If there exists a homomorphism of a semigroup $\mathcal{S}$ onto a semigroup $\mathcal{S}'$, $\mathcal{S}$ is decomposed into the class sum of mutually disjoint subsets $\{\mathcal{S}_i\}_{i \in \mathcal{S}'}$, each of which is a inverse image of some element $x_i$ of $\mathcal{S}'$; i.e., $\mathcal{S} = \bigoplus \mathcal{S}_i$. ( $\bigoplus$ is meant the direct sum of sets.) In this case, clearly $\mathcal{S}_i$ forms a factor algebraic system of $\mathcal{S}$ and is isomorphic to $\mathcal{S}'$. We call such a partition of $\mathcal{S}$ a decomposition of $\mathcal{S}$ to $\mathcal{S}'$, and each $\mathcal{S}_i$ a residue class of this decomposition. The decomposition to a semilattice is most important among others; i.e., $\mathcal{S}' = \mathcal{S}_1 \cup \mathcal{S}_2$, where every $\mathcal{S}_i$ is a semigroup and for any $\mathcal{S}_i$, $\mathcal{S}_j$, there exists a unique $\mathcal{S}_k$ such that $\mathcal{S}_j \mathcal{S}_k \subset \mathcal{S}_i$ and $\mathcal{S}_k \mathcal{S}_j \subset \mathcal{S}_i$. Henceforward we will call the decomposition of $\mathcal{S}$ to a semilattice the semilattice decomposition of $\mathcal{S}$.

Generally there exist many semilattice decompositions of a semigroup, but since it is proved that the collection of all semilattice decompositions of a semigroup forms a complete semilattice, there exists the greatest one.

In this paper, we shall determine the greatest semilattice decomposition of a semigroup. T. Tamura and N. Kimura showed that such the decomposition of a commutative semigroup is determined as the decomposition to the factor algebraic system under a congruence relation $(\sim_0$ introduced as follows (1).

$$a \sim_0 b \text{ if } a^n = x \text{ and } b^m = x \text{ for some positive integers } n, m \text{ and some elements } x, y.$$ 

In this paper, the author deals with general case. To abbreviate the terminology, from now on, $\mathcal{S}$ denotes a general semigroup and the symbol $\exists$ denotes the word 'exist'. Hence if we describe as $\exists x$; $\ldots$, it means that there exists an element $x$ which satisfies the relation $\ldots$.

### 31 Semilattice decomposition

If we define $a \sim b$ between elements $a, b$ of $\mathcal{S}$ to mean that $a, b$ are contained in a same residue class of a semilattice decomposition of $\mathcal{S}$, then the relation $(\sim)$ is a congruence relation of $\mathcal{S}$ which satisfies the following two conditions:

1. $a^2 \sim a$ for any $a \in \mathcal{S}$.
2. $ab \sim ba$ for any $a, b \in \mathcal{S}$.

Moreover this converse also holds good; i.e.,

Lemma 1. If a congruence relation $(\sim)$ which satisfies two conditions (1), (2) is defined on $\mathcal{S}$, then the factor algebraic system of $\mathcal{S}$ under the relation $(\sim)$ forms a semilattice.

Proof. Obvious by the definition of the congruence relation.

We turn our attention to a subsemigroup $\mathcal{S}'$ of $\mathcal{S}$ which has the following property (P):

(P) For any number of elements $a_1, a_2, \ldots, a_m \in \mathcal{S}'$, $\mathcal{S}' a_1 a_2 \ldots a_m \triangleq \{ \xi \in \mathcal{S}' \mid \text{there exist }\xi_1, \xi_2, \ldots, \xi_k \in \mathcal{S}' \text{ which satisfy the relation } \{\xi_1, \xi_2, \ldots, \xi_k\} = \{a_1 a_2 \ldots a_m\} \}$.

We call such a subsemigroup to be a $P$-subsemigroup of $\mathcal{S}$.

Lemma 2. If $\mathcal{S}'$ is a $P$-subsemigroup of $\mathcal{S}$, then

1. $x y \in \mathcal{S}'$ implies $y x \in \mathcal{S}'$ for any $x, y \in \mathcal{S}'$.
2. $x^k \in \mathcal{S}'$ implies $x \in \mathcal{S}'$ for any $x \in \mathcal{S}$ and for any positive integer $k$.

Proof. Since $\mathcal{S}'$ has the property (P), if we set $w = x_1 \ldots x_n$, $a_i = x_i$, $a_i = y_i$, $x_1 = y_1$ and $y_i = x_i$ in the above-mentioned property (P) the first part of this Lemma is proved. Similarly if we set $\mu = K$, $x = 1$, $a_i = x_i$, $y_1 = x_1$, $\ldots$, $K$ and $\xi_1 = x$ the second part follows.

We denote by $\mathcal{S}_\infty$ the collection of all $P$-subsemigroups of $\mathcal{S}$, and by $\mathcal{S}_1$, $\mathcal{S}_2$, $\ldots$, etc. elements of $\mathcal{S}_\infty$, that is, $P$-subsemigroups of $\mathcal{S}$, we introduce by a subcollection $\Gamma$ of $\mathcal{S}_\infty$ the following relation ($\mathcal{S}_\infty$), which is
closely related to one defined by Pierce (2).

If \( \{ x, y \} | x, y \in S, x \neq y \) for every element \( a \in S \), then \( a \sim b \) in \( S \).

It is easy to see \( (\bar{\Gamma}) \) to be an equivalence relation of \( S \).

Lemma 3. \( (\bar{\Gamma}) \) is congruence relation, and the factor algebraic system of \( S \) under \( (\bar{\Gamma}) \) forms a semilattice. Therefore, \( (\bar{\Gamma}) \) gives one of semilattice decompositions of \( S \).

Proof. We first show that \( a \sim b \) implies \( a \sim c \) as well as \( a \sim b \). Let \( \bar{\Gamma} \) be any element of \( \Gamma \). Then \( x \bar{a} (c, y) = x \bar{a} (R, y) \) implies \( x \bar{a} (c, y) = x \bar{a} (R, y) \). Conversely, \( x \bar{a} (c, y) = x \bar{a} (R, y) \) implies \( x \bar{a} (c, y) = x \bar{a} (R, y) \). Hence \( \{ x \bar{a} (c, y) = x \bar{a} (R, y) \} \) for any \( x \) and \( y \), and this implies \( a \sim c \). Similarly \( a \sim b \) is easy to prove. Next if \( a \sim b \) and \( b \sim c \) are assumed, then \( a \sim c \) follows. Hence \( a \sim b \) by transitivity. This implies \( (\bar{\Gamma}) \) to be a congruence relation. Since \( \bar{\Gamma} \) is a P-subsemigroup of \( S \), \( x \bar{a} (c, y) = x \bar{a} (R, y) \) is equivalent to \( x \bar{a} (c, y) = x \bar{a} (R, y) \) respectively. Therefore, \( a \sim b \) and \( b \sim c \). Accordingly, the remainder of Lemma follows from Lemma 1.

Lemma 4. Any semilattice decomposition of \( S \) is the decomposition to the factor algebraic system of \( S \) under a congruence relation \( (\bar{\Gamma}) \) introduced by some subcollection \( \Gamma \) of \( \Omega \).

Proof. Let \( S = \bigcup \mathcal{D}_k \) be a semilattice decomposition of \( S \). Since it is not hard to verify that each \( \mathcal{D}_k \) is a P-subsemigroup of \( S \), we set \( \Gamma = \{ \mathcal{D}_k \} \). Take up any two elements \( a, b \in S \) contained in a same residue class \( \mathcal{D}_k \). Then \( x \bar{a} (c, y) \) is a subsemigroup of \( S \), \( a \bar{c} (c, y) \) and \( a \bar{c} (R, y) \) are equivalent to \( x \bar{a} (c, y) \) respectively. Therefore, \( a \bar{c} (c, y) \) follows from the definition of \( (\bar{\Gamma}) \). On the other hand if \( a \sim b \), then \( a \bar{c} (c, y) \) and \( b \bar{c} (c, y) \) hold good because of \( a \bar{c} (c, y) \) and \( b \bar{c} (c, y) \) respectively. As \( \mathcal{D}_k \) and \( \mathcal{D}_l \) are a residue class such that it contains an element \( a \) or \( b \), respectively. As \( \mathcal{D}_k \) and \( \mathcal{D}_l \) are P-subsemigroups of \( S \), \( a \bar{c} (c, y) \) and \( b \bar{c} (c, y) \) are contained in both \( \mathcal{D}_k \) and \( \mathcal{D}_l \). Hence \( a \bar{c} (c, y) \) implies \( \mathcal{D}_k \). Therefore, \( \mathcal{D}_k \) is a semilattice decomposition of \( S \).

Summarizing the above-mentioned results, we obtain the following Theorem which will play an important part in the next paragraph.

Theorem 1. Any semilattice decomposition of \( S \) is the decomposition to the factor algebraic system of \( S \) under a congruence relation \( (\bar{\Gamma}) \) introduced by some subcollection \( \Gamma \) of \( \Omega \). Conversely, the decomposition to the factor algebraic system of \( S \) under a congruence relation \( (\bar{\Gamma}) \) introduced by any subcollection \( \Gamma \) of \( \Omega \) is a semilattice decomposition of \( S \).

§ 2 Greatest semilattice decomposition

Let \( \gamma : S = \bigcup \mathcal{D}_k \) be any two semilattice decompositions of \( S \). We define an ordering \( \gamma \geq \gamma \) between \( \gamma \) and \( \gamma \) to mean that for any \( \mathcal{D}_k \), there exists \( \mathcal{D}_k \) such that \( S \in \mathcal{D}_k \). Then the collection \( \mathcal{D} \) of all semilattice decompositions of \( S \) forms not only a partial ordered set but also a complete semilattice (1), and therefore there exists the greatest element, that is, the greatest semilattice decomposition of \( S \). Our first purpose of this paragraph is to show that the greatest semilattice decomposition of \( S \) is the decomposition to the factor algebraic system of \( S \) under the congruence relation \( (\bar{\Gamma}) \), and the second is to obtain a necessary and sufficient condition for each residue of the greatest semilattice decomposition to be either a nonpotent semigroup or a unipotent semigroup (3), (4).

Theorem 2. The greatest semilattice decomposition of \( S \) is the decomposition to the factor algebraic system of \( S \) under the congruence relation \( (\bar{\Gamma}) \).

Proof. Let \( \gamma : S = \bigcup \mathcal{D}_k \) be the greatest semilattice decomposition of \( S \). Then \( \gamma \geq \gamma \), because each residue class of any semilattice decomposition of \( S \) is a P-subsemigroup of \( S \). Hence for any \( a, b \in S \), \( a \sim b \) implies \( \mathcal{D}_k \). On the one hand if \( a \sim b \), then \( a \bar{c} (c, y) \) and \( b \bar{c} (c, y) \) hold good because of \( a \bar{c} (c, y) \) and \( b \bar{c} (c, y) \). As \( \mathcal{D}_k \) and \( \mathcal{D}_l \) are a residue class such that it contains an element \( a \) or \( b \), respectively. As \( \mathcal{D}_k \) and \( \mathcal{D}_l \) are P-subsemigroups of \( S \), \( a \bar{c} (c, y) \) and \( b \bar{c} (c, y) \) are contained in both \( \mathcal{D}_k \) and \( \mathcal{D}_l \). Hence \( a \bar{c} (c, y) \) implies \( \mathcal{D}_k \). Therefore \( (\bar{\Gamma}) \) is a semilattice decomposition of \( S \). Therefore \( (\bar{\Gamma}) \) is a semilattice decomposition of \( S \).
Corollary 1. The greatest semi-
lattice decomposition of a commutative
semigroup $S$ is the decomposition to
the factor algebraic system of $S$ under
a congruence-relation ($\approx$) introduced
as follows (1):

$$a \approx b \text{ if } a^m = bx \text{ and } b^m = ay$$

are satisfied for some positive integers
$m, n$ and some elements $x, y \in S$.

Proof. First of all, since $S$ is a
commutative semigroup, a relation $\{(x, y) |
\{x, y \in S\} \approx \{x, y \in S\}\}$ is equivalent
to a relation $\{x, y \in S\} \approx \{x, y \in S\}$
for any $S \in \Omega$. If we first show that
$a \approx b$ implies $a \approx b$ for any elements
$a, b \in S$. By the definition, $a \approx b$
means $\{x, y \in S\} \approx \{x, y \in S\}$ to be
satisfied for any $S \in \Omega$. If we
set $S' = \{x \in S | x \approx y\}$, since $a \in S'$,
the following results follow in order;
$e < a < e'$, $a \approx b$, $b \approx c$
and consequently $b \approx c$.
Thereforer, there exist positive integers
$m, n$ and elements $x, y$ such that

$$a^m = bx \text{ and } b^m = ay$$

Hence $a \approx b$. Next, $a \approx b$ implies
$a \approx b$ for any elements $a, b \in S$.
If $a \approx b$, there exist positive integers
$m, n$ and elements $x, y$ such that

$$a^m = bx \text{ and } b^m = ay$$

Take up any $S \in \Omega$. Then if $t \in \Omega$,
the following relation are satisfied
in order; $e < a < e'$, $a \approx b$
$e < a < e'$, and consequently $b \approx c$.
Hence $\approx$ implies $\approx$ for any element $t \in S$.
Similarly for any $t \in S$, the $S \approx t$
implies $\approx$ $t \in S$, thus $\approx$ $\approx$.
Therefore $(\approx)$ ($\approx$). This completes the proof of this corollary.

Theorem 3. In the greatest semi-
lattice decomposition of $S$, each of
residue classes is either a nonpotent
semigroup or a unipotent semigroup if
and only if, for each pair of mutually
different idempotent elements $e_1, e_2$, there exists a $P$-subsemigroup $S'$ of $S$
such that either $S' \nsubseteq S$ but $S' \ni e_1$ or
$S' \nsubseteq S$ but $S' \ni e_2$.

Proof. Since necessity of the condi-
tion is obvious, we may prove only
sufficiency. We assume that there
exists a pair of mutually different idempotent elements $e_1, e_2$ such that
$e_1 \not\approx e_2$. By the hypothesis, there exists a $P$-subsemigroup $S'$ of $S$ such
that either $S' \nsubseteq S$ but $S' \ni e_1$ or
$S' \nsubseteq S$ but $S' \ni e_1$. Without loss
of generality, we may assume $S' \ni e_1$ but
$S' \nsubseteq S$. Since $e_1 \not\approx e_2$ and $e_1, e_2 \in S'$,
$e_1, e_2 \in S'$. Hence $e_1, e_2 \in S'$ because $S'$ is a $P$-subsemigroup of $S$. Therefore
$e_1, e_2 \in S'$; hence $e_1, e_2 \in S'$; hence $e_1 \in S'$. Thus, there exist no pairs of mutually different idempotent elements $e_1, e_2$ such that $e_1 \not\approx e_2$.

Remark. In the greatest semi-
lattice decomposition of a general
semigroup, each of residue classes is
not necessarily a nonpotent or uni-
potent semigroup. This is obtained by
a simple example as follows.

Example. Let $S$ be a right singular
semigroup consisting of two or more
elements. Since $S$ contains no $P$
subsemigroups of $S$ except $S$ itself,
residue classes of the greatest semi-
lattice decomposition of $S$ are $S$
alones. However, $S$ is neither a non-
potent semigroup nor a unipotent semi-
group.

Corollary 2. In the greatest semi-
lattice decomposition of a commutative
semigroup, each of residue classes is
either a nonpotent semigroup or a
unipotent semigroup.

Proof. Let $S$ be a commutative
semigroup and $e_1, e_2$ be two mutually
different idempotent elements of $S$.
If we set $S' = \{t | \exists x, y \in S, x \approx y\}$, then
$S'$ is a $P$-subsemigroup of $S$. It is
obvious that $e_1 \in S'$ if $e_1$ is
also contained in $S'$, then there
exists elements $x, y$ such that $e_1 = x e_1$
and $e_2 = y e_2$. Therefore $e_1, e_2 = e_1, x e_1, x e_2$
and $e_2, e_1, y e_1, y e_2$, and consequently $e_1, e_2 \in S'$. Hence $e_1, e_2 \in S'$. By
Theorem 3, this completes the proof of this corollary.

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(*) Received June 30, 1955.
This book deals with the elementary differential geometry of curves and surfaces in two- and three-euclidean spaces.


The book covers the whole field of the elementary differential geometry, the vector notation being adopted throughout. Concise and clear explanations can be found passim. Both relevant remarks and rich examples in the book will help the reader in getting the ideal which the author wants to tell in the book. We may say, at the close of this short comments, the book is very handy for students.

(A. Kuribayasi)
The present work is formally the second edition of a book with the same title written by the first two of the authors and published in 1938. However, its contents are, compared with the former edition, so substantially revised throughout that it seems to be quite another new book. Attempting to make the reader familiar with new formulations and methods in the theory of integrals from classical as well as modern viewpoints, this book takes an intermediate situation. For instance, on the one hand, measures and integrals are dealt with in usual sense while, on the other hand, the theory of linear functionals is developed as an extension of Lebesgue integral.

The titles of contents listed in the following lines will well explain an extensive and profound character of this book:


Second part. Integrals by subdivision and \( \sigma \)-additive functions. Linear functionals. IV. Integral by subdivision belonging to a measure. V. Additive functions with arbitrary sign. VI. Linear continuous functionals. VII. Measures and integrals in product spaces. Multiple integrals.

Third part. Measures and integrals in topological spaces. VIII. Measures and contents adaptive to a topology. Integrals belonging to them.

Fourth part. Primitive functions. Indefinite integral. IX. \( \sigma \)-additive function as a primitive function. X. Additive function as a primitive function.

Fifth part. Some Applications. XI. Functions and surfaces of bounded dilatation in \( \mathbb{R}_n \).

Literature.

(Y. Komatu, Tokyo Institute of Technology.)