§ 1. Introduction.

Let \( C \) be a bounded closed convex subset of a Banach space \( E \) and let \( T \) be a nonexpansive mapping from \( C \) into itself. Browder [2] and Göhde [10] showed that if \( E \) is uniformly convex then \( T \) has a fixed point, while Kirk [13] proved that if \( E \) is reflexive and if \( C \) has normal structure then \( T \) has a fixed point. On the other hand, Goebel [7] defined the characteristic \( \varepsilon_0 \) of convexity of \( E \) and showed that \( E \) is uniformly convex if and only if \( \varepsilon_0 = 0 \), if \( \varepsilon_0 < 1 \) then \( E \) has normal structure and if \( \varepsilon_0 < 2 \) then \( E \) is reflexive. Also, Bynum [3] defined the normal structure coefficient \( N(E) \) of \( E \), and then Maluta [17] and Bae [1] proved that if \( N(E)^{-1} < 1 \) then \( E \) is reflexive and has normal structure. Using these coefficients, Goebel and Kirk [8], Goebel, Kirk and Thele [9] and Casini and Maluta [4] proved the fixed point theorems for uniformly \( k \)-lipschitzian mappings. (For the results on Hilbert space, see [5], [12], [14].) But it seems natural to define these coefficients for a convex set, since for any Banach space \( E \), a nonexpansive mapping has a fixed point if \( C \) is weakly compact and has normal structure.

In this paper, we introduce the modulus \( \delta(C, \varepsilon) \) of convexity, the characteristic \( \varepsilon_0(C) \) of convexity and the constant \( \tilde{N}(C) \) of uniformity of normal structure for a convex subset \( C \) of a Banach space and prove some results similar to [3], [7], [11], [17]. For example, we show that if \( \tilde{N}(C) < 1 \) then \( C \) is boundedly weakly compact. Further, by using these coefficients, we prove three fixed point theorems. All of these proofs are given by explicitly constructing a sequence which converges to a fixed point. We first show a fixed point theorem for nonexpansive semigroups. Secondly, we obtain a fixed point theorem for uniformly \( k \)-lipschitzian semigroups on \( C \) under \( k < \gamma \), where \( \gamma \) is determined by the modulus of convexity of \( C \). Also, using our results, we evaluate \( \gamma \) as \( 1 < \gamma \leq 1 + (1 - \varepsilon_0(C))/2 \). Finally, we prove that Casini and Maluta’s result [4] is valid under more general semigroups.

§ 2. Preliminaries.

Let \( E \) be a real Banach space and let \( B \) be a bounded subset of \( E \). For a
nonempty subset \( C \) of \( E \) define,
\[
R(B, x) = \sup \{ \| x - y \| : y \in B \};
\]
\[
R(B, C) = \inf \{ R(B, x) : x \in C \};
\]
\[
\mathcal{C}(B, C) = \{ x \in C : R(B, x) = R(B, C) \}.
\]
We call the number \( R(B, C) \) the **Chebyshev radius** of \( B \) in \( C \) and the set \( \mathcal{C}(C, B) \) the **Chebyshev center** of \( B \) in \( C \).

Let \( \{ B_\alpha : \alpha \in A \} \) be a decreasing net of bounded subsets of \( E \). For a nonempty subset \( C \) of \( E \) define,
\[
r(\{ B_\alpha \}, x) = \inf_\alpha R(B_\alpha, x);
\]
\[
r(\{ B_\alpha \}, C) = \inf \{ r(\{ B_\alpha \}, x) : x \in C \};
\]
\[
\mathcal{A}(\{ B_\alpha \}, C) = \{ x \in C : r(\{ B_\alpha \}, x) = r(\{ B_\alpha \}, C) \}.
\]
The number \( r(\{ B_\alpha \}, C) \) and the set \( \mathcal{A}(\{ B_\alpha \}, C) \) are called the **asymptotic radius** and the **asymptotic center** of \( \{ B_\alpha : \alpha \in A \} \) in \( C \), respectively. We also know that \( R(B, \cdot) \) and \( r(\{ B_\alpha \}, \cdot) \) are continuous convex functions on \( E \) which satisfy the following:
\[
|R(B, x) - R(B, y)| \leq \| x - y \| \leq R(B, x) + R(B, y);
\]
\[
|r(\{ B_\alpha \}, x) - r(\{ B_\alpha \}, y)| \leq \| x - y \| \leq r(\{ B_\alpha \}, x) + r(\{ B_\alpha \}, y)
\]
for each \( x, y \in E \), cf. \([16]\).

A nonempty subset \( C \) of \( E \) is **boundedly weakly compact** if its intersection with every closed ball is weakly compact. It is easy to see that if \( C \) is boundedly weakly compact and convex, then \( \mathcal{C}(B, C) \) and \( \mathcal{A}(\{ B_\alpha \}, C) \) are nonempty.

For a subset \( D \) of \( E \), we denote by \( d(D) \) the diameter of \( D \) and by \( \overline{co}D \) the closure of the convex hull of \( D \). A convex set \( C \) of \( E \) is said to have **normal structure** if each bounded convex subset \( D \) of \( C \) with \( d(D) > 0 \) contains a point \( y \) such that \( R(D, y) < d(D) \).

The **modulus of convexity** of \( E \) is the function
\[
\delta_E(\varepsilon) = \inf \left\{ 1 - \frac{\| x + y \|}{2} : \| x \| \leq 1, \| y \| \leq 1, \| x - y \| \geq \varepsilon \right\}
\]
defined for \( 0 \leq \varepsilon \leq 2 \).

Let \( S \) be a semitopological semigroup, i.e., \( S \) is a semigroup with a Hausdorff topology such that for each \( a \in S \) the mappings \( s \mapsto a \cdot s \) and \( s \mapsto s \cdot a \) from \( S \) to \( S \) are continuous. Let \( C \) be a nonempty closed convex subset of \( E \). Then a family \( \mathcal{S} = \{ T_t : t \in S \} \) of mappings from \( C \) into itself is said to be a **uniformly \( k \)-Lipschitzian semigroup** on \( C \) if \( \mathcal{S} \) satisfies the following:

1. \( T_t(x) = T_s T_t(x) \) for \( t, s \in S \) and \( x \in C \);
2. the mapping \( (s, x) \mapsto T_s(x) \) from \( S \times C \) into \( C \) is continuous when \( S \times C \) has
the product topology;

(3) \( \| T_s(x) - T_s(y) \| \leq k \| x - y \| \) for \( x, y \in C \) and \( s \in S \).

In particular, a uniformly 1-lipschitzian semigroup on \( C \) is said to be a nonexpansive semigroup on \( C \). A semitopological semigroup \( S \) is left reversible if any two closed right ideals of \( S \) have nonvoid intersection. In this case, \( (S, \leq) \) is a directed system when the binary relation "\( \leq \)" on \( S \) is defined by \( a \leq b \) if and only if \( \{ a \} \cup aS \supseteq \{ b \} \cup bS \).

§ 3. Modulus of convexity and characteristic of convexity.

We first define the modulus of convexity, the characteristic of convexity and the constant of uniformity of normal structure for a nonempty convex subset of a Banach space.

**Definition 3.1.** Let \( C \) be a nonempty convex subset of a real Banach space \( E \) with \( d(C) > 0 \). Then we define, for \( 0 \leq \varepsilon \leq 2 \),

\[
\delta(C, \varepsilon) = \inf \left\{ \frac{1}{r} \| z - \frac{x + y}{2} \| : x, y, z \in C, 0 < r \leq d(C), \right. \\
\left. \| z - x \| \leq r, \| z - y \| \leq r, \| x - y \| \leq r \varepsilon \right\};
\]

\[
\varepsilon_0(C) = \sup \{ \varepsilon : 0 \leq \varepsilon \leq 2, \delta(C, \varepsilon) = 0 \};
\]

\[
\tilde{N}(C) = \sup \left\{ \frac{R(D, D)}{d(D)} : D \text{ is a nonempty bounded convex subset of } C \right\}.
\]

**Remark 3.1.** It follows from Definition 3.1 that \( \delta(C, 0) = 0 \), \( 0 \leq \delta(C, \varepsilon) \leq 1 \), \( \delta(C, \varepsilon) \) is nondecreasing in \( \varepsilon \) and \( \delta(E, \varepsilon) = \delta_E(\varepsilon) \). Further for a nonempty convex subset \( D \) of \( C \) with \( d(D) > 0 \) it follows that \( \delta(C, \varepsilon) \leq \delta(D, \varepsilon), \varepsilon_0(D) \leq \varepsilon_0(C) \) and \( \tilde{N}(D) \leq \tilde{N}(C) \).

**Remark 3.2.** Let \( C \) and \( D \) be convex subsets of \( E \). For \( a \in E \), it is easy to see that \( \delta(C, \varepsilon) = \delta(C, a, \varepsilon), \delta(C + a, \varepsilon) = \delta(C, \varepsilon) \), and \( \delta(C \cap D, \varepsilon) = \max \{ \delta(C, \varepsilon), \delta(D, \varepsilon) \} \). Similarly we have \( \tilde{N}(C) = \tilde{N}(C), \tilde{N}(C + a) = \tilde{N}(C), \) and \( \tilde{N}(C \cap D) = \min \{ \tilde{N}(C), \tilde{N}(D) \} \).

**Example 3.1.** Let \( C[0, 1] \) be a Banach space of all continuous real functions on \( [0, 1] \) with supremum norm and let \( A \) be a subspace of all affine functions in \( C[0, 1] \). Since \( C[0, 1] \) is not reflexive, we have \( \tilde{N}(C[0, 1]) = 1 \). But it is easy to see that \( A \) is isomorphic to \( l^2_\infty = (R, \| \cdot \|_\infty) \) and hence \( \tilde{N}(A) = \tilde{N}(l^2_\infty) = \frac{1}{2} \), cf. [17], [1].
It is well known that $\delta_{E}(\varepsilon)$ is continuous on $[0, 2)$, cf. [11]. We can also prove an inequality concerning the continuity of $\delta(C, \varepsilon)$. Before proving it we need the following lemma.

**Lemma 3.1.** Let $C$ be a nonempty convex subset of a real Banach space $E$ with $d(C) > 0$, let $u, v \in E$ and let $0 < r \leq d(C)$. For $z \in C$ and $\varepsilon$ with $0 \leq \varepsilon \leq 2$ define a set $N_{r, u, v}(z)$ and a function $\delta_{r, u, v}(\varepsilon)$ as follows:

$$N_{r, u, v}(z) = \{(x, y) : x, y \in C, \|x - z\| \leq r, \|z - y\| \leq r, x - y = au, z - \frac{x + y}{2} = bv \text{ for some } a, b \geq 0\};$$

$$\delta_{r, u, v}(\varepsilon) = \inf \left\{ \frac{1}{r} \left\| z - \frac{x + y}{2} \right\| : z \in C, (x, y) \in N_{r, u, v}(z), \|x - y\| \geq r \varepsilon \right\}.$$

Then $\delta_{r, u, v}$ is a nondecreasing convex function from $[0, 2]$ to $[0, 1]$ with

$$\delta(C, \varepsilon) = \inf \{ \delta_{r, u, v}(\varepsilon) : u, v \in E, 0 < r \leq d(C) \}.$$

**Proof.** Since it is obvious that $\delta_{r, u, v}$ is nondecreasing and

$$\delta(C, \varepsilon) = \inf \{ \delta_{r, u, v}(\varepsilon) : u, v \in E, 0 < r \leq d(C) \},$$

we only prove that $\delta_{r, u, v}$ is convex.

For arbitrary $z_1, z_2 \in C$ and $(x_1, y_1) \in N_{r, u, v}(z_1)$ and $(x_2, y_2) \in N_{r, u, v}(z_2)$ with $\|x_1 - y_1\| \geq r \varepsilon_1$ and $\|x_2 - y_2\| \geq r \varepsilon_2$, there exist $a_1, a_2, b_1, b_2 \geq 0$ such that

$$x_1 - y_1 = a_1 u, z_1 - \frac{x_1 + y_1}{2} = b_1 v,$$

and

$$x_2 - y_2 = a_2 u, z_2 - \frac{x_2 + y_2}{2} = b_2 v.$$

For $\lambda$ with $0 \leq \lambda \leq 1$, define $x_3 = \lambda x_1 + (1 - \lambda)x_2, y_3 = \lambda y_1 + (1 - \lambda)y_2$ and $z_3 = \lambda z_1 + (1 - \lambda)z_2$. Then, we have

$$x_3 - y_3 = \lambda(x_1 - y_1) + (1 - \lambda)(x_2 - y_2) = (\lambda a_1 + (1 - \lambda)a_2)u,$$

$$z_3 - \frac{x_3 + y_3}{2} = \lambda(z_1 - \frac{x_1 + y_1}{2}) + (1 - \lambda)(z_2 - \frac{x_2 + y_2}{2})$$

$$= (\lambda b_1 + (1 - \lambda)b_2)v.$$

Since $\|z_3 - x_3\| \leq r$ and $\|z_3 - y_3\| \leq r$, we have $(x_3, y_3) \in N_{r, u, v}(z_3)$. We also obtain

$$\|x_3 - y_3\| = \lambda \|x_1 - y_1\| + (1 - \lambda)\|x_2 - y_2\| \geq \lambda \varepsilon_1 + (1 - \lambda)\varepsilon_2$$

and
\[ \delta_{r, u, v}(\lambda \varepsilon_1 + (1-\lambda)\varepsilon_2) \leq 1 - \frac{1}{r} \left\| z_2 - \frac{x_1 + y_1}{2} \right\| \]

\[ = \lambda \left(1 - \frac{1}{r} \left\| z_1 - \frac{x_1 + y_1}{2} \right\| \right) + (1-\lambda) \left(1 - \frac{1}{r} \left\| z_2 - \frac{x_2 + y_2}{2} \right\| \right) \]

for arbitrary \( z_1, z_2 \in C \), \( (x_1, y_1) \in N_{r, u, v}(z_1) \) and \( (x_2, y_2) \in N_{r, u, v}(z_2) \). Therefore we have

\[ \delta_{r, u, v}(\lambda \varepsilon_1 + (1-\lambda)\varepsilon_2) \leq \lambda \delta_{r, u, v}(\varepsilon_1) + (1-\lambda)\delta_{r, u, v}(\varepsilon_2). \]

**Theorem 3.1.** Let \( C \) be a nonempty convex subset of a real Banach space \( E \) with \( d(C) > 0 \). Then for all \( \varepsilon_1 \) and \( \varepsilon_2 \) with \( 0 < \varepsilon_1 < \varepsilon_2 \leq 2r \),

\[ \delta(C, \varepsilon_2) - \delta(C, \varepsilon_1) \leq \frac{\varepsilon_2 - \varepsilon_1}{2 - \varepsilon_1} (1 - \delta(C, \varepsilon_1)) \leq \frac{\varepsilon_2 - \varepsilon_1}{2 - \varepsilon_1}. \]

**Proof.** For any real number with \( \eta > 0 \), there exist \( u, v \in E \) and \( r \) with \( 0 < r < d(C) \) such that \( \delta_{r, u, v}(\varepsilon_1) \leq \delta(C, \varepsilon_1) + \eta \) and hence we obtain

\[ \delta_{r, u, v}(\varepsilon_2) = \delta_{r, u, v}\left(\frac{\varepsilon_2 - \varepsilon_1}{2 - \varepsilon_1} \right)^2 + \left(1 - \frac{\varepsilon_2 - \varepsilon_1}{2 - \varepsilon_1}\right) \delta_{r, u, v}(\varepsilon_1) \]

or

\[ \delta_{r, u, v}(\varepsilon_2) - \delta_{r, u, v}(\varepsilon_1) \leq \frac{\varepsilon_2 - \varepsilon_1}{2 - \varepsilon_1} \left( \delta_{r, u, v}(2) - \delta_{r, u, v}(\varepsilon_1) \right) \leq \frac{\varepsilon_2 - \varepsilon_1}{2 - \varepsilon_1} (1 - \delta(C, \varepsilon_1)). \]

Then we have

\[ \delta(C, \varepsilon_2) - \delta(C, \varepsilon_1) \leq \delta_{r, u, v}(\varepsilon_2) - \delta_{r, u, v}(\varepsilon_1) + \eta \leq \frac{\varepsilon_2 - \varepsilon_1}{2 - \varepsilon_1} (1 - \delta(C, \varepsilon_1)) + \eta. \]

Since \( \eta > 0 \) is arbitrary, we have

\[ \delta(C, \varepsilon_2) - \delta(C, \varepsilon_1) \leq \frac{\varepsilon_2 - \varepsilon_1}{2 - \varepsilon_1} (1 - \delta(C, \varepsilon_1)) \leq \frac{\varepsilon_2 - \varepsilon_1}{2 - \varepsilon_1}. \]

The following lemma can be proved as in [16].

**Lemma 3.2.** Let \( C \) be a convex subset of a real Banach space \( E \). Let \( B \) be a bounded subset of \( C \) and let \( \{B_\alpha : \alpha \in A\} \) be a decreasing net of bounded subsets of \( C \). For each \( x, y \in C \), if \( R(B, x) \leq t, R(B, y) \leq t \) and \( \|x - y\| \leq t \cdot \varepsilon \) then
and if \( r(\{ B_\alpha \}, x) \leq t, r(\{ B_\alpha \}, y) \leq t \) and \( \| x - y \| \geq t \varepsilon \) then

\[
R \left( B, \frac{x + y}{2} \right) \leq t(1 - \delta(C, \varepsilon))
\]

It was proved by Bynum [3] that \( \hat{N}(E) \leq 1 - \delta_\varepsilon(1) \). By using Theorem 3.1, Lemma 3.2 and the method of [3], we can also obtain the following: Let \( C \) be a nonempty convex subset of a real Banach space \( E \) with \( d(C) > 0 \). Then \( \hat{N}(C) \leq 1 - \delta(C, 1) \).

Maluta [17] and Bae [1] proved that if \( \hat{N}(E) < 1 \) then \( E \) is reflexive. We can prove the following:

**Theorem 3.2.** Let \( C \) be a nonempty convex subset of a real Banach space \( E \) with \( d(C) > 0 \). If \( \hat{N}(C) < 1 \) then \( C \) is boundedly weakly compact and has normal structure.

**Proof.** It is obvious from \( \hat{N}(C) < 1 \) that \( C \) has normal structure. We may assume that \( C \) is bounded. Let \( \{ C_n \} \) be an arbitrary decreasing sequence of nonempty closed convex subsets of \( C \). If we show \( \{ C_n \} \) has nonempty intersection then we complete the proof, cf. [p. 433, 4]. If \( d(C_n) = 0 \) for some \( n \geq 1 \) then it is obvious that \( \{ C_n \} \) has nonempty intersection. So we assume \( d(C_n) > 0 \) for all \( n \geq 1 \). Let \( \eta \) be a real number with \( \hat{N}(C) < \eta < 1 \) and define by induction:

\[
C_{n, 0} = C_n;
\]

\[
x_{n, m} \in C_{n, m} \quad \text{such that} \quad R(C_{n, m}, x_{n, m}) \leq \eta d(C_{n, m});
\]

\[
C_{n, m+1} = \overline{\text{co}} \{ x_{k, m} : k \geq n \}.
\]

Then, we have \( C_{n, m} \) is nonempty, \( C_{n, m} \supseteq C_{n+1, m}, C_{n, m} \supseteq C_{n, m+1} \) and

\[
d(C_{n, m}) = \sup \{ \| x_{i, m, -1} - x_{j, m, -1} \| : i, j \geq n \} = \sup_{i \equiv n} \sup_{j \equiv n} \| x_{i, m, -1} - x_{j, m, -1} \|
\]

\[
\leq \sup_{i \equiv n} R(C_{i, m, -1}, x_{i, m, -1}) \leq \sup_{i \equiv n} R(C_{i, m, -1}, x_{i, m, -1})
\]

\[
\leq \sup_{i \equiv n} \eta d(C_{i, m, -1}) \leq \eta d(C_{n, m-1}) \leq \eta^m d(C_n)
\]

for all \( n, m \geq 1 \). Hence \( \lim d(C_{n, m}) = 0 \). Since \( \bigcap_{m=1}^\infty C_{n, m} = \bigcap_{m=1}^\infty C_{n, m-1} \) for all \( n \geq 1 \), there exists \( y \in E \) such that \( \bigcap_{n=1}^\infty C_{n, m} = \{ y \} \) for all \( n \geq 1 \). Therefore \( \bigcap_{n=1}^\infty C_n \) is nonempty.

**Corollary 3.1** (Maluta [17] and Bae [1]). Let \( E \) be a real Banach space with \( \hat{N}(E) < 1 \). Then \( E \) is reflexive and has normal structure.
§ 4. Fixed point theorems.

In this section, we prove three fixed point theorems by using the results obtained in section 3. The following lemma is crucial in the proofs.

**Lemma 4.1.** Let C be a convex subset of a real Banach space E. Let \( \{B_a : a \in A\} \) be a decreasing net of bounded subsets of C and let D be a boundedly weakly compact convex subset of C. Let r be the asymptotic radius and A be the asymptotic center of \( \{B_a\} \) in D. Then

\[
d(A) \leq \varepsilon_0(C)r.
\]

Further let \( \varepsilon_0(C) < 1 \) and let \( \gamma \) be a real number such that \( \gamma(1 - \delta(C, 1/\gamma)) = 1 \). For a real number \( k \) with \( 1 \leq k < \gamma \), define \( A_k = \{x \in D : r(\{B_a\}, x) \leq kr\} \). Then

\[
d(A_k) \leq \frac{k}{\gamma} r.
\]

**Proof.** In case \( r = 0 \), the inequality is true. In fact, if \( x, y \in A \) then

\[
\|x - y\| \leq r(\{B_a\}, x) + r(\{B_a\}, y) = 0
\]

and hence \( d(A) = 0 \). So we assume \( r > 0 \) and \( d(A) > 0 \). For any real number \( \eta \) with \( 0 < \eta < d(A) \), there exist \( x, y \in A \) such that \( \|x - y\| = d(A) - \eta \). By Lemma 3.2 and convexity of A, we have

\[
r = r(\{B_a\}, \frac{x + y}{2}) \leq r(1 - \delta(C, \frac{d(A) - \eta}{r})).
\]

This implies

\[
\delta(C, \frac{d(A) - \eta}{r}) = 0
\]

and hence \( d(A) \leq \varepsilon_0(C)r \).

We may also assume \( r > 0 \) and \( d(A_k) > 0 \). For any real number \( \eta \) with \( 0 < \eta < d(A_k) \), there exist \( x, y \in A_k \) such that \( \|x - y\| \geq d(A_k) - \eta \). Then, we have

\[
r \leq r(\{B_a\}, \frac{x + y}{2}) \leq kr(1 - \delta(C, \frac{d(A_k) - \eta}{kr})).
\]

Since \( \eta > 0 \) is arbitrary and \( \delta \) is continuous, it follows that

\[
\delta(C, \frac{d(A_k)}{kr}) \leq 1 - \frac{1}{k}.
\]

Suppose that \( \frac{1}{\gamma} \leq \frac{d(A_k)}{kr} \). Then we have

\[
1 - \frac{1}{\gamma} = \delta(C, \frac{1}{\gamma}) \leq \delta(C, \frac{d(A_k)}{kr}) \leq 1 - \frac{1}{k} < 1 - \frac{1}{\gamma}.
\]
This is a contradiction.

**Remark 4.1.** From Lemma 4.1, we have immediately the similar inequality concerning the Chebyshev radius and center. In fact, putting $B_n = B$, we have

$$d(C(B, D)) \leq \varepsilon_0(C) R(B, C).$$

The following theorem is a special case of results of Lim [15] and Takahashi [18], while the proof is constructive.

**Theorem 4.1.** Let $C$ be a closed convex subset of a real Banach space $E$ with $\varepsilon_0(C) < 1$ and let $S = \{T_t : t \in S\}$ be a nonexpansive semigroup on $C$. Suppose that $S$ is left reversible and $\{T_t y : t \in S\}$ is bounded for some $y \in C$. Then there exists a $z \in C$ such that $T_t z = z$ for all $t \in S$.

**Proof.** Let $B_s(x) = \{T_t x : t \geq s\}$ for $s \in S$ and $x \in C$. Define $\{x_n : n \geq 0\}$ by induction as follows:

$$x_0 = y;$$

$$x_n \in A(\{B_s(x_{n-1})\}, C) \quad \text{for } n \geq 1.$$ Let $r_n(x) = r(\{B_s(x_{n-1})\}, x)$, $r_n = r(\{B_s(x_{n-1})\}, C)$ and $A_n = A(\{B_s(x_{n-1})\}, C)$ for $n \geq 1$. Then we have

$$r_n(T_t x_n) = \limsup_s \|T_s x_n - T_t x_n\| \leq \limsup_s \|T_s T_t x_{n-1} - T_t x_n\|$$

$$\leq \limsup_s \|T_s x_{n-1} - x_n\| = r_n$$

for all $t \in S$ and $n \geq 1$ and hence $T_t A_n \subseteq A_n$ for $t \in S$ and $n \geq 1$. By Lemma 4.1, we obtain

$$r_{n+1} = r_{n+1}(x_{n+1}) \leq r_{n+1}(x_n) \leq \sup_s \|T_s x_n - x_n\|$$

$$\leq d(A_n) \leq \varepsilon_0(C) r_n \leq (\varepsilon_0(C))^n r_1$$

and hence

$$\|x_{n+1} - x_n\| \leq r(\{B_s(x_n)\}, x_{n+1}) + r(\{B_s(x_n)\}, x_n) = r_{n+1} + r_{n+1}(x_n)$$

$$\leq 2(\varepsilon_0(C))^n r_1$$

for all $n \geq 1$. So, $\{x_n\}$ is a Cauchy sequence and hence $\{x_n\}$ converges to a point $z \in C$. Therefore we have

$$\|z - T_s z\| = \lim_{n \to \infty} \|x_n - T_s x_n\| \leq \lim_{n \to \infty} (r_n(x_n) + r_n(T_s x_n))$$

$$\leq \lim_{n \to \infty} 2(\varepsilon_0(C))^{n} r_1 = 0$$

for all $s \in S$.

By the method of Theorem 4.1, we can prove the following fixed point theorem which is slightly different from [9].
**Theorem 4.2.** Let \( C \) be a closed convex subset of a real Banach space \( E \) with \( \varepsilon_0(C) < 1 \) and let \( \gamma \) be a real number such that \( \gamma(1-\delta(C, 1/\gamma)) = 1 \). Let \( S = \{ T_t : t \in S \} \) be a uniformly \( k \)-lipschitzian semigroup on \( C \) with \( 1 \leq k < \gamma \). Suppose that \( S \) is left reversible and \( \{ T_{ty} : t \in S \} \) is bounded for some \( y \in C \). Then there exists a \( z \in C \) such that \( T_sz = z \) for all \( s \in S \).

**Proof.** Let \( B_s(x) = \{ T_t x : t \geq s \} \) for \( s \in S \) and \( x \in C \). Define \( \{ x_n : n \geq 0 \} \) by induction as follows:

\[
x_0 = y;
\]
\[
x_n \in A(\{ B_s(x_{n-1}) \}, C) \quad \text{for} \quad n \geq 1.
\]

Let \( r_n(x) = r(\{ B_s(x_{n-1}) \}, x) \), \( r_n = r(\{ B_s(x_{n-1}) \}, C) \) and \( A_n = \{ x \in C : r_n(x) \leq kr_n \} \) for \( n \geq 1 \). Then since \( r_n(x_n) = r_n \leq kr_n \) and

\[
r_n(T_s x_n) = \lim_{s \to 0} \sup_s \| T_{s} x_{n-1} - T_s x_n \| \leq k \lim_{s \to 0} \sup_s \| T_s x_{n-1} - x_n \| = kr_n
\]

for all \( t \in S \) and \( n \geq 1 \), we have \( x_n, T_t x_n \in A_n \) for all \( t \in S \) and \( n \geq 1 \). By Lemma 4.1, we obtain

\[
r_{n+1} = r_{n+1}(x_{n+1}) \leq r_{n+1}(x_n) \leq \sup_s \| T_s x_n - x_n \| \leq d(A_n) \leq \frac{k}{\gamma} r_n \leq \left( \frac{k}{\gamma} \right)^n r_1
\]

for all \( n \geq 1 \). Therefore, as in the proof of Theorem 4.1, \( \{ x_n \} \) converges to a point \( z \in C \). So, we have

\[
\| z - T_s z \| = \lim_{n \to \infty} \| x_n - T_s x_n \| \leq \lim_{n \to \infty} (r_n(x_n) + r_n(T_s x_n)) \leq \lim_{n \to \infty} (1 + k) r_n = 0
\]

for all \( s \in S \).

**Remark 4.2.** Let \( C \) and \( \gamma \) be defined as in Theorem 4.2. Then we have

\[
1 < \gamma \leq 1 + \frac{1 - \varepsilon_0(C)}{2}.
\]

In fact, let \( \eta = 1/\gamma \) and if \( \delta(C, \eta) = 1 - \eta = 0 \). Then we have \( 1 > \varepsilon_0(C) \geq \eta = 1 \). This is a contradiction. Hence \( \varepsilon_0(C) \leq \eta < 1 \). So, from Theorem 3.1,

\[
1 - \eta = \delta(C, \eta) \leq \frac{\eta - \varepsilon_0(C)}{2 - \varepsilon_0(C)}.
\]

Therefore we have

\[
1 < \gamma \leq 1 + \frac{1 - \varepsilon_0(C)}{2}.
\]
We can also obtain a generalization of Casini and Maluta’s fixed point theorem [4].

**Lemma 4.2.** Let $C$ be a boundedly weakly compact convex subset of a real Banach space $E$. Let $\{B_a : a \in A\}$ be a decreasing net of nonempty bounded closed convex subsets of $C$ and let $B = \bigcap_a B_a$. Then

$$r(\{B_a\}, B) \leq \hat{N}(C) \inf_{a} d(B_a).$$

**Proof.** Let $u_\beta \in C(B_\beta, B_\beta)$ for each $\beta \in A$. Then we have

$$r(\{B_a\}, u_\beta) \leq R(B_\beta, u_\beta) = R(B_\beta, B_\beta) \leq \hat{N}(C) d(B_\beta).$$

Let $\{u_\beta\}$ be a subnet of $\{u_\beta\}$ which converges weakly to a point $u_0 \in B$. By weakly lower semicontinuity of $r$ and monotonicity of $d(B_\beta)$, we have

$$r(\{B_a\}, B) \leq r(\{B_a\}, u_0) \leq \limsup_{r} r(\{B_a\}, u_\beta) \leq \limsup_{r} \hat{N}(C) d(B_\beta) = \hat{N}(C) \inf_{a} d(B_a).$$

**Theorem 4.3.** Let $C$ be a closed convex subset of a real Banach space $E$ with $\hat{N}(C) < 1$ and let $S = \{T_t : t \in S\}$ be a uniformly $k$-Lipschitzian semigroup on $C$ with $k < \hat{N}(C)^{-1/2}$. Suppose that $S$ is left reversible and $\{T_t y : t \in S\}$ is bounded for some $y \in C$. Then there exists a $z \in C$ such that $T_s z = z$ for all $s \in S$.

**Proof.** Let $B_s(x) = \overline{co} \{T_t x : t \geq s\}$ and let $B(x) = \bigcap_s B_s(x)$ for $s \in S$ and $x \in C$. Define $\{x_n : n \geq 0\}$ by induction as follows:

$$x_0 = y,$$

$$x_n \in \mathcal{A}(\{B_s(x_{n-1})\}, B(x_{n-1})) \quad \text{for} \quad n \geq 1.$$

Let $r_n(x) = r(\{B_s(x_{n-1})\}, x)$ and $r_n = r(\{B_s(x_{n-1})\}, B(x_{n-1}))$ for $n \geq 1$. Then from $x_n \in B(x_{n-1}) = \bigcap_s B_s(x_{n-1})$ for $n \geq 1$, we have

$$r_{n+1}(x_n) = \limsup_{s} \|T_s x_n - x_n\| \leq \limsup_{s} (\inf_s R(B_s(x_{n-1}), T_s x_n))$$

$$= \limsup_{s} r_n(T_s x_n) = \limsup_{s} (\limsup_{s} \|T_s x_{n-1} - T_s x_n\|)$$

$$\leq k \limsup_{s} (\inf_{s} \|T_s x_{n-1} - x_n\|) = k r_n$$

and

$$\inf_{s} d(B_s(x_{n-1})) \leq \inf_{s} \{\|T_0 x_{n-1} - T_s x_{n-1}\| : a, b \geq s\}$$
Hence we have
\[ r_{n+1}(x_n) \leq kr_n(x_{n-1}) \leq (k^2 \bar{N}(C))^n r_1(x_0). \]
Therefore, as in the proof of Theorem 4.2, \( \{ x_n \} \) converges to a common fixed point.

REFERENCES


Department of Information Science, Tokyo Institute of Technology
Oh-okayama, Meguro-ku, Tokyo 152, Japan