ON CONFORMAL CHANGES OF RIEMANNIAN METRICS

Dedicated to Professor Yūsaku Komatu on his sixtieth birthday

BY KENTARO YANO AND HITOSI HIRAMATU

§ 1. Introduction.

Let $M$ be an $n$-dimensional connected differentiable manifold and $g$ a Riemannian metric tensor field on $M$. We denote by $(M, g)$ a Riemannian manifold with the metric tensor field $g$. Let there be given two metric tensor fields $g$ and $g^*$ on $M$. If $g^*$ is conformal to $g$, that is, if there exists a function $\rho$ on $M$ such that $g^* = e^{\rho} g$, then we call such a change of metric tensor field $g \rightarrow g^*$ a conformal change of metric. In particular, if $\rho = \text{constant}$ then the conformal change of metric is said to be homothetic and if $\rho = 0$ then the conformal change of metric is said to be isometric.

Let $(M, g)$ and $(M', g')$ be two Riemannian manifolds and $f: M \rightarrow M'$ a diffeomorphism. Then $g^* = f^* g$ is a Riemannian metric tensor field on $M$. When $g^*$ is conformal to $g$, that is, when there exists a function $\rho$ on $M$ such that $g^* = e^{\rho} g$, we call $f: (M, g) \rightarrow (M', g')$ a conformal transformation. In particular, if $\rho = \text{constant}$ then $f$ is called a homothetic transformation or a homothety and if $\rho = 0$ then $f$ is called an isometric transformation or an isometry.

If a vector field $v$ on $M$ satisfies

$$L_v g = 2\phi g,$$

where $L_v$ denotes the Lie derivation with respect to $v$ and $\phi$ a function on $M$, then $v$ is called an infinitesimal conformal transformation. The $v$ is said to be homothetic or isometric according as $\phi$ is a constant or zero.

Given a Riemannian manifold $(M, g)$, we denote by $g_{ij}$, $\{i^j\}$, $\nabla_v$, $K_{ij}^h$, $K_{ij}$ and $K$, respectively, the components of the metric tensor field $g$, the Christoffel symbols formed with $g_{ij}$, the operator of covariant differentiation with respect to $\{i^j\}$, the components of the curvature tensor field, the components of the Ricci tensor field and the scalar curvature of $(M, g)$, where and in the sequel, indices $i, j, k, \cdots$ run over the range $\{1, 2, \cdots, n\}$. Hereafter we assume that functions under consideration are always differentiable.

When we consider a conformal change of metric
if $\Omega$ is a quantity formed with $g$ then we denote by $\Omega^*$ the quantity formed with $g^*$ by the same rule as that $\Omega$ is formed with $g$.

Recently, Goldberg and Yano [2] studied non-homothetic conformal changes of metrics and obtained the following

**Theorem A.** Let $(M, g)$ be a compact orientable Riemannian manifold of dimension $n>3$ with constant scalar curvature $K$ and admitting a non-homothetic conformal change of metric $g^*=e^{\theta g}$ such that $K^*=K$. Then if

$$\int_M u^{-\frac{n+1}{2}}G_{ji}u^i u^j dV \geq 0,$$

where

$$G_{ji} = K_{ji} - \frac{K}{n}g_{ji},$$

and $u=e^{-\theta}$, $u_i = \nabla_i u$, $u^* = u_g^{th}$ and $dV$ denotes the volume element of $(M, g)$, then $(M, g)$ is isometric to a sphere.


**Theorem B.** If a compact orientable Riemannian manifold $(M, g)$ of dimension $n>2$ admits a non-homothetic conformal change of metric $g^*=e^{\theta g}$ such that

$$\int_M (\Delta u)K dV = 0,$$

where $\Delta u = g^{ij}\nabla_i \nabla_j u$, then $(M, g)$ is conformal to a sphere.

**Theorem C.** If a compact orientable Riemannian manifold $(M, g)$ of dimension $n>2$ and with $K$=constant admits a non-homothetic conformal change of metric $g^*=e^{\theta g}$ such that

$$G^*_{ji}G^*_{ji} = u^*G_{ji}G^*_{ji},$$

then $(M, g)$ is isometric to a sphere.

**Theorem D.** If a compact orientable Riemannian manifold $(M, g)$ of dimension $n>2$ admits a non-homothetic conformal change of metric $g^*=e^{\theta g}$ such that

$$\int_M (\Delta u)K dV = 0,$$

where

$$Z_{kji} = K_{kji} - \frac{K}{n(n-1)} (\partial g_{ji} - \partial g_{ki}),$$

then $(M, g)$ is conformal to a sphere.

**Theorem E.** If a compact orientable Riemannian manifold $(M, g)$ of dimension $n>2$ and with $K$=constant admits a non-homothetic conformal change of metric $g^*=e^{\theta g}$ such that

$$Z_{kji} = u^4 Z_{kji} Z^{kji},$$

then $(M, g)$ is conformal to a sphere.
then \((M, g)\) is isometric to a sphere.

**Theorem F.** If a compact orientable Riemannian manifold \((M, g)\) of dimension \(n>2\) admits a non-homothetic conformal change of metric \(g^*=e^{a+bu}g\) such that
\[
\int_M (du) K dV = 0,
\]
\[
W^{*}_{kjh} W^{*}_{kjish} = u^4 W_{kjh} W^{kjish}, \quad a+(n-2)b \neq 0,
\]
where
\[
W^{*}_{kjh} = a Z^{*}_{kjh} + b (\partial^i G_{ji} - \partial^j G_{ki}) + G^*_k g^*_i - G^*_j g^*_k,
\]
a and \(b\) being constants, then \((M, g)\) is conformal to a sphere.

**Theorem G.** If a compact orientable Riemannian manifold \((M, g)\) of dimension \(n>2\) and with \(K=\)constant admits a non-homothetic conformal change of metric \(g^*=e^{a+bu}g\) such that
\[
W^{*}_{kjh} W^{*}_{kjish} = u^4 W_{kjh} W^{kjish}, \quad a+(n-2)b \neq 0,
\]
then \((M, g)\) is isometric to a sphere.

**Theorem H.** If a compact orientable Riemannian manifold \((M, g)\) of dimension \(n\geq 2\) admits a non-homothetic conformal change of metric \(g^*=e^{a+bu}g\) such that
\[
K^* = K, \quad L_{du} K = 0, \quad \int_M u^{-n+1} G_{ji} u^i u^j dV = 0,
\]
where \(L_{du}\) denotes the Lie derivation with respect to a vector field \(u^h = g^{ih} F_i u\), then \((M, g)\) is isometric to a sphere.

**Theorem I.** If a compact orientable Riemannian manifold \((M, g)\) of dimension \(n>2\) admits a non-homothetic conformal change of metric \(g^*=e^{a+bu}g\) such that
\[
K^* = K, \quad L_{du} K = 0, \quad G^*_{ji} G^*_{kj} = G_{ji} G_{kj},
\]
then \((M, g)\) is isometric to a sphere.

(See also Barbance [1].)

**Theorem J.** If a compact orientable Riemannian manifold \((M, g)\) of dimension \(n>2\) admits a non-homothetic conformal change of metric \(g^*=e^{a+bu}g\) such that
\[
K^* = K, \quad L_{du} K = 0,
\]
then \((M, g)\) is conformal to a sphere.

(See also Hsiung and Liu [3].)

**Theorem K.** If a compact orientable Riemannian manifold \((M, g)\) of dimension \(n>2\) admits a non-homothetic conformal change of metric \(g^*=e^{a+bu}g\) such that
\[
K^* = K, \quad L_{du} K = 0,
\]
\[
W^*_{kjh} W^*_{kjish} = W_{kjh} W^{kjish}, \quad a+(n-2)b \neq 0,
\]
then \((M, g)\) is isometric to a sphere.
Yano and Sawaki [14] proved following theorems.

**THEOREM L.** If a compact orientable Riemannian manifold \( (M, g) \) of dimension \( n > 2 \) admits a non-homothetic conformal change of metric \( g^* = e^{2p} g \) such that

\[
L_{du} K = 0, \quad L_{du} K^* = 0, \quad u^p G_{ij} G^{ij} = (u-1)\varphi + 1 \quad G_{ij} G^{ij},
\]

where \( p \) is a real number such that \( p \leq 4 \) and \( \varphi \) a non-negative function on \( M \), then \( (M, g) \) is isometric to a sphere.

**THEOREM M.** If a compact orientable Riemannian manifold \( (M, g) \) of dimension \( n > 2 \) admits a non-homothetic conformal change of metric \( g^* = e^{2p} g \) such that

\[
L_{du} K = 0, \quad L_{du} K^* = 0, \quad u^p Z_{kijh} Z^{kijh} = (u-1)\varphi + 1 \quad Z_{kijh} Z^{kijh},
\]

where \( p \) and \( \varphi \) are the same as in Theorem L, then \( (M, g) \) is isometric to a sphere.

**THEOREM N.** If a compact orientable Riemannian manifold \( (M, g) \) of dimension \( n > 2 \) admits a non-homothetic conformal change of metric \( g^* = e^{2p} g \) such that

\[
L_{du} K = 0, \quad L_{du} K^* = 0, \quad u^p W_{kijh} W^{kijh} = (u-1)\varphi + 1 \quad W_{kijh} W^{kijh}, \quad a + (n-2)b \neq 0,
\]

where \( p \) and \( \varphi \) are the same as in Theorem L, then \( (M, g) \) is isometric to a sphere.

The purpose of the present paper is to prove generalizations of Theorems L~N.

In the sequel, we need the following two theorems.

**THEOREM O (Tashiro [8]).** If a compact Riemannian manifold \( (M, g) \) of dimension \( n \geq 2 \) admits a non-constant function \( u \) on \( M \) such that

\[
\varphi_f \varphi, u - \frac{1}{n} \Delta u^f = 0,
\]

then \( (M, g) \) is conformal to a sphere in an \( (n+1) \)-dimensional Euclidean space.

(See also Ishihara [4], Ishihara and Tashiro [5].)

**THEOREM P (Yano and Obata [13]).** If a complete Riemannian manifold \( (M, g) \) of dimension \( n \geq 2 \) admits a non-constant function \( u \) on \( M \) such that

\[
\varphi_f \varphi, u - \frac{1}{n} \Delta u^f = 0, \quad L_{du} K = 0,
\]

then \( (M, g) \) is isometric to a sphere in an \( (n+1) \)-dimensional Euclidean space.
§ 2. Preliminaries.

We consider a conformal change of metric

\[ g_{ji}^* = e^{\rho} g_{ji}. \]  

First of all, we have

\[ \{ \frac{h}{j} \}_{i} = \{ \frac{h}{j} \}_{i} + \partial_{j}^h \rho_i + \partial_i^h \rho_j - g_{ji} \rho^h, \]

where

\[ \rho_i = \nabla_i \rho, \quad \rho^h = g^{ih} \rho_h, \]

from which

\[ K^*_{,h} = K_{,h} + \partial_{h} \rho_{j} - \partial_{j} \rho_{h} + \rho^h g_{ji} + \rho^i g_{ji}, \]

where

\[ \rho_{ji} = \nabla_j \rho_i + \frac{1}{2} \rho_i \rho_j g_{ji}, \quad \rho^h = \rho_{ji} g^{ih}, \]

and consequently

\[ K_{,h}^* = K_{,h} - (n-2) \rho_{ji} - \rho^i g_{ji}, \]

and

\[ e^{\rho} K^* = K - (n-1) \rho^i, \]

where

\[ \rho^i = \Delta \rho + \frac{n-2}{2} \rho_i \rho^i, \quad \Delta \rho = g^{ij} \nabla_j \rho_i. \]

From (2.3), (2.4) and (2.5) and the definitions of \( G_{ji}, Z_{,h} \) and \( W_{,h} \), we have

\[ G_{ji}^* = G_{ji} - (n-2) (\nabla_j \rho_i - \rho_j \rho_i) + \frac{n-2}{n} \left( \Delta \rho - \rho_i \rho_i \right) g_{ji}, \]

\[ Z_{,h}^* = Z_{,h} - \partial_{,h} (\nabla_j \rho_i - \rho_j \rho_i) + \partial_{,h} (\nabla_i \rho_j - \rho_i \rho_j) \]

\[ - (\nabla_i \rho^h - \rho_i \rho^h) g_{ji} + (\nabla_j \rho^h - \rho_j \rho^h) g_{hi} + \frac{2}{n} (\Delta \rho - \rho_i \rho^i) (\partial_{,h} g_{ji} - \partial_{,h} g_{hi}), \]

and

\[ W_{,h}^* = W_{,h} + \{ a + (n-2) b \} \left\{ - \partial_{,h} (\nabla_j \rho_i - \rho_j \rho_i) + \partial_{,h} (\nabla_i \rho_j - \rho_i \rho_j) \]

\[ - (\nabla_i \rho^h - \rho_i \rho^h) g_{ji} + (\nabla_j \rho^h - \rho_j \rho^h) g_{hi} \]

\[ + \frac{2}{n} (\Delta \rho - \rho_i \rho^i) (\partial_{,h} g_{ji} - \partial_{,h} g_{hi}) \}. \]

If we put
then we have

\( V_j u_i = -u(V_j \rho_i - \rho_j \rho_i) \)

and

\( \Delta u = -u(\Delta \rho - \rho \rho^t) \),

and consequently

\( K^* = u^* K + 2(n-1)u \Delta u - n(n-1)u \rho \rho^t \),

\( G^*_{ji} = G_{ji} + (n-2)P_{ji} \),

\( Z^*_{kji} = Z_{kji} + Q_{kji} \)

and

\( W^*_{kji} = W_{kji} + \{a + (n-2)b\} Q_{kji} \),

where

\( P_{ji} = u^{-1}(V_j u_i - \frac{1}{n} \Delta u g_{ji}) \),

\( Q_{kji} = \delta_k^j P_{ji} - \delta_k^j P_{kj} + P_{k}^h g_{ji} - P_{j}^h g_{ki} \)

and

\( P_j^h = P_{ji} g_{ih} \).

From (2.16) and (2.17), we obtain

\( P_{ji} P_{ji} = u^{-1} \{(V_j u_i)(V_j u_i) - \frac{1}{n} (\Delta u)^2 \} \)

and

\( Q_{kji} Q_{kji} = 4(n-2)P_{ji} P_{ji} \)

respectively.

We also have, from (2.13), (2.14) and (2.15),

\( G^*_{ji} G^*_{ji} = u^* \{G_{ji} G_{ji} + 2(n-2)G_{ji} P_{ji} + (n-2)^2 P_{ji} P_{ji} \} \),

\( Z^*_{kji} Z^*_{kji} = u^* \{Z_{kji} Z_{kji} + 8G_{ji} P_{ji} + 4(n-2)P_{ji} P_{ji} \} \)

and

\( W^*_{kji} W^*_{kji} = u^* \{W_{kji} W_{kji} + 8 \{a + (n-2)b\} G_{ji} P_{ji} + 4(n-2)\{a + (n-2)b\} G_{ji} P_{ji} \} \)

respectively. For the expression \( G_{ji} P_{ji} \) in (2.20), (2.21) and (2.22), we have, from (2.16),
For a vector field $\nu^h$ on a compact orientable Riemannian manifold $(M, g)$, we have

\begin{equation}
\int_M \left( G^{ji} \nabla_j \nu^h + K^i \nu^h + \frac{n-2}{n} \nu^h \nabla_i \nu^j \right) \nu_h dV
\end{equation}

Proof. By a straightforward computation, we have

\begin{align*}
G^{ji} \nabla_j \nu^h + K^i \nu^h + &\frac{n-2}{n} \nu^h \nabla_i \nu^j \\
= &\left( G^{ji} \nabla_j \nu^h + K^i \nu^h + \frac{n-2}{n} \nu^h \nabla_i \nu^j \right) \nu_h \\
+ &\frac{1}{2} \left( \nabla_j \nu^i + \nabla_i \nu^j - \frac{2}{n} \nu^i \nabla_j g_{ij} \right) \\
\times &\left( \nabla_j \nu^i + \nabla_i \nu^j - \frac{2}{n} \nu^i \nabla_j g_{ij} \right),
\end{align*}

and consequently, integrating over $M$, we have \( \text{(3.1)} \).

Remark. If a vector field $\nu^h$ defines an infinitesimal conformal transformation, then we have

\begin{equation}
L_\nu g_{ij} = 2 \rho g_{ij},
\end{equation}

that is,

\begin{equation}
\nabla_j \nu_i + \nabla_i \nu_j - \frac{2}{n} \nu^i \nabla_j g_{ij} = 0.
\end{equation}

From this, we can deduce

\begin{equation}
G^{ji} \nabla_j \nu^h + K^i \nu^h + \frac{n-2}{n} \nu^h \nabla_i \nu^j = 0.
\end{equation}

Formula \( \text{(3.1)} \) shows that this is not only necessary but also sufficient in order that the vector field $\nu^h$ defines an infinitesimal conformal transformation in a compact orientable Riemannian manifold.

Lemma 2 (Yano [10]). For a function $u$ on a compact orientable Riemannian manifold $(M, g)$, we have

\begin{equation}
G^{ji} \nabla_j u = u^{-1} G_{ji} \nabla^j u,
\end{equation}

where $\nabla^i = g^{ij} \nabla_j$.
\begin{align}
\int_M (g^{ij} \nabla_i u^j + K_{ij}^+ u_i + \frac{n-2}{n} \nabla^h Au) u_h dV \\
+ 2\int_M (\nabla_i u^i - \frac{1}{n} \Delta u g^{ij}) (\nabla_j u_i - \frac{1}{n} \Delta u g_{ji}) dV = 0
\end{align}

and
\begin{align}
\int_M \left\{ (g^{ij} \nabla_i u^j + K_{ij}^+ u_i) u_h - \frac{n-2}{n} (\Delta u)^2 \right\} dV \\
+ 2\int_M (\nabla_i u^i - \frac{1}{n} \Delta u g^{ij}) (\nabla_j u_i - \frac{1}{n} \Delta u g_{ji}) dV = 0,
\end{align}

where $u_i = \nabla_i u$, $u^h = u_i g^{ih}$ and $\Delta u = g^{ij} \nabla_j \nabla_i u$.

**Proof.** Putting $v^h = u^h$ in (3.1) and using $\nabla_j u^i = \nabla^i u^j$, we obtain (3.3). (3.4) follows from (3.3) because of
\begin{align}
\int_M (\nabla_i \Delta u) u_i dV = -\int_M (\Delta u)^2 dV.
\end{align}

**Lemma 3** (Yano [10]). For a function $u$ on a Riemannian manifold $(M, g)$, we have
\begin{align}
\nabla^h \Delta u = g^{ij} \nabla_j u^i - K_{ij}^+ u^i,
\end{align}
that is,
\begin{align}
g^{ij} \nabla_j \nabla_i u = \nabla^h \Delta u + K_{ij}^+ u^i.
\end{align}

**Proof.** We have
\begin{align}
\nabla^h (\Delta u) &= \nabla^h (g^{ij} \nabla_j u_i) = g^{ij} \nabla^h \nabla_j u_i \\
&= g^{ij} (\nabla_j u_i - K_{ij}^+ u_i) \\
&= g^{ij} \nabla_j \nabla_i u - K_{ij}^+ u_i,
\end{align}
from which (3.5) follows.

**Lemma 4.** For a function $u$ on a compact orientable Riemannian manifold $(M, g)$, we have
\begin{align}
\int_M (K_{ij} u^i u^j + \frac{n-1}{n} u^h \nabla_h \Delta u) dV \\
+ \int_M (\nabla_i u^i - \frac{1}{n} \Delta u g^{ij}) (\nabla_j u_i - \frac{1}{n} \Delta u g_{ji}) dV = 0
\end{align}

and
\begin{align}
\int_M \left\{ K_{ij} u^i u^j - \frac{n-1}{n} (\Delta u)^2 \right\} dV \\
+ \int_M (\nabla_i u^i - \frac{1}{n} \Delta u g^{ij}) (\nabla_j u_i - \frac{1}{n} \Delta u g_{ji}) dV = 0.
\end{align}
Proof. Substituting (3.6) into (3.3), we have (3.7), and substituting (3.6) into (3.4), we have (3.8).

Lemma 5. If a compact orientable Riemannian manifold \((M, g)\) of dimension \(n \geq 2\) admits a conformal change of metric \(g^* = e^{2p}g\), then, for any real number \(p\), we have

\[
\int_M u^{p-1} G_{ji} u^i u^j dV
+ (p+n-2) \int_M u^{p-2}(\nabla_i u_i) u^i u^j dV + \frac{1}{2n} \int_M (u^{p-1} L_{du} K^* - u^p L_{du} K) dV
\]

\[
- \frac{p+n-2}{2} \int_M u^{p-2}(u_i u^i) dV - \frac{p+n-2}{2n(n-1)} \int_M u^{p-2} u_i u_i K dV
\]

\[
+ \frac{p+n-2}{2n(n-1)} \int_M u^{p-1} u_i u_i K dV + \int_M u^{p+1} P_{ji} P^i dV = 0
\]

and

\[
\int_M u^{p-1} K_{ji} u^i u^j dV
+ \frac{p+n-2}{n} \int_M u^{p-1} \langle \nabla_i u_i \rangle dV + \frac{p-1}{n(n-1)} \int_M u^{p-2} u_i u_i K dV
\]

\[
- \frac{p-1}{2n(n-1)} \int_M (u^{p-1} L_{du} K^* - u^p L_{du} K) dV + \int_M u^{p+1} P_{ji} P^i dV = 0
\]

In particular, if \(p = -n+2\) then

\[
\int_M u^{-n+1} G_{ji} u^i u^j dV
+ \frac{1}{2n} \int_M (u^{-n} L_{du} K^* - u^{-n+2} L_{du} K) dV + \int_M u^{-n+3} P_{ji} P^i dV = 0
\]

and if \(p = 1\) then

\[
\int_M K_{ji} u^i u^j dV - \frac{1}{n} \int_M (\Delta u)^2 dV + \int_M u^2 P_{ji} P^i dV = 0
\]

and

\[
\int_M K_{ji} u^i u^j dV + \frac{n-1}{n} \int_M u^i \nabla_i (\Delta u) dV + \int_M u^2 P_{ji} P^i dV = 0
\]
Proof. We first have
\[
\mathcal{F}_j(u^{p-1}u_i\mathcal{F}^{j}u^i) = (p-1)u^{p-2}(\mathcal{F}^i u^i)u_j u_i + u^{p-1}(\mathcal{F}_j u_i)(\mathcal{F}^j u^i) + u^{p-1}u_i \mathcal{F}_j \mathcal{F}^i u^i
\]
\[
= (p-1)u^{p-2}(\mathcal{F}^i u^i)u_j u_i + u^{p-1}(\mathcal{F}_j u_i)(\mathcal{F}^j u^i)
+ u^{p-1}K_{ji}u^j u^i + u^{p-1}u_i \mathcal{F}^i (\Delta u),
\]
where we have used (3.6), that is,
\[
\mathcal{F}_j \mathcal{F}^i u^i = K_{ji}u^i + \mathcal{F}^i \Delta u,
\]
and consequently, integrating over \( M \), we have
\[
\int_M u^{p-1}(\mathcal{F}_j u_i)(\mathcal{F}^j u^i)dV + (p-1)\int_M u^{p-2}(\mathcal{F}^i u^i)u_j u_i dV
+ \int_M u^{p-1}K_{ji}u^j u^i dV
+ \int_M u^{p-1}u_i \mathcal{F}^i (\Delta u)dV = 0.
\]
Similarly, computing \( \mathcal{F}_i(u^{p-1}u^i \Delta u) \) and integrating over \( M \), we have
\[
(p-1)\int_M u^{p-2}u_i u^i \Delta u dV + \int_M u^{p-1}(\Delta u)^2 dV
+ \int_M u^{p-1}u^i \mathcal{F}_i (\Delta u)dV = 0.
\]
By using (2.18), (3.15) and (3.16), we get
\[
\int_M u^{p+1}P_{ji}P^{j}dV = \int_M u^{p-1}(\mathcal{F}_j u_i)(\mathcal{F}^j u^i)dV - \frac{1}{n} \int_M u^{p-1}(\Delta u)^2 dV
- (p-1)\int_M u^{p-2}(\mathcal{F}^i u^i)u_j u_i dV - \int_M u^{p-1}K_{ji}u^j u^i dV
+ \frac{p-1}{n} \int_M u^{p-2}u_i u^i \Delta u dV - \frac{n-1}{n} \int_M u^{p-1}u^i \mathcal{F}_i (\Delta u)dV.
\]
On the other hand, from (2.12), we have
\[
\Delta u = \frac{1}{2(n-1)}(u^{-1}K^* - uK) + \frac{n}{2}u^{-1}u_i u^i,
\]
from which
\[
\mathcal{F}_i(\Delta u) = -\frac{1}{2(n-1)}(u^{-2}u^i K^* + u_i K) + \frac{1}{2(n-1)}(u^{-1}\mathcal{F}_i K^* - u\mathcal{F}_i K)
- \frac{n}{2}u^{-2}u_i u^i u^i + nu^{-1}(\mathcal{F}_i u_i)u^i.
\]
Substituting (3.18) and (3.19) into (3.17) and using
\[
K_{ji} = G_{ji} + \frac{K}{n}g_{ji},
\]
we have (3.9).

Substituting
\[ u^{p-2}u_i u^i K^* = 2(n-1)u^{p-2}u_i u^i \Delta u - n(n-1)u^{p-3}(u_i u^i)^2 + u^{p-1}u_i u^i K \]
which can be obtained from (3.18) into
\[
\int_M u^{p-1}u^i \Phi_i(\Delta u) dV = n\int_M u^{p-2}(\Phi_j u_i) u^j u^i dV - \frac{1}{2(n-1)} \int_M u^{p-1}u_i u^i K dV + \frac{1}{2(n-1)} \int_M (u^{p-2}L_{\Delta u} K^* - u^p L_{\Delta u} K) dV - \frac{n}{2} \int_M u^{p-2}(u_i u^i)^2 dV
\]
which follows from (3.19), we have
\[
\int_M u^{p-1}u^i \Phi_i(\Delta u) dV = n\int_M u^{p-2}(\Phi_j u_i) u^j u^i dV - \frac{1}{n-1} \int_M u^{p-1}u_i u^i K dV + \frac{1}{2(n-1)} \int_M (u^{p-2}L_{\Delta u} K^* - u^p L_{\Delta u} K) dV,
\]
and consequently, by using
\[
(3.20) \quad \int_M u^{p-1}u^i \Phi_i(\Delta u) dV = -(p-1)\int_M u^{p-2}u_i u^i \Delta u dV - \int_M u^{p-1}(\Delta u)^2 dV
\]
which is equivalent to (3.16), we obtain
\[
(3.21) \quad \int_M u^{p-2}(\Phi_j u_i) u^j u^i dV = -\frac{p-2}{n} \int_M u^{p-2}u_i u^i \Delta u dV - \frac{1}{n} \int_M u^{p-1}(\Delta u)^2 dV + \frac{1}{n(n-1)} \int_M u^{p-1}u_i u^i K dV
\]
and
\[
\int_M u^{p-2}(\Phi_j u_i) u^j u^i dV = -\frac{p-2}{n} \int_M u^{p-2}u_i u^i \Delta u dV - \frac{1}{n} \int_M u^{p-1}(\Delta u)^2 dV + \frac{1}{n(n-1)} \int_M u^{p-1}u_i u^i K dV - \frac{1}{2n(n-1)} \int_M (u^{p-2}L_{\Delta u} K^* - u^p L_{\Delta u} K) dV.
\]
Substituting (3.20) and (3.21) into (3.17), we get (3.10). From (3.16) and (3.20), we have (3.11) immediately.

**Lemma 6.** If a compact orientable Riemannian manifold \((M, g)\) admits a conformal change of metric \(g^* = e^{2p}g\), then, for any real number \(p\),
\[
(3.22) \quad \int_M (u^{p-2}G*_{ji}G*_{ji} - u^{p+1}G_{ji}G_{ji}) dV
+ 2(n-2)p \int_M u^{p-1}G_{ji} u^j u^i dV + \frac{(n-2)}{n} \int_M u^p L_{\Delta u} K dV
- (n-2)^2 \int_M u^{p+1}P_{ji} P^{ji} dV = 0.
\]
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In particular, if \( p = -n + 2 \) then

\[
\int_M (u^{-n} G^*_{ij} G^{*ij} - u^{-n+3} G^*_{ij} G^{ij}) dV
- 2(n-2) \int_M u^{-n+3} G^*_{ij} u^i u^j dV + \frac{(n-2)^2}{n} \int_M u^{-n+3} L_{a\kappa} K dV
- (n-2)^2 \int_M u^{-n+3} P_{ij} P^{ij} dV = 0,
\]

and if \( p = 0 \) then

\[
\int_M (u^{-3} G^*_{ij} G^{*ij} - u G^*_{ij} G^{ij}) dV
+ \frac{(n-2)^2}{n} \int_M L_{a\kappa} K dV - (n-2)^2 \int_M u P_{ij} P^{ij} dV = 0.
\]

Proof. Using (2.20) and (2.23), we have

\[
\int_M (u^{-3} G^*_{ij} G^{*ij} - u G^*_{ij} G^{ij}) dV
= 2(n-2) \int_M u^p G^*_{ij} P^i u^j dV + (n-2)^2 \int_M u^{p+i} P^{ij} dV.
\]

On the other hand, calculating \( P^i(u^p G^*_{ij} u^j) \) and using

\[ P^i G_{ij} = \frac{n-2}{2n} P_i K, \]

we have

\[ P^i(u^p G^*_{ij} u^j) = p u^{p-1} G^*_{ij} u^i u^j + \frac{n-2}{2n} u^p u^i P_i K + u^p G_{ij} P^j u^i, \]

and consequently, integrating over \( M \), we have

\[
\int_M u^p G_{ij} P^i u^j dV = -p \int_M u^{p-1} G_{ij} u^i u^j dV - \frac{n-2}{2n} \int_M u^p u^i P_i K dV.
\]

Substituting this into (3.25), we have (3.22) to be proved.

**Lemma 7.** If a compact orientable Riemannian manifold \((M, g)\) of dimension \( n \geq 2 \) admits a conformal change of metric \( g^* = e^{2\phi} g \), then

\[
\int_M (u^{-n-1} G^*_{ij} G^{*ij} - u^{-n+3} G^*_{ij} G^{ij}) dV
+ \frac{(n-2)^2}{n} \int_M u^{-n} L_{a\kappa} K^* dV + (n-2)^2 \int_M u^{-n+1} P_{ij} P^{ij} dV = 0.
\]

Proof. Adding (3.12) \( \times 2(n-2)^2 \) and (3.23), we have (3.27).

**Lemma 8.** If a compact orientable Riemannian manifold \((M, g)\) of dimension \( n \geq 2 \) admits a conformal change of metric \( g^* = e^{2\phi} g \), then, for any real number \( p \),
\[ (3.28) \int_M (u^{p-1} Z^{k_i h}_j - u^{p+1} Z^{k_j h}_i) dV + 8p \int_M u^{p-1} G_{ji} u'^j u^i dV + \frac{4(n-2)}{n} \int_M u^p L_{du} K dV
- 4(n-2) \int_M u^{p+1} P_{ji} P^{ji} dV = 0. \]

In particular, if \( p = -n+2 \) then
\[ (3.29) \int_M (u^{-n+1} Z^{k_i h}_j - u^{-n+3} Z^{k_j h}_i) dV - 8(n-2) \int_M u^{-n+1} G_{ji} u'^j u^i dV + \frac{4(n-2)}{n} \int_M u^{-n+2} L_{du} K dV
- 4(n-2) \int_M u^{-n+3} P_{ji} P^{ji} dV = 0, \]

and if \( p = 0 \) then
\[ (3.30) \int_M (u^{-n} Z^{k_i h}_j - u Z^{k_j h}_i) dV + \frac{4(n-2)}{n} \int_M L_{du} K dV - 4(n-2) \int_M u P_{ji} P^{ji} dV = 0. \]

**Proof.** Using (2.21) and (2.23), we have
\[ (3.31) \int_M (u^{p-1} Z^{k_i h}_j - u^{p+1} Z^{k_j h}_i) dV
- 8 \int_M u^p G_{ji} u'^j u^i dV - 4(n-2) \int_M u^{p+1} P_{ji} P^{ji} dV = 0. \]
Substituting (3.26) into (3.31), we have (3.28).

**Lemma 9.** If a compact orientable Riemannian manifold \((M, g)\) of dimension \(n \geq 2\) admits a conformal change of metric \(g^* = e^{2p} g\), then
\[ (3.32) \int_M (u^{-n+1} Z^{k_i h}_j - u^{-n+3} Z^{k_j h}_i) dV + \frac{4(n-2)}{n} \int_M u^{-n} L_{du} K dV + 4(n-2) \int_M u^{-n+3} P_{ji} P^{ji} dV = 0. \]

**Proof.** (3.32) follows from (3.12) and (3.29).

**Lemma 10.** If a compact orientable Riemannian manifold \((M, g)\) of dimension \(n \geq 2\) admits a conformal change of metric \(g^* = e^{2p} g\), then, for any real number \(p\),
\[ (3.33) \int_M (u^{p-1} W^{k_i h}_j - u^{p+1} W^{k_j h}_i) dV
+ 8 \{a + (n-2)b\} \int_M u^{p-1} G_{ji} u'^j u^i dV \]
In particular, if $p = -n+2$ then

$$
\int_M \left( u^{-n+1} W_{kjth} W_{kjth} - u^{-n+3} W_{kjth} W_{kjth} \right) dV
- 8(n-2) \{ a + (n-2)b \} \int_M u^{-n+1} G_{ji} u^i u^j dV
+ \frac{4(n-2)}{n} \{ a + (n-2)b \} \int_M u^{-n+2} L du K dV
- 4(n-2) \{ a + (n-2)b \} \int_M u^{-n+1} P_{ji} P_{ji} dV = 0,
$$

and if $p = 0$ then

$$
\int_M \left( u^{-2} W_{kjth} W_{kjth} - u W_{kjth} W_{kjth} \right) dV
+ \frac{4(n-2)}{n} \{ a + (n-2)b \} \int_M L du K dV
- 4(n-2) \{ a + (n-2)b \} \int_M u P_{ji} P_{ji} dV = 0.
$$

Proof. Using (2.22) and (2.23), we have

$$
\int_M \left( u^{p-3} W_{kjth} W_{kjth} - u^{p+1} W_{kjth} W_{kjth} \right) dV
- 8 \{ a + (n-2)b \} \int_M u^p G_{ji} P_{ji} u^i dV
- 4(n-2) \{ a + (n-2)b \} \int_M u^{p+1} P_{ji} P_{ji} dV = 0.
$$

Substituting (3.26) into (3.36), we have (3.33).

**Lemma 11.** If a compact orientable Riemannian manifold $(M, g)$ of dimension $n \geq 2$ admits a conformal change of metric $g^* = e^{2\phi} g$, then

$$
\int_M \left( u^{-n+1} W_{kjth} W_{kjth} - u^{-n+3} W_{kjth} W_{kjth} \right) dV
+ \frac{4(n-2)}{n} \{ a + (n-2)b \} \int_M u^{-n} L du K^* dV
+ 4(n-2) \{ a + (n-2)b \} \int_M u^{-n+3} P_{ji} P_{ji} dV = 0.
$$

Proof. (3.37) follows from (3.12) and (3.34).
**Lemma 12.** Suppose that a Riemannian manifold \((M, g)\) of dimension \(n \geq 2\) admits a conformal change of metric \(g^* = e^{2\phi}g\) and \(f\) and \(f^*\) are non-negative functions on \(M\) such that

\[
    u^pf = \{u^q + (u^r - 1)\phi\}f^*,
\]

where \(p\) is a real number such that \(p \leq 4\), and \(q\) and \(r\) non-negative numbers and \(\phi\) a non-negative function on \(M\). Then

\[
    (u^{-n-1}f^* - u^{-n+3}f) - (u^{-p}f^* - uf) \geq 0.
\]

**Proof.** We have

\[
    (u^{-n-1}f^* - u^{-n+3}f) - (u^{-p}f^* - uf) = u^{-n-1}(1-u^{n-5})(f^* - uf).
\]

We can easily prove that

\[
    (1-u^{n-5})(1-u^p) \geq 0, \quad (1-u^{n-3})(1-u^r) \geq 0, \quad (1-u^{n-5})(1-u^r) \geq 0,
\]

and consequently that (3.39) holds.

§ 4. **Propositions.**

**Proposition 1.** If a compact Riemannian manifold \((M, g)\) of dimension \(n \geq 2\) admits a non-constant function \(u\) on \(M\), then

\[
    (\nabla_j u_i)(\nabla^j u^i) \geq \frac{1}{n} (Du)^2,
\]

equality holding if and only if \((M, g)\) is conformal to a sphere. If moreover \(L_{du}K = 0\) or \(K = \text{constant}\), then the equality holds if and only if \((M, g)\) is isometric to a sphere.

**Proof.** (4.1) is equivalent to

\[
    (\nabla_j u_i - \frac{1}{n} Du g_{ji})(\nabla^j u^i - \frac{1}{n} Du g^{ji}) \geq 0,
\]

and consequently equality in (4.1) holds if and only if

\[
    \nabla_j u_i - \frac{1}{n} Du g_{ji} = 0,
\]

that is, by Theorem O, if and only if \((M, g)\) is conformal to a sphere. The latter part of this proposition follows from Theorem P.
PROPOSITION 2. If a compact orientable Riemannian manifold \((M, g)\) of dimension \(n \geq 2\) admits a non-constant function \(u\) on \(M\) such that
\[
K^i_i u^i + \frac{n-1}{n} F^h \Delta u = 0,
\]
then \((M, g)\) is conformal to a sphere. If moreover \(L_{du}K = 0\) or \(K = \text{constant}\), then \((M, g)\) is isometric to a sphere.

Proof. From (3.5), we have
\[
g^{ij} P_j u^i - K^i_i u^i - F^h \Delta u = 0.
\]
Adding (4.2) \(\times 2\) and this relation, we have
\[
g^{ij} P_j u^i + K^i_i u^i + \frac{n-2}{n} F^h \Delta u = 0.
\]
Thus, by the Remark to Lemma 1, we see that the vector field \(u^h\) on \(M\) defines an infinitesimal conformal transformation and consequently that
\[
P_j u^i - \frac{1}{n} \Delta u g_{ji} = 0.
\]
Thus, by Theorem O, \((M, g)\) is conformal to a sphere. The latter part of the proposition follows from Theorem P.

PROPOSITION 3. If a compact orientable Riemannian manifold \((M, g)\) of dimension \(n \geq 2\) admits a non-homothetic conformal change of metric \(g^* = e^{2\varphi} g\) such that
\[
K^i_i u^i + \frac{n-1}{n} F^h \Delta u = 0,
\]
then \((M, g)\) is conformal to a sphere. If moreover \(L_{du}K = 0\) or \(K = \text{constant}\), then \((M, g)\) is isometric to a sphere.

Proof. This is an immediate consequence of Proposition 2. But, an another proof is as follows. From (3.14) and (4.2), we have \(P_{ji} = 0\), that is,
\[
P_j u^i - \frac{1}{n} \Delta u g_{ji} = 0,
\]
and consequently, by Theorem O, \((M, g)\) is conformal to a sphere. The latter part of the proposition follows from Theorem P.

PROPOSITION 4. If a compact orientable Riemannian manifold \((M, g)\) of dimension \(n \geq 2\) admits a non-constant function \(u\) on \(M\) such that
\[
\int_M K^j_j u^j u^i dV \geq \frac{n-1}{n} \int_M (\Delta u)^2 dV,
\]
then \((M, g)\) is conformal to a sphere. If moreover \(L_{du}K = 0\) or \(K = \text{constant}\), then \((M, g)\) is isometric to a sphere.
Proof. From (3.8) and (4.3), we have
\[ \nabla_j u_i - \frac{1}{n} \Delta u g_{ji} = 0, \]
and consequently, by Theorem O, \((M, g)\) is conformal to a sphere. The latter part of the proposition follows from Theorem P.

**Proposition 5.** If a compact orientable Riemannian manifold \((M, g)\) of dimension \(n \geq 2\) admits a non-homothetic conformal change of metric \(g^* = e^{2\phi} g\) such that
\[ \int_M K_{ji} u^i u^j dV \geq \frac{n-1}{n} \int_M (\Delta u)^2 dV, \]
then \((M, g)\) is conformal to a sphere. If moreover \(L_{au}K = 0\) or \(K = \text{constant}\), then \((M, g)\) is isometric to a sphere.

Proof. This is an immediate consequence of Proposition above. But, we can give an another proof. From (3.13) and the above relation, we find \(P_{ji} = 0\), that is,
\[ \nabla_j u_i - \frac{1}{n} \Delta u g_{ji} = 0, \]
and consequently, by Theorem O, \((M, g)\) is conformal to a sphere. The latter part of the proposition follows from Theorem P.

(For Propositions 2~5, see Yano and Hiramatu [12].)

**Proposition 6.** If a compact orientable Riemannian manifold \((M, g)\) of dimension \(n \geq 2\) admits a non-homothetic conformal change of metric \(g^* = e^{2\phi} g\) such that
\[ \int_M u^{-n+1} G_{ji} u^i u^j dV + \frac{1}{2n} \int_M (u^{-n} L_{au} K^* - u^{-n+2} L_{au} K) dV \geq 0, \]
then \((M, g)\) is conformal to a sphere. If moreover \(L_{au}K = 0\) or \(K = \text{constant}\), then \((M, g)\) is isometric to a sphere.

Proof. By using (3.12) and (4.4), we have \(P_{ji} = 0\), that is,
\[ \nabla_j u_i - \frac{1}{n} \Delta u g_{ji} = 0, \]
and consequently, by Theorem O, \((M, g)\) is conformal to a sphere. We have the latter part of the proposition by Theorem P.

The latter part of the proposition above is a generalization of Theorems A and H.

**Proposition 7.** If a compact orientable Riemannian manifold \((M, g)\) of dimension \(n > 2\) admits a non-homothetic conformal change of metric \(g^* = e^{2\phi} g\) such that
\( \int_M \left( u^{n-3} G^{*j} G^{*j} - u G^{*j} G^{*j} \right) dV + \frac{(n-2)^2}{n} \int_M L_{d_u} K dV \leq 0, \)

then \((M, g)\) is conformal to a sphere. If moreover \(L_{d_u} K = 0\) or \(K = \text{constant}\), then \((M, g)\) is isometric to a sphere.

**Proof.** By using (3.24) and (4.5), we have \( P_j = 0 \), that is,

\[ \sum_{j} \Delta u \chi_j = 0, \]

and consequently, by Theorem O, \((M, g)\) is conformal to a sphere. Using Theorem P, we can prove the latter part of the proposition.

The first part of Proposition 7 is a generalization of Theorem B because of

\[ \int_M (\Delta u) K dV = -\int_M L_{d_u} K dV, \]

and the latter part a generalization of Theorem C.

**Proposition 8.** If a compact orientable Riemannian manifold \((M, g)\) of dimension \(n > 2\) admits a non-homothetic conformal change of metric \(g^* = e^{\varphi} g\) such that

\[ \int_M \left( u^{n-3} G^{*j} G^{*j} - u G^{*j} G^{*j} \right) dV + \frac{(n-2)^2}{n} \int_M u^{-n} L_{d_u} K^* dV \geq 0, \]

then \((M, g)\) is conformal to a sphere. If moreover \(L_{d_u} K = 0\) or \(K = \text{constant}\), then \((M, g)\) is isometric to a sphere.

**Proof.** This follows from (3.27) and Theorems O and P.

**Proposition 9.** If a compact orientable Riemannian manifold \((M, g)\) of dimension \(n > 2\) admits a non-homothetic conformal change of metric \(g^* = e^{\varphi} g\) such that

\[ \int_M \left( u^{-3} Z^{*k} Z^{*k} - u Z^{*j} Z^{*j} \right) dV + \frac{4(n-2)}{n} \int_M u^{-n} L_{d_u} K^* dV \leq 0, \]

then \((M, g)\) is conformal to a sphere. If moreover \(L_{d_u} K = 0\) or \(K = \text{constant}\), then \((M, g)\) is isometric to a sphere.

**Proof.** This follows from (3.30) and Theorems O and P.

The first part of this proposition is a generalization of Theorem D and the latter part is a generalization of Theorem E.

**Proposition 10.** If a compact orientable Riemannian manifold \((M, g)\) of dimension \(n > 2\) admits a non-homothetic conformal change of metric \(g^* = e^{\varphi} g\) such that

\[ \int_M \left( u^{-n-1} Z^{*k} Z^{*k} - u^{-n+3} Z^{*j} Z^{*j} \right) dV + \frac{4(n-2)}{n} \int_M u^{-n} L_{d_u} K^* dV \geq 0, \]
then \((M, g)\) is conformal to a sphere. If moreover \(L_{da}K = 0\) or \(K = \text{constant}\), then \((M, g)\) is isometric to a sphere.

**Proof.** This follows from (3.32) and Theorems O and P.

**Proposition 11.** If a compact orientable Riemannian manifold \((M, g)\) of dimension \(n > 2\) admits a non-homothetic conformal change of metric \(g^* = e^{2u}g\) such that

\[
\int_M (u^{-n} W^*_{k,j} W^*_{k,j} - u W^*_{k,j} W^*_{k,j}) dV
+ \frac{4(n-2)}{n} \{a + (n-2)b\}^2 \int_M L_{da}K dV \leq 0
\]

then \((M, g)\) is conformal to a sphere. If moreover \(L_{da}K = 0\) or \(K = \text{constant}\), then \((M, g)\) is isometric to a sphere.

**Proof.** This follows from (3.35) and Theorems O and P.

The first part of Proposition 11 generalizes Theorem F and the latter part generalizes Theorem G.

**Proposition 12.** If a compact orientable Riemannian manifold \((M, g)\) of dimension \(n > 2\) admits a non-homothetic conformal change of metric \(g^* = e^{2u}g\) such that

\[
\int_M (u^{-n-2} W^*_{k,j} W^*_{k,j} - u^{-n+3} W^*_{k,j} W^*_{k,j}) dV
+ \frac{4(n-2)}{n} \{a + (n-2)b\}^2 \int_M u^{-n} L_{da}K^* dV \geq 0
\]

then \((M, g)\) is conformal to a sphere. If moreover \(L_{da}K = 0\) or \(K = \text{constant}\), then \((M, g)\) is isometric to a sphere.

**Proof.** This follows from (3.37) and Theorems O and P.

**Proposition 13.** If a compact orientable Riemannian manifold \((M, g)\) of dimension \(n > 2\) admits a non-homothetic conformal change of metric \(g^* = e^{2u}g\) such that

\[
u_p G_{ji} = \{u^q + (u^p-1)\varphi\} G^{*}_{ji} G^{*}_{ji}
\]

and

\[
\int_M (u^{-n} L_{da}K^* - L_{da}K) dV \geq 0
\]

where \(p\) is a real number such that \(p \leq 4\), \(q\) and \(r\) non-negative numbers and \(\varphi\) a
non-negative function on $M$, then $(M, g)$ is conformal to a sphere. If moreover $L_{du}K=0$ or $K=\text{constant}$, then $(M, g)$ is isometric to a sphere.

Proof. Subtracting (3.24) from (3.27), we obtain

\begin{equation}
\int_M \left\{ \left( u^{n-1}G_{ji}G^{*ji} - u^{-n+3}G_{ji}G^{ji} \right) - \left( u^{-3}G_{ji}G^{*ji} - uG_{ji}G^{ji} \right) \right\} dV \\
\quad + \frac{(n-2)^2}{n} \int_M (u^nL_{du}K* - L_{du}K) dV \\
\quad + (n-2)^2 \int_M (u^{-n} + u) P_{ji}P^{ji} dV = 0.
\end{equation}

By Lemma 12, from (4.11), (4.12) and (4.13), we have $P_{ji} = 0$, that is,

$$P_{ji} = -\frac{1}{n} f_{ij} = 0$$

and consequently, by Theorem O, $(M, g)$ is conformal to a sphere. By using Theorem P, we can prove the latter part of this proposition.

The latter part of Proposition 13 is a generalization of Theorem L.

**Corollary 1.** If a compact orientable Riemannian manifold $(M, g)$ of dimension $n>2$ admits a non-homothetic conformal change of metric $g^* = e^{2p}g$ such that

\begin{equation}
G_{ji}G^{*ji} = G_{ji}G^{*ji}
\end{equation}

and

$$\int_M (u^{-n}L_{du}K* - L_{du}K) dV \geq 0,$$

then $(M, g)$ is conformal to a sphere. If moreover $L_{du}K = 0$ or $K = \text{constant}$, then $(M, g)$ is isometric to a sphere.

Proof. Putting $p=q=r=0$ in (4.11), we have (4.14), and consequently this corollary follows immediately from Proposition 13.

The latter part of this corollary is a generalization of Theorem I.

**Proposition 14.** If a compact orientable Riemannian manifold $(M, g)$ of dimension $n>2$ admits a non-homothetic conformal change of metric $g^* = e^{2p}g$ such that

\begin{equation}
u^p Z_{kjsh}Z^{*kjsh} = \{u^p + (u^r - 1)\varphi\} Z_{kjsh}Z^{*kjsh}
\end{equation}

and

$$\int_M (u^{-n}L_{du}K* - L_{du}K) dV \geq 0,$$

where $p, q, r$ and $\varphi$ are the same as in Proposition 13, then $(M, g)$ is conformal to a sphere. If moreover $L_{du}K = 0$ or $K = \text{constant}$, then $(M, g)$ is isometric to a
sphere.

**Proof.** Subtracting (3.30) from (3.32), we have

\begin{equation}
(4.16)\quad \int_M \left( (u^{-n+1} Z_{kfl} Z_{k^{jfl}} - u^{-n+1} Z_{jfl} Z^{k^{jfl}}) 
- (u^{-3} Z_{kfl} Z_{k^{jfl}} - u Z_{jfl} Z^{k^{jfl}}) \right) dV
\end{equation}

\[ + \frac{4(n-2)}{n} \int_M (u^{-n} L_d K^* - L_d u K) dV \]

\[ + 4(n-2) \int_M (u^{-n+3} + u) P_{ji} P_{ji} dV = 0. \]

Using Lemma 12, (4.12), (4.15) and (4.16), we have \( P^i = 0 \), that is,

\[ \nabla_j u_i - \frac{1}{n} \Delta u g_{ji} = 0, \]

and consequently, by Theorem O, \((M, g)\) is conformal to a sphere. By using Theorem P, we can prove the latter part of the proposition.

The latter part of Proposition 14 is a generalization of Theorem M.

**COROLLARY 2.** If a compact orientable Riemannian manifold \((M, g)\) of dimension \(n > 2\) admits a non-homothetic conformal change of metric \(g^* = e^{2\varphi} g\) such that

\begin{equation}
(4.17)\quad Z_{kfl} Z_{k^{jfl}} = Z_{jfl} Z^{k^{jfl}}
\end{equation}

and

\[ \int_M (u^{-n} L_d K^* - L_d u K) dV \geq 0, \]

then \((M, g)\) is conformal to a sphere. If moreover \(L_d K = 0\) or \(K = \text{constant}\), then \((M, g)\) is isometric to a sphere.

**Proof.** Putting \(p = q = r = 0\) in (4.15), we get (4.17), and consequently Corollary 2 follows immediately from Proposition 14.

The latter part of Corollary 2 generalizes Theorem J.

**PROPOSITION 15.** If a compact orientable Riemannian manifold \((M, g)\) of dimension \(n > 2\) admits a non-homothetic conformal change of metric \(g^* = e^{2\varphi} g\) such that

\begin{equation}
(4.18)\quad u^p W_{kfl} W_{k^{jfl}} = \{u^q + (u^r - 1) \varphi\} W_{kfl} W_{k^{jfl}},
\end{equation}

\[ a + (n-2)b \neq 0 \]

and

\[ \int_M (u^{-n} L_d K^* - L_d u K) dV \geq 0, \]
where \( p, q, r \) and \( \varphi \) are the same as in Proposition 13, then \((M, g)\) is conformal to a sphere. If moreover \( L_{du}K=0 \) or \( K=\text{constant} \), then \((M, g)\) is isometric to a sphere.

**Proof.** Subtracting (3.35) from (3.37), we have

\[
(4.19) \quad \int_M \left( (u-^{n-1}W^*_{kjih}W^{kjih} - u^{-n+3}W_{kjih}W^{kjih}) \right. \\
- (u^{-3}W^*_{kjih}W^{kjih} - uW_{kjih}W^{kjih}) \left) dV \\
+ \frac{4(n-2)}{n} \left\{ a + (n-2)b \right\} \int_M (u^{-n}L_{du}K* - L_{du}K) dV \\
+ 4(n-2) \left\{ a + (n-2)b \right\} \int_M (u^{-n}+u) P_jP^j dV = 0.
\]

By using Lemma 12, from (4.12), (4.18) and (4.19), we have \( P_{ji}=0 \), that is,

\[
P_{ji} = 0,
\]

and consequently, by Theorem O, \((M, g)\) is conformal to a sphere. By using Theorem P, we can prove the latter part of Proposition 15.

The latter part of Proposition 15 is a generalization of Theorem N.

**COROLLARY 3.** If a compact orientable Riemannian manifold \((M, g)\) of dimension \( n>2 \) admits a non-homothetic conformal change of metric \( g*=e^{2\rho}g \) such that

\[
(4.20) \quad W^*_{kjih}W^{kjih} = W_{kjih}W^{kjih}, \quad a + (n-2)b = 0
\]

and

\[
\int_M (u^{-n}L_{du}K* - L_{du}K) dV \geq 0,
\]

then \((M, g)\) is conformal to a sphere. If moreover \( L_{du}K=0 \) or \( K=\text{constant} \), then \((M, g)\) is isometric to a sphere.

**Proof.** Putting \( p=q=r=0 \) in (4.18), we get (4.20), and consequently Corollary 3 follows immediately from Proposition 15.

The latter part of Corollary 3 is a generalization of Theorem K.

**BIBLIOGRAPHY**


