ON THE NORMALIZATION OF BI-QUADRATIC FORM

By Masato UENO

(Communicated by Y. Komatsu)

We know that any quadratic form can be normalized by an orthogonal transformation. We now investigate whether a bi-quadratic form may be normalized or not, and if it is possible we attempt to find out what are conditions necessary and sufficient.

Let a bi-quadratic form be given:

\[ f(x, y) = \sum_{i+j=k} c_{ij} x_i y_j \]

summation being taken with respect to \( i, j \) and \( k \) and

\[ c_{ij} = c_{ji} = c_{ij} \]

being real numbers.

Our present problem is to see that the above form can be transformed into

\[ g(x', y') = \sum_{i+j=k} c_{ij} x'_ix'_j \]

by transformations

\[ x_i = \sum x'_i \alpha_i \]

where the determinants \( |\alpha_i| \) and \( |\beta_{ij}| \) are both different from zero.

We take the matrix of degree \( n^2 \) of coefficients, of transformation and of the normalized form respectively

\[ C = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{pmatrix} \]

\[ R = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{pmatrix} \]

\[ D = \begin{pmatrix} \varepsilon_1 & 0 & \cdots & 0 \\ 0 & \varepsilon_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \varepsilon_n \end{pmatrix} \]

then it must be

\[ R^* C R = D \]

\( R^* \) denoting the complementary matrix of \( R \). In this case \( R \) is the Kronecker products of \( P \) and \( Q \) where \( P = (\alpha_{ij}) \) and \( Q = (\beta_{ij}) \) are respectively \( x_i \)'s and \( y_j \)'s transformation matrices, i.e. \( R = P \times Q \).

Therefore, the problem of normalization of bi-quadratic form is reduced to normalization of matrix.

We next consider that the matrices are divided into \( n^2 \) small matrices of degree \( n \):

\[ C_{ab} = \begin{pmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{nn} \end{pmatrix}, \quad R_{ab} = \begin{pmatrix} r_{11} & \cdots & r_{1n} \\ \vdots & \ddots & \vdots \\ r_{n1} & \cdots & r_{nn} \end{pmatrix} \]

\[ D_{ab} = \begin{pmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{pmatrix} \]

so that

\[ C = \begin{pmatrix} C_{11} & \cdots & C_{nn} \end{pmatrix}, \quad R = \begin{pmatrix} R_{11} & \cdots & R_{nn} \end{pmatrix}, \quad R^* = \begin{pmatrix} R_{11}^* & \cdots & R_{nn}^* \end{pmatrix}, \quad D = \begin{pmatrix} D_{11} & 0 \\ 0 & D_{22} \end{pmatrix} \]

and

\[ \begin{pmatrix} R_{11}^* & R_{12}^* \\ R_{21}^* & R_{22}^* \end{pmatrix} \begin{pmatrix} C_{11} & \cdots & C_{nn} \end{pmatrix} \begin{pmatrix} R_{11} & \cdots & R_{nn} \end{pmatrix} \]

\[ = \begin{pmatrix} D_{11} & 0 \\ 0 & D_{22} \end{pmatrix} \]

where \( x, y = \sum_{n} \alpha_{ij} x_i y_j \).
It then follows that
\[ \sum_{\mu=1}^{n} R_{\mu}^* C_{\mu} R_{\mu} = D, \]
\[ = D, \quad (\forall \mu), \]
consequently
\[ \sum_{\mu=1}^{n} P_{\mu}^* C_{\mu} P_{\mu} \gamma_{\mu} \gamma_{\mu} \begin{bmatrix} \alpha \end{bmatrix} = \circ, \quad (\forall \mu), \]
\[ \sum_{\mu=1}^{n} P_{\mu}^* C_{\mu} P_{\mu} \gamma_{\mu} \gamma_{\mu} = D. \]
Moreover, it becomes
\[ P_{\mu}^* C_{\mu} P = F_{\mu}, \quad (1) \]
where $F_{\mu}$ is a diagonal matrix for all $\mu$ and $\nu$.

Proof. Let the element of $l$th row and $m$th column of $P_{\mu}^* C_{\mu} P$ be $x_{l,m}^{l,m}$, then it follows that if $l \neq m$,
\[ \sum_{\mu=1}^{n} \gamma_{\mu} \gamma_{\mu} x_{l,m}^{l,m} = \circ \]
for all $i$ and $\Psi$;
since the determinant of the coefficients $\gamma_{\mu} \gamma_{\mu}$ does not vanish, $x_{l,m}^{l,m}$ must be zero for all $\mu$ and $\nu$. In other words, it is necessary that all the $C_{\mu}$ are transformed into diagonal matrices by the same matrix $P$.

Corollary. If several symmetric matrices $A$, $B$, $C$, $\ldots$, of the same degree are transformed by the same orthogonal matrix into diagonal matrices simultaneously, then $A$, $B$, $C$, $\ldots$ are commutable.

Conversely, if matrices $A$, $B$, $C$, $\ldots$ are commutable then there exists an orthogonal matrix $P$ which transforms all the matrices into diagonal matrices.

Proof. Let $P$ be an orthogonal matrix, $P^* = P^{-1}$, such that
\[ P_1^* A P = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}, \quad P_1^* B P = \begin{bmatrix} \beta & 0 \\ 0 & \beta \end{bmatrix}, \quad P_1^* C P = \begin{bmatrix} \gamma & 0 \\ 0 & \gamma \end{bmatrix}, \]
Then, we have
\[ P_1^* A P \begin{bmatrix} \beta & 0 \\ 0 & \beta \end{bmatrix} P = P_1^* B P P_1^{-1} A P, \]
that is
\[ AB = BA \]
and so on.

Conversely, if $AB = BA$, $AC = CA$, $BC = CB$, then there exists an orthogonal matrix $P$ transforming $A$ into a diagonal matrix:
\[ P_1^* A P = \begin{bmatrix} \alpha \end{bmatrix}, \quad (\forall \mu), \]
From the assumption, it must be
\[ P_1^* B P = P_1^* B P (\alpha), \]
Without loss of generality we may assume $\alpha = \alpha_1 = \ldots = \alpha_t$, $\alpha = \alpha_r = \ldots = \alpha_t$, and hence $P_1^* B P$ must be of the form
\[ \begin{bmatrix} B_1 & B_2 & 0 \\ 0 & \end{bmatrix}, \]
$B_1$, $B_2$, $\ldots$, being of degree $t$, $s$, $\ldots$, respectively. Since $B_1$, $B_2$, $\ldots$ are also symmetric, we can take orthogonal matrices $P_1$, $P_2$, $\ldots$, such that
\[ P_1^* B_1 P_1 = \begin{bmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{bmatrix}, \quad P_2^* B_2 P_2 = \begin{bmatrix} \beta_r & 0 \\ 0 & \beta_r \end{bmatrix}, \]
If we put
\[ \begin{bmatrix} P_1^* B_1 P_1 & 0 \\ 0 & \end{bmatrix} = Q, \]
then
\[ Q_1^* P_1^* B_1 P_1 Q = (\beta_1), \]
and, of course, $Q_1^* P_1^* B_1 P_1 Q = (\alpha)$. If we assume
\[ \beta_1 = \beta_1', \beta_2 = \beta_2', \ldots, \beta_r = \beta_r', \ldots, \]
then $Q_1^* P_1^* B_1 P_1 Q$ must be of the form
\[ \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix}, \]
By continuing this operation, all the matrices $A$, $B$, $C$, $\ldots$ can be transformed by the matrix $R = PQ$, into diagonal matrices.

Theorem. A necessary and sufficient condition that the bi-
quadratic form \( f(x, y) \) can be normalized by two orthogonal transformations of \( x \) and \( y \) is

1) \( C_{\mu\nu} \) are mutually commutable;

and

11) \( C'_{\mu\nu} \) are also mutually commutable,

\( C_{\mu\nu} \) being a small matrix of degree \( n \) in the \( \mu \) th row and \( \nu \) th column contained in the coefficient matrix \( C \), and \( C'_{\mu\nu} \) the corresponding one contained in the coefficient matrix \( C' \) whose constitution is as follows:

\[
C' = \begin{pmatrix}
C_{11} & C_{12} & \cdots & C_{1n} \\
C_{21} & C_{22} & \cdots & C_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
C_{n1} & C_{n2} & \cdots & C_{nn}
\end{pmatrix}
\]

Proof. It is evident that the condition is necessary. We shall show that it is also sufficient.

In view of i) there exists a matrix \( P \) such that

\[
P^T C_{\mu\nu} P = \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\
C_{21} & C_{22} & \cdots & C_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
C_{n1} & C_{n2} & \cdots & C_{nn}
\end{pmatrix}
\]

where the \( F_{i\sigma} \) are all diagonal matrices. Now,

\[
F = \begin{pmatrix} F_{11} & \cdots & F_{1n} \\
\vdots & \ddots & \vdots \\
F_{n1} & \cdots & F_{nn}
\end{pmatrix}
\]

can be transformed by a proper orthogonal matrix \( R \) into a matrix of the form

\[
G = \begin{pmatrix} G_1 & 0 \\
0 & G_n
\end{pmatrix}
\]

where \( G_i \) are all symmetric. Let \( G_j = (f_{ij}^{''}) \); \( f_{ij}^{''} \) representing the element of \( i \) th row and \( j \) th column.

If all the \( G_i \) could not be transformed simultaneously by an orthogonal matrix into diagonal matrices, then there exist a pair of matrices \( G_j \) and \( G_j \) being not commutable.

\[
G_j G_j^T \neq G_j G_j^T
\]

It follows

\[
\sum_i f_{ij}^{''} f_{ij}^{''} = \sum_i f_{ij}^{''''} f_{ij}^{''''}
\]

for some \( i, j \).

Since

\[
f_{ij}^{''''} = \sum_{r} p_{ij} C_{\mu\nu} p_{rj}
\]

and so on;

\[
\sum_{r} \left( \sum_{\mu} p_{ij} C_{\mu\nu} p_{rj} \right) (\sum_{\nu} p_{ij} C_{\mu\nu} p_{rj})
\]

that is

\[
\sum_{r} p_{ij} p_{ij} p_{ij} p_{ij} \sum_{\mu} C_{\mu\nu} C_{\mu\nu}
\]

Since \( C_{\mu\nu} = C'_{\mu\nu} \), where \( C'_{\mu\nu} \) is an element of \( i \) th row and \( \mu \) th column of \( C' \), and since \( C'_{\mu\nu} C'_{\mu\nu} = C'_{\mu\nu} C'_{\mu\nu} \), the condition ii) implies that

\[
\sum_{\mu} C_{\mu\nu} C_{\mu\nu} C_{\mu\nu} = \sum_{\mu} C_{\mu\nu} C_{\mu\nu} C_{\mu\nu}
\]

for all \( i, j, k, \mu, \rho, \sigma \).

This contradicts to the above inequality, and the proof is completed.

Example.

\[
f(x, y) = 4 x_1^2 y_1^2 + 9 x_2^2 y_2^2 + 1 x_3^2 y_3^2 + 20 x_1^2 y_1^2 + 11 x_1^2 y_1^2 + 4 x_3^2 y_3^2 + 12 x_2^2 y_2^2 + 27 x_2^2 y_2^2 + 7 x_3^2 y_3^2
\]

- 47 -


\[
\begin{pmatrix}
4 & 2 & -4 \\
2 & 19 & -6 \\
-14 & -16 & 13
\end{pmatrix}, \quad \begin{pmatrix}
20 & -2 & 8 \\
-2 & 11 & 10 \\
8 & 10 & 14
\end{pmatrix}, \quad \begin{pmatrix}
12 & -6 & -18 \\
-6 & 27 & -12 \\
-18 & -12 & 15
\end{pmatrix},
\]

\[
\begin{pmatrix}
-1 & 4 & 8 \\
4 & 7 & 4 \\
8 & 4 & -1
\end{pmatrix}, \quad \begin{pmatrix}
-5 & 8 & 10 \\
8 & -11 & 2 \\
10 & 2 & -2
\end{pmatrix},
\]

\[
\begin{pmatrix}
-5 & 4 & -20 \\
-4 & 13 & -4 \\
20 & -16 & 1
\end{pmatrix}.
\]

These are all commutable each other, and

\[
\begin{pmatrix}
4 & -1 & -5 \\
-1 & 20 & -5 \\
-5 & -5 & 12
\end{pmatrix}, \quad \begin{pmatrix}
-16 & 4 & 2 \\
4 & 10 & -16 \\
2 & -16 & 12
\end{pmatrix}
\]

are also commutable. Therefore, the normalization must be possible. In fact, it takes place as follows: From

\[
\begin{pmatrix}
4 & -1 & -5 \\
2 & 19 & -6 \\
-14 & -16 & 13
\end{pmatrix} = \mathbf{0}, \quad \lambda = \pm 9, \; 36 ;
\]

\[
P = \begin{pmatrix}
\frac{1}{3} & -\frac{1}{2} & \frac{2}{3} \\
\frac{1}{3} & \frac{5}{6} & \frac{1}{3} \\
-\frac{2}{3} & \frac{5}{6} & \frac{1}{3}
\end{pmatrix}.
\]

\[
\lambda = \lambda_1, \lambda_2, \lambda_3 \quad (1 < \lambda_1 < 2, \; 12 < \lambda_2 < 13, \; 22 < \lambda_3 < 23) ;
\]

\[
Q = \begin{pmatrix}
-5\lambda_1 + 105 & -5\lambda_1 + 105 & -5\lambda_1 + 105 \\
(5(7\lambda_1^2 - 192\lambda_1 + 2286) - 2\lambda_1\lambda_2 + 192\lambda_2 - 192\lambda_1^2) & (5(7\lambda_1^2 - 192\lambda_1 + 2286) - 2\lambda_1\lambda_2 + 192\lambda_2 - 192\lambda_1^2) & (5(7\lambda_1^2 - 192\lambda_1 + 2286) - 2\lambda_1\lambda_2 + 192\lambda_2 - 192\lambda_1^2) \\
-5\lambda_1 + 105 & -5\lambda_1 + 105 & -5\lambda_1 + 105 \\
-5\lambda_1 + 105 & -5\lambda_1 + 105 & -5\lambda_1 + 105 \\
\lambda_1^2 - 24\lambda_1 + 77 & \lambda_1^2 - 24\lambda_1 + 77 & \lambda_1^2 - 24\lambda_1 + 77
\end{pmatrix}
\]

\[
\begin{pmatrix}
\mathbf{F}_0(\lambda_1) & \mathbf{F}_2(\lambda_1) & \mathbf{F}_4(\lambda_1) \\
\mathbf{F_0}(\lambda_2) & \mathbf{F}_2(\lambda_2) & \mathbf{F}_4(\lambda_2) \\
\mathbf{F_0}(\lambda_3) & \mathbf{F}_2(\lambda_3) & \mathbf{F}_4(\lambda_3)
\end{pmatrix}
\]

where

\[
\mathbf{F}_0(\lambda) = \frac{\lambda}{\lambda^4 + 17\lambda + 46}, \quad \mathbf{F}_2(\lambda) = \frac{\lambda}{\lambda^4 + 7\lambda - 102}, \quad \mathbf{F}_4(\lambda) = \frac{1}{\lambda^4 - 9\lambda + 14}.
\]

Remarks.

1. We can apply the theorem to cases:

1) \( n \) variables \( x \) and \( n' \) variables \( y \), \( n \) and \( n' \) being different;

2) poly-quadratic form, for example;

\[
\sum_{i,j} C_{ij} x_i y_j = \mathbf{0} \quad (x, y, z, \ldots);
\]

3) Hermitian form of complex coefficients satisfying

\[
C_{ij} = \overline{C_{ji}} = \overline{C_{ij}} = \overline{C_{ij}}.
\]

2. It seems to be difficult to find conditions for the case that the form may be normalized by non-orthogonal transformations whose determinants are not extinguished.

3. A condition for a form to be positive-definite, that is, all \( C_{ij} > 0 \) is given by,

\[
\begin{cases}
C_{ii} > 0, & C_{ii} > 0, \\
|C_{ij}| > 0.
\end{cases}
\]

(*) Received June 9, 1951.

Musashi University, Tokyo.