SOME PROPERTIES OF CANONICAL PRODUCTS
OF FINITE GENUS

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Introduction. Let \( f(z) \) be a canonical product of finite order with only negative zeros. If \( \lambda > 1 \), then

\[
\delta(0, f) > \frac{A}{1+A}
\]

with an absolute constant \( A > 0 \). This result is due to Edrei, Fuchs and Hellerstein [1]; for \( \delta(0, f) \) and other standard terminology and notations used below, see [2].

Recently Ozawa obtained a fairly improved bound of the above constant \( A \) [3]. But it still remains open to find the best possible bound of \( A \).

We now set

\[
h(\lambda) = \inf \delta(0, f)
\]

\[
l(\lambda) = \sup \lim_{r \to \infty} \frac{N(r, 1/f)}{\log M(r, f)},
\]

where \( f \) ranges over all canonical products of finite order \( \lambda \), with only negative zeros. Then the above problem reduces to get the exact value of \( h(\lambda) \).

In this note we shall prove first the following

**Theorem 1.** If \( 1 \leq q \leq \lambda < q+1 \), then we obtain

\[
h(\lambda) \leq 1 - \frac{1}{B(q)},
\]

where

\[
B(q) = 2(2q+1)(2+\log (q+1)).
\]

From the definitions it is clear that

\[
1 - h(\lambda) \geq l(\lambda).
\]

Hence Theorem 1 is contained in the following

**Theorem 2.** If \( 1 \leq q \leq \lambda < q+1 \), then

\[
l(\lambda) \geq 1/B(q).
\]

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Our proof of Theorem 2 depends on the construction of a canonical product $f(z)$ of order $\lambda$ satisfying

$$\lim_{r \to \infty} \frac{N(r, \frac{1}{f})}{\log M(r, f)} \geq 1/B(q).$$

On the other hand Shea conjectured [4] that for entire functions of order $\lambda > 1$

$$\lim_{r \to \infty} \frac{N(r, \frac{1}{f})}{\log M(r, f)} \geq \frac{\sin \pi \lambda}{\pi \lambda}$$

and Williamson showed [5] that for canonical products with only negative zeros (1) is valid under suitable hypotheses. In this connection Williamson asked if canonical products $f(re^{i\theta})$ of genus $q \geq 2$ with only negative zeros asymptotically attain their maximum modulus for $|\theta| \leq \pi/2$. It will be shown here that this is not in general the case. In fact for a canonical product $f(re^{i\theta})$ if we denote by $S(\alpha)$ a set of $r$ such that the maximum modulus of $f(z)$ on $|z|=r$ only attains for $|\theta - \pi| < \alpha$, our third result is

**THEOREM 3.** There exists a canonical product $f(re^{i\theta})$ of genus $q \geq 1$ with only negative zeros such that for an arbitrarily given number $\varepsilon > 0$ $S(\varepsilon)$ has upper density 1.

1. **Constructions of functions $n(r)$ and $N(r)$**. Consider a decreasing sequence $\{\varepsilon_n\} (n=1, 2, 3, \ldots)$ such that

$$\varepsilon_n \to 0 \quad \text{as} \quad n \to \infty$$

and define an increasing unbounded sequence $\{r_n\} (n=1, 2, 3, \ldots)$ satisfying

$$\eta r_n^{1+\varepsilon} < r_{n+1}$$

where $\lambda$ is a positive constant. From the sequence $\{r_n\}$ we construct three sequences $\{t_n\}$, $\{s_n\}$ and $\{u_n\}$ by

$$u_n = \frac{r_n}{(\log r_n)^{1-p}}, \quad t_n = \frac{r_n}{(\log r_n)^{1-p}}, \quad s_n = \frac{r_n}{\log r_n},$$

for $r_n > \varepsilon$, respectively.

Denoting by $\lceil X \rceil$ the integral part of $X$, we define

$$L(n, p) = \lceil \left( \frac{r_n}{u_n} \right)^{1+p} \rceil = \lceil (\log r_n)^{p(1+p)} \rceil$$

with a positive constant $p$.

Let

$$u_{n,k} = u_n^{1+1/p} \quad (k=1, 2, 3, \ldots, L(n, p))$$

and
We may assume, by renumbering of \( \{r_n\} \) if necessary, that the following relations are satisfied:

\[
\begin{align*}
&u_{n,1} = u_n < t_n < s_n < u_{n,L(n,p)} \leq r_n, \\
&r_n < u_{n+1}, \\
&L(n,p) \geq 2, \\
&m_{n,p} \geq 2
\end{align*}
\]

for \( n = 1, 2, 3, \ldots \).

We now put

\[
n(r) = 0, \quad (0 \leq r < u_1)
\]

for \( n \geq 2 \)

\[
n(r) = \begin{cases} \frac{km_{1,p}}{L(1,p)m_{1,p}} & \text{if } r \leq u_{1,k+1} : k = 1, 2, 3, \ldots, L(1,p) - 1 \\
\frac{L(n,p)m_{n,p}}{(u_{1,L(n,p)} - r < u_2)} & \text{if } r \leq u_{n+1}.
\end{cases}
\]

Then we deduce from (1.1), (1.2) and (1.4) that

\[
n(r) = n(r_{n-1}) + km_{n,p} \quad (u_{n,k} \leq r < u_{n,k+1} : k = 1, 2, 3, \ldots, L(n,p) - 1)
\]

and

\[
n(r) = n(r_{n-1}) + L(n,p)m_{n,p} = (1 + p)r_n^{-1} + o(1) \quad (n \to \infty).
\]

We next notice that if \( t_n \leq r \leq u_{n,L(n,p)} \), there is a \( k \) such that

\[
u_{n,k} \leq r < u_{n,k+1}
\]

namely,

\[
k \leq \left( \frac{r}{u_n} \right)^{1+p} < k + 1 \quad (1 \leq k \leq L(n,p))
\]

and then, in view of (1.3) and (1.6),

\[
\left[ \left( \frac{r}{u_n} \right)^{1+p} \right] m_{n,p} + n(r_{n-1}) \leq n(r) < \left( \frac{r}{u_n} \right)^{1+p} m_{n,p} + n(r_{n-1}).
\]

By (1.2), (1.4) and (1.6) we obtain for \( t_n \leq r \leq r_n \)

\[
\left[ \left( \frac{r}{u_n} \right)^{1+p} \right] m_{n,p} = \left( \frac{r}{r_n} \right)^{1+p} m_{n,p} = 1 + o(1) \quad (n \to \infty)
\]

and by (1.4) and (1.8)

\[
\left( \frac{r}{r_n} \right)^{1+p} m_{n,p} = n(r_n) = \left( \frac{r}{r_n} \right)^{1+p} (1 + o(1)) \quad (n \to \infty).
\]
Hence, by (1.7) and (1.9)-(1.11)
\[(1.12)\quad n(r) = n(r_n) \left( \frac{r}{r_n} \right)^{1+p} (1 + o(1)) \quad (n \to \infty) \]
for \( t_n \leq r \leq r_n \).
We now set
\[ N(r) = \int_0^r \frac{n(t)}{t} \, dt. \]

We deduce from (1.6) and (1.7) that if \( s_n \leq r \leq r_n \),
\[ N(r) = N(t_n) + \int_{t_n}^r \frac{n(t)}{t} \, dt \]
\[ = N(t_n) + (1 + o(1)) n(r_n) \int_{t_n}^r \left( \frac{t}{r_n} \right)^{1+p} \, dt \]
\[ = N(t_n) + (1 + o(1)) \frac{n(r_n)}{1+p} \left( \frac{r}{r_n} \right)^{1+p} \quad (n \to \infty). \]

On the other hand, if \( s_n \leq r \leq s_n \),
\[ N(t_n) = \int_0^{t_n} \frac{n(t)}{t} \, dt \leq n(t_n) \log t_n \]
\[ = (1 + o(1)) n(r_n) \left( \frac{r}{r_n} \right)^{1+p} \left( \frac{t_n}{r} \right)^{1+p} \log t_n \quad (n \to \infty) \]
\[ = o \left( n(r_n) \left( \frac{r}{r_n} \right)^{1+p} \right) \quad (n \to \infty) \]
by (1.2) and (1.12).
Hence we obtain
\[(1.13)\quad N(r) = (1 + o(1)) \frac{n(r_n)}{1+p} \left( \frac{r}{r_n} \right)^{1+p} \quad (n \to \infty) \]
for \( s_n \leq r \leq s_n \)
and
\[(1.14)\quad N(r_n) = (1 + o(1)) \frac{n(r_n)}{1+p} \quad (n \to \infty). \]

Finally we notice that both \( n(r) \) and \( N(r) \) have the same order \( \lambda \).

2. Proof of Theorem 2. Let \( q(\geq 1) \) be an integer. Put
\[ E(u, q) = (1-u) \exp \left( u + \frac{u^2}{2} + \cdots + \frac{u^q}{q} \right). \]

Let \( \lambda(\geq 1) \) be a positive number and choose the integer \( q \) satisfying \( q+1 > \lambda \geq q \).
We consider the canonical product
\[(2.1)\quad \prod_{n=1}^\infty \prod_{k=1}^{L(n,p)} E \left( -\frac{z}{u_{n,k}}, q \right)^{m_{n,k}} = f(z), \]
where \( p \) is a positive constant and \( L(n, p), u_{n,k} \) and \( m_{n,p} \) are the ones defined in Section 1. By the construction of \( n(r) \) we have

\[
\sum_{n=1}^{\infty} \sum_{k=1}^{L(n, p)} \frac{m_{n,p}}{u_{n,k}^{q+1}} = \int_0^{\infty} \frac{d(n(r))}{r^{q+1}} = (q+1) \int_0^{r_n} \frac{n(r)}{r^{q+1}} dr < \infty
\]

and, since \( u_{n,L(n,p)} \leq r_n \),

\[
\sum_{k=1}^{L(n, p)} \frac{m_{n,p}}{u_{n,k}^{q+1}} \leq L(n, p) \frac{m_{n,p}}{u_{n,L(n,p)}^{q+1}} \leq \frac{(1+p)r_{n}^{q+1}}{r_n^{q+1}} (1+o(1)) \quad (n \to \infty).
\]

In view of (1.1) these inequalities show that the product in (2.1) converges absolutely and uniformly in any bounded part of the plane to an integral function \( f(z) \) having the order \( \lambda \) and the genus \( q \). Further \( f(z) \) satisfies an inequality

\[
\log |f(z)| \leq (q+1)A(q+1) \left\{ \int_0^{s_n} \frac{n(t)}{t^{q+1}} dt + \int_{s_n}^{s_n + r_n^{q+1}} \frac{n(t)}{t^{q+1}} dt \right\}
\]

where \( A(q+1) = 2(2+\log(q+1)) \) and \( |z| = r \) [2].

We now put \( p = 2q \) in (1.4) and \( |z| = r = r_n \) in (2.2). To estimate the first integral part of (2.2) we set

\[
r_n \int_0^{s_n} \frac{n(t)}{t^{q+1}} dt = \int_0^{s_n} \frac{n(t)}{t^{q+1}} dt + \int_{s_n}^{s_n + r_n^{q+1}} \frac{n(t)}{t^{q+1}} dt.
\]

By (1.12) we find

\[
r_n \int_0^{s_n} \frac{n(t)}{t^{q+1}} dt = (1+o(1))r_n^{2q} n(r_n) \int_{s_n}^{s_n + r_n^{q+1}} \left( \frac{t}{r_n} \right)^{1+2q} \frac{dt}{t^{q+1}}
\]

\[
= (1+o(1)) \frac{n(r_n)}{q+1} \quad (n \to \infty).
\]

Suppose that

\[
r_n \int_0^{s_n} \frac{n(t)}{t^{q+1}} dt = r_n^{s_n} \int_0^{s_n} \frac{n(t)}{t^{q+1}} dt + r_n^{t_n} \int_{s_n}^{t_n} \frac{n(t)}{t^{q+1}} dt + r_n^{s_n} \int_{s_n}^{t_n} \frac{n(t)}{t^{q+1}} dt
\]

\[
= I_1 + I_2 + I_3, \quad \text{say}.
\]

Then, we have

\[
I_1 = r_n^{u_n} \int_0^{u_n} \frac{n(t)}{t^{q+1}} dt \leq r_n^{s_n} \int_0^{s_n} \frac{n(t)}{t^{q+1}} dt + r_n^{s_n} \int_{s_n}^{u_n} \frac{n(t)}{t^{q+1}} dt
\]

\[
= o(n(r_n)) \quad (n \to \infty)
\]

in view of (1.1) and (1.8). Similarly, by (1.12)
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(2.6)
\[ I_n = r_n \int_{u_n}^{t_n} \frac{n(t)}{t^{q+1}} dt \leq n(t_n) r_n \frac{1}{q} \left( \frac{1}{u_n^q} - \frac{1}{t_n^q} \right) \]
\[ = o(n(r_n)) \quad (n \to \infty) \]

and

(2.7)
\[ I_s = r_n \int_{t_n}^{r_n} \frac{n(t)}{t^{q+1}} dt \leq n(s_n) r_n \frac{1}{q} \left( \frac{1}{s_n^q} - \frac{1}{t_n^q} \right) \]
\[ = o(n(r_n)) \quad (n \to \infty) \]

Hence we deduce from (2.4)-(2.7) that
\[ r_n \int_0^{t_n} \frac{n(t)}{t^{q+1}} dt = (1 + o(1)) \frac{n(r_n)}{q+1} \quad (n \to \infty). \]

We find next that, since the order of \( n(r) \) is \( \lambda \), for all sufficiently large \( n \)
\[ r_n^{q+1} \int_{r_n}^{u_{n+1}} \frac{n(t)}{t^{q+2}} dt = r_n^{q+1} \int_{r_n}^{u_{n+1}} \frac{n(t)}{t^{q+2}} dt + r_n^{q+1} \int_{u_{n+1}}^{u_{n+2}} \frac{n(t)}{t^{q+2}} dt \]
\[ \leq n(r_n) + \frac{r_n^{q+1}}{q+1 - (\lambda + \epsilon)} \]
where \( \epsilon < q+1 - \lambda \) is a positive constant. Hence, (1.1) yields
\[ r_n^{q+1} \int_{r_n}^{u_n} \frac{n(t)}{t^{q+2}} dt = \frac{n(r_n)}{q+1} (1 + o(1)) \quad (n \to \infty). \]

We now obtain from (2.2), (2.8), (2.9) and (1.14)
\[ \log M(r_n, f) \leq (q+1) A(q+1)(1 + o(1)) N(r_n)(2q+1) \frac{1}{q+1} \quad (n \to \infty) \]
which leads to
\[ \lim_{r \to \infty} \frac{N(r, 1/f)}{\log M(r, f)} \geq \frac{1}{2(2q+1)A(q+1)}. \]
The assertion of Theorem 2 now follows from this inequality.

3. Proof of Theorem 3. We shall adopt the functions and the notations of Section 2. For \( m \) sufficiently large we consider \( R \) such that
\[ r_m \log r_m \leq R \leq r_m (\log r_m)^2 < u_{n+1}. \]

Let
\[ G_q(R) = \begin{cases} \sum_{n=1}^{m} m_n \rho_{n, p} \left\{ \frac{1}{2} \left( \frac{R}{u_{n, k}} \right)^2 + \cdots + \frac{1}{q} \left( \frac{R}{u_{n, k}} \right)^q \right\} & (q \neq 2) \\ \log \left( 1 + \frac{R}{u_{n, k}} \right)^2 - 2 \frac{R}{u_{n, k}} \cos \theta \right| - \frac{R}{u_{n, k}} \cos \theta \right| & (q = 1) \end{cases} \]

and
\[ F(R, \theta) = \sum_{n=0}^{m} m_n \rho_{n, p} \sum_{k=1}^{q} \left\{ \frac{1}{2} \log \left( 1 + \left( \frac{R}{u_{n, k}} \right)^2 \right) + \frac{R}{u_{n, k}} \cos \theta \right| - \frac{R}{u_{n, k}} \cos \theta \right| \]
Since, by (1.1), \( R/u_{m+1} < 1/2 \) for sufficiently large \( R \), we have
\[
\sum_{n,m,p} m_{n,p} \sum_{k=1}^{L(n,p)} \left| \log E\left( \frac{Re^{i\theta}}{u_{n,k}}, q \right) \right| \leq 2 \sum_{n,m,p} m_{n,p} \sum_{k=1}^{L(n,p)} \left( \frac{R}{u_{n,k}} \right)^{q+1}
\]
\[
\leq 4n(r_m),
\]
which yields
\[
\log |f(Re^{i\theta})| = \sum_{n=1}^{m} m_{n,p} \sum_{k=1}^{L(n,p)} \log \left| E\left( -\frac{Re^{i\pi}}{u_{n,k}}, q \right) \right|
\]
\[
+ \sum_{n,m,p} m_{n,p} \sum_{k=1}^{L(n,p)} \log \left| E\left( -\frac{Re^{i\pi}}{u_{n,k}}, q \right) \right|
\]
(3.2)
\[
\geq F(R, \pi) + G_q(R) - 4n(r_m) \quad (R > R_0)
\]
and
\[
\log |f(Re^{i\theta})| \leq F(R, \theta) + G_q(R) + 4n(r_m) \quad (R > R_0).
\]
Now the construction of \( n(r) \) implies that
\[
G_q(R) = \sum_{i=2}^{q} \left\{ R^i \int_{0}^{\frac{R}{r_m}} \frac{n(t)}{t^{i+1}} dt + n(r_m) \frac{1}{l} \left( \frac{R}{r_m} \right)^{l} \right\}
\]
and hence, by (2.4)–(2.7)
(3.4)
\[
G_q(R) = n(r_m)(1 + o(1)) \sum_{i=2}^{q} \left( \frac{1}{2q-l+1} + \frac{1}{l} \right) \left( \frac{R}{r_m} \right)^{l} \quad (R \to \infty).
\]
On the other hand, since
\[
F(R, \theta) = \sum_{n=1}^{m} m_{n,p} \sum_{k=1}^{L(n,p)} \left\{ \log \left( R \left[ \frac{1}{u_{n,k}} + \frac{1}{2} \log \left( 1 + \frac{u_{n,k} R}{R} \left( 2 \cos \theta + \frac{u_{n,k}}{R} \right) \right) \right] \right) \right\},
\]
we have
\[
\left| F(R, \theta) - \left( R \int_{0}^{\frac{R}{r_m}} \log \frac{R}{t} \, dt - R \cos \theta \int_{0}^{\frac{R}{r_m}} \frac{dn(t)}{l} \right) \right| < n(R) \log 2 \quad (R > R_0)
\]
which implies that
(3.5)
\[
\left| F(R, \theta) - \left( R \int_{0}^{\frac{R}{r_m}} \log \frac{R}{t} \, dt - R \cos \theta \int_{0}^{\frac{R}{r_m}} \frac{dn(t)}{l} \right) \right| < n(r_m) \log 2 \quad (R > R_0).
\]
If \( q \geq 2 \), by (3.2), (3.4), (3.5) and (2.4)–(2.7) we obtain
(3.6)
\[
\log |f(Re^{i\theta})| \geq N(R) + G_q(R) + (1 + o(1)) \left( n(r_m) \frac{R}{r_m} + R \int_{0}^{\frac{R}{r_m}} \frac{n(t)}{l^2} \, dt \right) \quad (R \to \infty).
\]
Similarly (3.3) and (3.5) yield
(3.7)
\[
\log |f(Re^{i\theta})| \leq N(R) + G_q(R) - (\cos \theta + o(1)) \left( n(r_m) \frac{R}{r_m} + R \int_{0}^{\frac{R}{r_m}} \frac{n(t)}{l^2} \, dt \right) \quad (R \to \infty).
By (3.4)
\[ n(r_m)\frac{R}{r_m}/G_q(R) \to 0 \quad (R \to \infty) \]
and hence, from (3.6) and (3.7) we deduce that
\[ \frac{\log |f(Re^{i\theta})|}{\log |f(Re^{i\theta})|} \leq 1 - (1 + \cos \theta + o(1))C(R) \quad (R \to \infty) \]
where
\[ C(R) = \frac{\int_0^{\pi} n(t) dt + n(r_m)\frac{R}{r_m}}{N(R) + G_q(R)} < 0. \]

This inequality holds for each \( R \) satisfying (3.1) if \( m \) is sufficiently large and the assertions of Theorem 3 become obvious for \( q \geq 2 \).

If \( q = 1 \), \( G_q(R) = o \) in (3.6) and (3.7), and hence, since
\[ \frac{N(R)/n(r_m)\frac{R}{r_m}}{N(R) + G_q(R)} \to 0 \quad (R \to \infty), \]
we obtain
\[ \frac{\log |f(Re^{i\theta})|}{\log |f(Re^{i\theta})|} \leq -\cos \theta + o(1) \quad (R \to \infty). \]

This completes the proof of Theorem 3.

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**References**


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