \to \infty} \frac{\phi(r)}{\log M(r, f)} = \lim_{r \to \infty} \left( \int_{r_0}^{r} \frac{\phi(t)}{t} \, dt \right) = \lim_{r \to \infty} \frac{\phi(r)}{\phi(r)} = 0, \]

which implies (2.7). Next taking (2.4) and (2.5) into account, we get

\[ \Phi(r)^2 \leq \frac{1}{K} \log \log r \cdot \int_{r_0}^{r} \frac{\phi(t)}{t} \, dt \cdot \int_{r_0}^{r} \frac{\phi(t)}{t} \, dt \]

with a suitable constant \( K \). Hence (2.8) follows from (2.2), (2.3) and the above inequality.

Now, the definition of \( \Phi(r) \) and (2.6) yield that \( \Phi(r) \) is increasing and convex in \( \log r \) and \( \Phi(r) = O(\log r) \quad (r \to \infty) \). Hence Clunie's theorem [2] asserts that there exists an entire function \( g(z) \) such that

\[ M(r, g) = \max_{|z|=r} \text{Re } g(z) \]

and

\[ \log M(r, g) \sim \Phi(r) \quad (r \to \infty). \]

and consequently

\[ \log M(r, \exp g) = \exp(\log M(r, g)) \]

\[ \geq \frac{1}{2} (\log M(r, g))^2 \sim \frac{1}{2} \Phi(r)^2 \quad (r \to \infty). \]
Therefore (2.1) follows from (2.7), (2.8), (2.9) and (2.10).

3. Proof of Theorem 1. Choose \( \rho > 0 \) such that \( \alpha > 1/\rho > 1 \) and assume, for convenience, that \( f(0) = 0 \). The case \( f(0) \neq 0 \) can be dealt with as in the proof of Theorem 1 in [3]. Then

\[
\lim_{r \to \infty} \frac{\log M(r, g)}{\log M(r, f)} = \lim_{r \to \infty} \frac{\log M(r, g)}{\log M(r, f)} = \lim_{r \to \infty} \frac{\log M(r, g)}{\log M(r, f)} + \log c(\rho)
\]

and so from the condition (1.2)

\[
\lim_{r \to \infty} \frac{\log M(r, g)}{\log c(\rho)M(r, f)} = 0.
\]

Hence there is \( r_0 > 0 \) such that for all \( r > r_0 \)

\[
(3.2)
\]

\[
M(r, g) < c(\rho)M(r, f).
\]

\( \log M(r, h) \) is an increasing convex function of \( \log r \), so that \( \log M(r, h)/\log r \) is finally increasing and hence Lemma 1 and (3.2) yield

\[
\frac{\log M(r, h_0g)}{\log M(r, h_0f)} \leq \frac{\log M(M(r, g), h_0)}{\log M(c(\rho)M(r, f), h_0)} \leq \frac{\log M(r, g)}{\log c(\rho)M(r, f)}
\]

for all large \( r \). Therefore Theorem 1 follows from this inequality and (3.1).

4. Proof of Theorem 2. The condition (1.3) implies

\[
\lim_{r \to \infty} \frac{\log M(r, g)}{\log M(r, f)/2} = 0
\]

and so there exists \( r_0 > 0 \) such that for all \( r > r_0 \)

\[
(4.2)
\]

\[
M(r, g) < \frac{M(r, f)}{2}.
\]

It follows from Lemma 2 that there is a set \( E \) of finite logarithmic measure such that

\[
\log M(r, h_0f) \geq \log M((1+o(1))M(r, f), h_0) \quad r \to \infty; \quad r \notin E.
\]

Hence using (4.1) and (4.2) and noting that \( \log M(r, h)/\log r \) is increasing we obtain

\[
\frac{\log M(r, h_0g)}{\log M(r, h_0f)} \leq \frac{\log M(r, g), h_0}{\log M((1+o(1))M(r, f), h_0)} \leq \frac{\log M(r, g)}{\log (1+o(1))M(r, f)}
\]

\[
\rightarrow 0 \quad \text{as} \quad r \to \infty; \quad r \notin E
\]
and consequently
\[
\lim_{r \to \infty} \frac{\log M(r, h \circ g)}{\log M(r, h \circ f)} = 0,
\]
which is the desired result.

5. **Proof of Theorem 3.** We may suppose, without loss of generality, that 
\( f(0) = 0 \) (cf. [3]). Let \( \lambda \) be the order of \( f(z) \). Take \( \beta \) such that \( \beta > \lambda - 1 \). Since 
\( \log M(r, f) \) is convex in \( \log r \), we get

\[
\log M(r, f) \sim \log M(r - r^\beta, f) \quad (r \to \infty)
\]
(cf. [3]). Hence (1.3) and (5.1) imply

\[
\lim_{r \to \infty} \frac{\log M(r, g)}{\log M(r - r^\beta, f)} = 0.
\]

We put \( \rho = (r - r^\beta)/r \). Then we have \( c(\rho) > r^{-2(\lambda + \beta)/4} \). Hence it follows from Lemma 1 and (5.2) that

\[
\frac{\log M(r, h \circ g)}{\log M(r, h \circ f)} \leq \frac{\log M(r, g, h)}{\log M(r - r^\beta, f, h)} \leq \frac{\log M(r, g) - \log M(r - r^\beta, f, h)}{\log (r^{-2(\lambda + \beta)} M(r - r^\beta, f, h) / 4 \log (r^{-2(\lambda + \beta)} M(r - r^\beta, f, h) / 4 \to 0 \quad (r \to \infty)),
\]
from which Theorem 3 follows.

6. **Proof of Theorem 4.** Let \( f(z) \) be a transcendental entire function such that

\[
\lim_{r \to \infty} \frac{\log M(r, f)}{\log M(r, \exp \circ f)} = \infty.
\]

The existence of such a function \( f(z) \) was shown by Clunie [3]. For the entire function \( f(z) \), from Lemma 3, there exists an entire function \( g(z) \) satisfying (2.1). The entire functions \( f(z) \) and \( g(z) \) are our desired functions. In fact, (2.1) and (6.1) imply

\[
\lim_{r \to \infty} \frac{\log M(r, \exp \circ g)}{\log M(r, \exp \circ f)} \geq \lim_{r \to \infty} \frac{\log M(r, \exp \circ g)}{\log M(r, f)} \lim_{r \to \infty} \frac{\log M(r, f)}{\log M(r, \exp \circ f)} = \infty.
\]

Thus the proof of Theorem 4 is complete.

**REFERENCES**


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