ON THE GROWTH OF MEROMORPHIC FUNCTIONS OF ORDER LESS THAN 1/2; IV

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Introduction. This note is a continuation of [10]. The notational conventions of [10] are adopted without modifications and strictly adhered to. We supplement Theorems 1, 2 and 3 of [10] by the information contained in the theorems of the present note.

In everything that follows
(i) \( p \) and \( \delta \) are numbers such that \( 0 < p < \frac{1}{2} \) and \( 1 - \cos \pi \rho < \delta \leq 1 \);
(ii) \( \alpha(f) = \limsup_{r \to \infty} T(r, f)/r^p \), \( \beta(f) = \liminf_{r \to \infty} T(r, f)/r^p \), where \( f(z) \) is a meromorphic function of order \( \rho \).

We first prove in § 1

THEOREM 6. Let \( f(z) \in \mathcal{M}_{\rho, \delta} \) be of minimal type. Then there is an \( h(r) \in S_1 \) such that

\[
\log m^*(r, f) > -\frac{\pi \rho}{\sin \pi \rho} (\cos \pi \rho - 1 + \delta)/(1 + h(r)) T(r, f)
\]

for certain arbitrarily large values of \( r \).

Our second result, which is proved in § 2, is the following

THEOREM 7. Let \( h(r) \in S_1 \) be given. If \( f(z) \in \mathcal{M}_{\rho, \delta} \) satisfies \( \beta(f) = 0 \), then the estimate (1) holds for a sequence of \( r \to \infty \).

Remarks. (i) Theorem 3 of [10] is contained in the above Theorem 7.
(ii) Modifying a part of the proof of Theorem 7, we are able to show the following

THEOREM 8. Let \( k = k(\rho) \) and \( K_1 = K_1(\rho) \) be positive constants which appear in Lemma 13 and (2.14), respectively. If \( f(z) \in \mathcal{M}_{\rho, \delta} \) satisfies \( 0 < \beta(f) < (k/K_1) \alpha(f) \leq +\infty \), then the estimate (1) holds with any \( h(r) \in S_1 \) on an unbounded sequence of \( r \).

In § 3, we use our results stated above and in [10] to refine the estimate
(2) \[ \log m^\varepsilon(r, f) > -\frac{\pi \rho}{\sin \pi \rho} (\cos \pi \rho - 1 + \varepsilon) T(r, f) \quad (\varepsilon > 0, \ r = r_n \to \infty) \]

for all \( f(z) \in \mathcal{M}_{\rho, \delta} \) whose characteristics vary regularly with order \( \rho \). It was this refinement that provided the impetus for the previous and the present works.

We say, according to Baernstein [2], that the function \( \phi(r) \) varies regularly with order \( \rho \) if \( \phi(r) \sim r^\rho L(r) \ (r \to \infty) \) for some slowly varying function \( L(r) \).

1. We start by showing the following

**Lemma 12.** Given \( G(r) \) positive and continuous for \( r \geq r_0, G(r) \to \infty (r \to \infty) \), there exists a function \( h(r) \in S_2 \) such that

\[ \int_r^s h(t) \frac{dt}{t} \leq G(r) + C \quad (r \geq 1), \]

where \( C \) is a positive constant depending only on \( G(r) \).

**Proof.** By assumptions on \( G(r) \), we find a positive integer \( n_0 \) and an increasing unbounded sequence \( \{r_n\}^\infty_{n_0} \) with the property that \( G(r) > G(r_n) = n \ (r > r_n) \).

Choose \( \{R_n\}^\infty_{n_0} \) such that

\[ R_{n_0} = r_0, \quad R_{n_0+1} = r_{n_0+1}, \]
\[ R_n \geq r_n \ (n \geq n_0 + 2), \]
\[ R_{n+1}/R_n \geq (R_{n+1}/R_n)^{2} \quad (n \geq n_0). \]

Define a function \( h_1(r) \) \((r \geq r_{n_0})\) by

\[ h_1(r) = \{\log(R_{n+1}/R_n)\}^{-1} \quad (R_n \leq r < R_{n+1}, \ n \geq n_0). \]

Then \( h_1(r) \) is positive, decreasing, and tends to 0 as \( r \to \infty \). We define

\[ H_1(r) = n_0 - 1 + \int_{r_{n_0}}^r h_1(t) t^{-1} dt. \]

Then if \( R_n \leq r < R_{n+1} \ (n \geq n_0), \)

\[ H_1(r) \leq H_1(R_{n+1}) = n_0 - 1 + (n - n_0 + 1) = n = G(r_n) \leq G(r). \]

Now, define \( h(r) \) \((r \geq 0)\) by

\[ h(r) = \{\log(R_{n_0+1}/R_n)\}^{-1} \quad (0 \leq r \leq \sqrt{R_{n_0}R_{n_0+1}}), \]
\[ h(r) = \{\log(R_{n+1}/R_n)\}^{-1} \quad (R_n \leq r \leq \sqrt{R_nR_{n+1}}, \ n \geq n_0 + 1), \]

and by linear interpolation otherwise. Clearly \( h(r) \in S_2 \), and if we put

\[ H(r) = n_0 - 1 + \int_{r_{n_0}}^r h(t) t^{-1} dt, \]
then \( H(r) \leq H_0(r) \leq G(r) (r \geq r_0) \). Thus, with a suitable positive constant \( C \) 
\( \left( \leq \int_{r_0}^{r_n} h(t) t^{-1} dt - n_0 + 1 \right) \), we obtain (1.1).

The proof of Theorem 6 is a combination of Lemma 12 and Theorem 2 in [10].

**Proof of Theorem 6.** Let \( f(z) \in \mathcal{M}_{p, \delta} \) be of minimal type, and set

\[
G(r) = \log(r^p / T(r, f)) \quad (r > 0).
\]

Then \( G(r) \) satisfies the assumptions of Lemma 12, so we find a function \( h(r) \in S_p \) satisfying (1.1) with a suitable positive constant \( C \). Now, choose a positive number \( K < C(p, \delta) \) arbitrarily, where \( C(p, \delta) \) is defined by (5) in [10], and put \( h_1(r) = Kh(r) \in S_p \). Then in view of (1.2)

\[
T(r, f) = r^p \exp{-G(r)} \leq e^{C(r + \int_{1}^{r} h_1(t)^{-1} dt)}.
\]

Hence from Theorem 2 we deduce (1) with \( h(r) \) replaced by \( h_1(r) \) for certain arbitrarily large values of \( r \).

2. Let \( f(z) \in \mathcal{M}_{p, \delta} \) be given, and let \( a \) be a complex number satisfying

\[
f(0) \neq a \quad \text{and} \quad (2.1) \quad N(r, \infty, f) < (1 - \delta) N(r, a, f) + O(1) \quad (r \to \infty).
\]

We set

\[
(2.2) \quad F(z) = f(z) - a = cz^{-p} \frac{\Pi(1-z/a_n)}{\Pi(1-z/b_n)} = cz^{-p} \frac{P(z)}{Q(z)} = cz^{-p} \hat{F}_1(z),
\]

where \( c \) is a nonzero constant and \( p \) is a nonnegative integer. It is convenient to introduce the notation

\[
(2.3) \quad \hat{P}(z) = \Pi(1+z/|a_n|), \quad \hat{Q}(z) = \Pi(1-z/|b_n|), \quad \hat{F}_1(z) = \hat{P}(z)/\hat{Q}(z).
\]

Our proofs of Theorems 7 and 8 make use of the following

**Lemma 13.** (See [1, Lemma 1] ) Let \( F_1(z) \) be defined by (2.2). Then there exist constants \( K = K(p) \), \( k = k(p) \) depending only on \( p \) satisfying \( 0 < k < K < 4\pi + 2\pi^2/\log 2 \), such that for any \( r_1 > r > 0 \),

\[
\left\{ \begin{array}{l}
\frac{r}{r_1} \pi \rho N(t, \infty, \hat{F}_1) + \sin \pi \rho \log m^{\#}(t, \hat{F}_1) - \pi \rho \cos \pi \rho N(t, 0, \hat{F}_1) \mid t^{-1-p} dt \\
> k T(r_1, \hat{F}_1) r_1^{-p} - K T(2r_2, \hat{F}_1) r_2^{-p}.
\end{array} \right.
\]

Now choose \( R \) sufficiently large so that \( F_1(z) \) has \( N \) zeros and \( M \) poles in \( |z| < R \), where \( \max(M, N) > 0 \). Let
and define \( f_3(z) \) by \( f_3(z) = f_2(z) f_4(z) \). Using a result of Edrei [4, Lemma A] we have for \( r < R/2 \)
\[
T(r, F_1) \leq T(r, f_2) + T(r, f_3) \leq T(r, f_2) + \frac{14r}{R} T(2R, F_1).
\]

Here we apply Lemma 13 to \( f_2(z) \) to obtain for any \( r_1, r_2, 0 < r_1 < r_2 < R \)
\[
\int_{r_1}^{r_2} \{ \pi p N(t, \infty, F_1) + \sin \pi p \log m^*(t, F_2) - \pi p \cos \pi p N(t, 0, F_1) \} t^{-1-\rho} dt
\geq kT(r_1, f_2) r_1^{-\rho} - KT(2r_2, f_2) r_2^{-\rho}.
\]

**Proof of Theorem 7.** Suppose that \( f(z) \in S_{\rho, \delta} \) satisfies \( 0 = \beta(f) \leq \alpha(f) \leq +\infty \) and
\[
\pi p N(r, \infty, F) + \sin \pi p \log m^*(r, F) - \pi p \cos \pi p N(r, 0, F)
\leq \pi p (\cos \pi p - 1 + \delta) h(r) T(r, F) + K_2 \log r \quad (r \geq r_0 = r_0(K_2)),
\]
where \( F(z) \) is defined by (2.2) and \( K_2 \) is any fixed positive number. By (2.2) and (2.3) we have
\[
N(r, \infty, F) = N(r, \infty, F_1) + p \log r,
\]
\[
\log m^*(r, F) = \log |c| - p \log r + \log m^*(r, F_1),
\]
\[
N(r, 0, F) = N(r, 0, F_1).
\]

Substituting (2.7) into (2.6), we obtain
\[
\pi p N(r, \infty, F_1) + \sin \pi p \log m^*(r, F_1) - \pi p \cos \pi p N(r, 0, F_1)
\leq \pi p (\cos \pi p - 1 + \delta) h(r) T(r, F) + \{ K_2 - p(\pi p - \sin \pi p) \} \log r
- \sin \pi p \log |c| \quad (r \geq r_0).
\]

Hence from (2.5) and (2.8) it follows that for any \( r_1, r_2, r_0 < r_1 < r_2 < R \)
\[
\pi p (\cos \pi p - 1 + \delta) \int_{r_1}^{r_2} h(t) T(t, F) t^{-1-\rho} dt + K_2 \int_{r_1}^{r_2} (\log t) t^{-1-\rho} dt
\]
\[
+ \sin \pi p \int_{r_1}^{r_2} \{ \log m^*(t, F_2) - \log m^*(t, F_1) \} t^{-1-\rho} dt
\geq kT(r_1, f_2) r_1^{-\rho} - KT(2r_2, f_2) r_2^{-\rho},
\]
where \( K_3 \geq K_2 \) is a suitable constant. Using a result of Edrei [4, Lemma A] again, we have for \( 0 < t < R/2 \)
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\[
\log m*(t, F) \equiv \log m*(t, f) + \log m*(t, f_1)
\]

(2.10)

By (2.4)

(2.11)

\[
T(r_1, f) = T(r_1, F) - 14T(2R, F_1) - 2T(2R, F_1)(2R)^\rho.
\]

Also, if we choose \( r_1 = R/2 \), we have

\[
T(2r_1, f) = T(R, f) - N(R, 0, \dot{P}) + N(R, 0, \dot{Q}) + \log \dot{P}(R) + \log \dot{Q}(-R)
\]

\[
\leq 2T(R, F) + n(R, 0, \dot{P})\log 2 + N(R, 0, \dot{P}) + n(R, 0, \dot{Q})\log 2 + N(R, 0, \dot{Q})
\]

\[
\leq 2T(R, F) + 2T(2R, F) + T(R, F_1) \leq 6T(2R, F_1)
\]

so that

(2.12)

\[
T(2r_1, f) = 6^{-\rho}T(2R, F_1)(2R)^{-\rho}.
\]

Further, with \( r_1 = R/2 \) (1) we have

(2.13)

\[
\int_{r_1}^{R/2} (\log t) t^{-1-\rho} dt = -\rho^{-1}(\log r_1) r_1^{-\rho} + \rho^{-1}(\log r_1) r_1^{-\rho}
\]

\[
-\rho^{-2} r_1^{-\rho} + \rho^{-1} r_1^{-\rho} < \rho^{-1}(\log r_1 + 1) r_1^{-\rho}.
\]

Incorporating (2.10)-(2.13) into (2.9), it follows that for \( r_1 < r_1 < R/2 \)

\[
\pi \rho \cos \rho - 1 + \delta \int_{r_1}^{R/2} h(t) T(t, F) t^{-1-\rho} dt + K_3 \rho^{-1}(\log r_1) r_1^{-\rho} + K_3 \rho^{-2} r_1^{-\rho}
\]

\[
+ 14(1 - \rho)^{-2\rho - 1} \sin \rho \rho T(2R, F_1)(2R)^{-\rho} > kT(r_1, F_1) r_1^{-\rho}
\]

\[
- 7.4^\rho k T(2R, F_1)(2R)^{-\rho} - 6.4^\rho k T(2R, F_1)(2R)^{-\rho} \quad \text{i.e.,}
\]

\[
\pi \rho \cos \rho - 1 + \delta \int_{r_1}^{R/2} h(t) T(t, F) t^{-1-\rho} dt + K_3 \rho^{-1}(\log r_1) r_1^{-\rho} + K_3 \rho^{-2} r_1^{-\rho}
\]

\[
> kT(r_1, F_1) r_1^{-\rho} - K_3 T(2R, F_1)(2R)^{-\rho}
\]

with a suitable positive constant \( K_1 = K_1(\rho) \).

Case 1. Assume first that \( \alpha(f) = 0 \). Let \( R \to \infty \) in (2.14) to get

(2.15)

\[
\pi \rho \cos \rho - 1 + \delta \int_{r_1}^{\infty} h(t) T(t, F) t^{-1-\rho} dt + K_3 \rho^{-1}(\log r_1) r_1^{-\rho}
\]

\[
+ K_3 \rho^{-2} r_1^{-\rho} \geq kT(r_1, F_1) r_1^{-\rho}.
\]

Choose a sequence \( \{r_1\}_n \to \infty \) such that

\[
T(t, F) t^{-\rho} < T((r_1)_n, F)((r_1)_n)^{-\rho} \quad (t > (r_1)_n).
\]

Then we deduce from (2.15) and (2.2) that for \( n \geq n_0 \).
\[
\pi \rho (\cos \rho - 1 + \delta) \int_{(r_{1})_{n}}^{\infty} h(t) t^{-1} dt + K_{3} \rho^{-1} \log (r_{1})_{n} / T((r_{1})_{n}, F) \\
+ K_{3} \rho^{-2} / T((r_{1})_{n}, F) \geq k T((r_{1})_{n}, F_{1}) / T((r_{1})_{n}, F) > k / 2.
\]

Since \( h(r) \subseteq S_{1} \), the left hand side of (2.16) \( \to 0 \) \( (n \to \infty) \). This is a contradiction.

Case 2. Next we consider the case \( \alpha = \alpha(f) \in (0, +\infty) \). Given \( \varepsilon > 0 \), there is a number \( R_{0} (\geq r_{0}) \) such that \( t \geq R_{0} \) implies \( T(t, F) t^{-\rho} < \alpha + \varepsilon \). Hence by (2.14) we have for \( R_{n} < r_{1} < R / 2 \)

\[
\pi \rho (\cos \rho - 1 + \delta) (\alpha + \varepsilon) \int_{r_{1}}^{R / 2} h(t) t^{-1} dt + K_{3} \rho^{-1} (\log r_{1})_{n}^{\rho} + K_{3} \rho^{-2} r_{1}^{\rho} \\
> k T(r_{1}, F_{1}) r_{1}^{\rho} - K_{1} T(2R, F_{1})(2R)^{-\rho}.
\]

Choose \( \{(r_{1})_{n}\} \to \infty, \{2R_{n}\} \to \infty \) such that \( R_{n} < (r_{1})_{n} < R_{n} / 2 (n = 1, 2, \ldots) \) and \( T((r_{1})_{n}, F_{1})(r_{1})_{n}^{\rho} \to \alpha, T(2R_{n}, F_{1})(2R_{n})^{-\rho} \to 0 \) \( (n \to \infty) \). Then from (2.17) it follows that for \( n \geq n_{0} = n_{0}(\varepsilon) \)

\[
\pi \rho (\cos \rho - 1 + \delta) (\alpha + \varepsilon) \int_{(r_{1})_{n}}^{R / 2} h(t) t^{-1} dt + \varepsilon > (\alpha - \varepsilon) k - \varepsilon K_{1}.
\]

Now, let \( n \to \infty \) to get \( \varepsilon \geq (\alpha - \varepsilon) k - \varepsilon K_{1} \). Since \( \varepsilon > 0 \) was arbitrary, this implies \( k \leq 0 \), a contradiction.

Case 3. It remains to consider the case \( \alpha(f) = +\infty \). First, choose \( \{2R_{n}\} \to \infty \) such that \( R_{n} > 2 \), and

\[
T(2R_{n}, F_{1})(2R_{n})^{-\rho} \to 0 \quad (n \to \infty).
\]

Next, define \( \{(r_{1})_{n}\} (1 \leq (r_{1})_{n} \leq R_{n} / 2) \) by

\[
\max_{1 \leq t \leq R_{n} / 2} T(t, F_{1}) t^{-\rho} = T((r_{1})_{n}, F_{1}) (r_{1})_{n}^{-\rho}.
\]

Then the fact that \( \alpha(f) = +\infty \) and (2.19) give

\[
T((r_{1})_{n}, F_{1})(r_{1})_{n}^{-\rho} \to \infty \quad (n \to \infty),
\]

which, in particular, implies \( \{(r_{1})_{n}\} \to \infty \). Further, in view of (2.18) and (2.20) we see that \( (r_{1})_{n} < R_{n} / 2 (n \geq n_{0}) \). Now, we use (2.14) with \( r_{1} = (r_{1})_{n} \) and \( R = R_{n} (n \geq n_{0}) \). Taking (2.19) into consideration, we have

\[
\pi \rho (\cos \rho - 1 + \delta) T((r_{1})_{n}, F_{1})(r_{1})_{n}^{-\rho} \int_{(r_{1})_{n}}^{R_{n} / 2} h(t) t^{-1} dt + K_{3} \rho^{-1} (\log (r_{1})_{n})(r_{1})_{n}^{\rho} \\
+ K_{3} \rho^{-2} (r_{1})_{n}^{\rho} > k T((r_{1})_{n}, F_{1})(r_{1})_{n}^{\rho} - K_{1} T(2R_{n}, F_{1})(2R_{n})^{-\rho}.
\]

Since \( h(r) \subseteq S_{1} \), we deduce from (2.21), (2.18) and (2.2) that

\[
T((r_{1})_{n}, F_{1})(r_{1})_{n}^{\rho} \to 0 \quad (n \to \infty),
\]
which contradicts (2.20).

Thus we see that (2.6) is not valid. Hence there is a sequence \( \{r_n\} \to \infty \) such that

\[
\pi \rho N(r, \infty, F) + \sin \pi \rho \log m^*(r, F) = \pi \rho \cos \pi \rho N(0, F) + K_2 \log r \quad (r=r_n),
\]

where \( K_2 \) is any fixed positive number. As in the proof of Theorem 1 of [9], we deduce from (2.22) that

\[
\sin \pi \rho \log m^*(r, F) > \pi \rho (\cos \pi \rho - 1 + \delta)(1 + h(r)) T(r, F) + K_2 \log r - O(1) \quad (r=r_n).
\]

From this and (2.2) it follows that

\[
\sin \pi \rho \log m^*(r, f) > \pi \rho (\cos \pi \rho - 1 + \delta)(1 + h(r)) T(r, f) + K_2 \log r - O(1) \quad (r=r_n).
\]

3. Edrei proved the following Theorem A in [5].

**Theorem A.** Assume that \( f(z) \in M_{\rho, s} \) satisfies the relation

\[
\lim_{r \to \infty} \frac{\log m^*(r, f)}{T(r, f)} = \frac{\pi \rho}{\sin \pi \rho} (\cos \pi \rho - 1 + \delta).
\]

Then there exist three positive sequences \( \{r_n\} \to \infty, \{r'_n\} \to \infty, \{r''_n\} \to \infty \) having all the following properties.

(i) \( r'_n < r_n < r''_n < r'_{n+1} \) (\( n=1, 2, 3, \ldots \)).

(ii) \( r_n/r'_n \to \infty, r''_n/r_n \to \infty \) as \( n \to \infty \).

(iii) \( \lim_{n \to \infty} \frac{\log m^*(r_n, f)}{T(r_n, f)} = \frac{\pi \rho}{\sin \pi \rho} (\cos \pi \rho - 1 + \delta) \).

(iv) Put \( L(r) = T(r, f)/r^\rho \) (\( r>0 \)), and let \( A = \bigcup_{n=1}^{\infty} (r'_n, r''_n) \). Then

\[
\lim_{r \to \infty, r \in A} \frac{L(\sigma r)}{L(r)} = 1 \quad (\sigma > 0)
\]

and

\[
\lim_{r \to \infty, r \in A} \frac{N(r, \infty, f)}{N(r, a, f)} = 1 - \delta
\]

hold, where \( a \in \mathbb{C} \) is any number satisfying \( f(0) \neq a \) and (1) of [10].

(v) Let \( s>0 \) and \( \varepsilon>0 \) be given. Consider the annuli \( A_n(s) = \{z = re^{i\theta}; e^{-\varepsilon} < r/r_n < e^\varepsilon\} \), the sectors \( S_n(s; \varphi-\varepsilon, \varphi+\varepsilon) = \{z = re^{i\theta} \in A_n(s); \varphi-\varepsilon < \theta < \varphi + \varepsilon\} \), and let \( \{\omega_n\} \) be any real sequence defined by the conditions \( m^*(r_n, f-a) = |f(r_n e^{i\omega_n})-a| \) (\( k=1, 2, 3, \ldots \)). Let \( \nu_\nu(a) \) be the number of zeros of \( f(z)-a \) in the sector \( A_n(s) \).
\(-S_n(s; \omega_n - \varepsilon, \omega_n + \varepsilon), \text{ and } \nu_n(\infty) \text{ the number of poles of } f(z) \text{ in } A_n(s) - S_n(s; \omega_n + \pi - \varepsilon, \omega_n + \pi + \varepsilon). \text{ Then}

\[
\lim_{n \to \infty} \frac{\nu_n(a) + \nu_n(\infty)}{T(r_n, f)} = 0.
\]

The above Edrei's result implies that the extremal functions \(f(z)\) for the estimate (2) satisfy the relation \(T(r, f) \sim r^pL(r)\) (with slowly varying functions \(L(r)\)) at least locally as \(r \to \infty\).

In this section we first prove the following

**Theorem 9.** Let \(f(z)\) be a meromorphic function of the form

\[
f(z) = \frac{\Pi(1+z/a_n)}{\Pi(1-z/b_n)} = \frac{P(z)}{Q(z)} \quad (0 < a_n \leq a_{n+1}, 0 < b_n \leq b_{n+1}),
\]

and let \(L(r)\) be a slowly varying function. Then

(3.1) \(T(r, f) \sim r^pL(r)\) \((r \to \infty, 0 < p < 1/2)\)

and

(3.2) \(N(r, \infty, f) \sim (1-\delta)N(r, 0, f)\) \((r \to \infty, 1-\cos \pi \rho < \delta < 1)\)

or

(3.2)' \(N(r, \infty, f) = 0\) \((r \geq 0, \delta = 1)\)

imply that for \(\varepsilon > 0\)

(3.3) \(\log m^*(r, f) \leq \frac{\pi \rho}{\sin \pi \rho} (\cos \pi \rho - 1 + \delta)(1+\varepsilon)T(r, f)\) \((r \geq r_\varepsilon(\delta))\).

**Proof.** Let \(\{r_n\}\) be any positive, increasing, unbounded sequence. Then the hypothesis (3.1) implies that \(\{r_n\}\) is a sequence of Pólya peaks of order \(p\) for \(T(r, f)\). (See [2, p 94].) Using the assumption (3.2) or (3.2)', we easily deduce that

(3.4) \(\delta(\infty, f) \geq \delta > 1 - \cos \pi \rho\).

Now, put \(J(r) = \{\theta \in (-\pi, +\pi] ; |f(re^{i\theta})| \geq 1\}. \text{ Then the spread relation (See \[3\].) and (3.4) yield}

\[
\lim_{n \to \infty} \inf \\text{meas} \ J(r_n) \geq \min \left\{\frac{4}{\rho} \sin^{-1} \sqrt{\frac{\delta(\infty, f)}{2}}, 2\pi\right\} = 2\pi, \text{ i.e.}
\]

(3.5) \(\lim_{n \to \infty} \text{meas} \ J(r_n) = 2\pi.\)

From the first fundamental theorem and the Edrei-Fuchs Lemma (See [6, p 322]), it follows that
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\[ T(r, f) - N(r, 0, f) = m(r, 0, f) - \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |f(\rho e^{i\theta})| d\theta \]

(3.6)

\[ \leq 11T(2r, f) \text{ meas } J(r) \left\{ 1 + \log \left( \frac{1}{\text{meas } J(r)} \right) \right\}. \]

In view of (3.1) we have

(3.7) \[ T(2r, f) \sim 2^p T(r, f) \quad (n \to \infty). \]

Substituting (3.5) and (3.7) into (3.6), we deduce that

(3.8) \[ T(r, f) - N(r, 0, f) = o(T(r, f)) \quad (n \to \infty). \]

Since the sequence \( \{r_n\} \) was arbitrary, (3.8) gives

(3.9) \[ N(r, 0, f) \sim T(r, f) \quad (r \to \infty), \]

and so by (3.1) and (3.2)

(3.10) \[ N(r, 0, f) \sim r^p L(r) \quad (r \to \infty), \]

(3.11) \[ N(r, \infty, f) \sim (1-\delta)r^p L(r) \quad (r \to \infty, 1-\cos \pi \rho < \delta < 1). \]

Then an abelian argument (See, for example, [7, Theorem 2].) may be used to prove

(3.12) \[ \log |P(\rho e^{i\theta})| = -\frac{\pi \rho}{\sin \pi \rho} \{ \cos \theta \rho + o(1) \} r^p L(r) \quad (r \to \infty, |\theta| < \pi), \]

and

(3.13) \[ \log |Q(\rho e^{i\theta})| = -\frac{\pi \rho}{\sin \pi \rho} (1-\delta) \{ \cos(\pi - \theta) \rho + o(1) \} r^p L(r) \quad (r \to \infty, 0 < \theta < 2\pi). \]

Given \( \varepsilon > 0 \), choose \( \eta > 0 \) with the property that \( \cos(\pi - \eta) \rho - 1 + \delta < (\cos \pi \rho - 1 + \delta) (1 + \epsilon/2) \). Then (3.12), (3.13) and (3.1) give

\[ \log m^*(r, f) = \log |P(-r)| - \log Q(-r) < \log |P(\rho e^{i(\pi - \eta)})| - \log Q(-r) \]

\[ < -\frac{\pi \rho}{\sin \pi \rho} \{ \cos(\pi - \eta) \rho - (1-\delta) + o(1) \} r^p L(r) \]

\[ < -\frac{\pi \rho}{\sin \pi \rho} \{ \cos \pi \rho - 1 + \delta \} (1+\epsilon) T(r, f) \quad (r \geq r_0(\varepsilon)). \]

This completes the proof of Theorem 9.

We conclude from Theorems A and 9 that the simplest and the most typical growth of the characteristic functions of \( f(z) \in \mathcal{M}_{\rho, \delta} \) satisfying (3.3) is regular variation of order \( \rho \).
Now, we refine the estimate (2) for all $f(z) \in M_{\rho, \delta}$ whose characteristics vary regularly with order $\rho$.

Case 1. $\alpha(f) = 0$. Choose $h(r) \in S_2$ arbitrarily satisfying

$$T(r, f) = O\left( r^\rho \exp \left\{ \frac{1}{1-\varepsilon} C(\rho, \delta) \int_1^r \frac{h(t)}{t} \, dt \right\} \right) \quad (r \to \infty)$$

with some $\varepsilon > 0$. Such an $h(r) \in S_2$ certainly exists. (See Lemma 12.) Then the estimate (1) holds on an unbounded sequence of $r$. (See Theorem 2.)

Case 2. $\beta(f) = 0$ or $0 < \beta(f) < \frac{1}{K_1} - \alpha(f) \leq +\infty$. In these cases, for any $h(r) \in S_2$, we have the estimate (1) for certain arbitrarily large values of $r$. (For the proof, see Theorems 7 and 8.)

Case 3. $0 < \beta(f) \leq \alpha(f) \leq \frac{1}{k} \beta(f) < +\infty$. Let $h(r) \in S_2$ be given. Then the estimate

$$\log m^*(r, f) > \frac{\pi \rho}{\sin \pi \rho} (\cos \pi \rho - 1 + \delta)(1 - h(r)) T(r, f)$$

holds for a sequence of $r \to \infty$. (See Corollary 1 of [10].)

Case 4. $\beta(f) = +\infty$. Choose $h(r) \in S_2$ arbitrarily such that

$$T(r, f) = O\left( r^\rho \exp \left\{ \frac{1}{1+\varepsilon} C(\rho, \delta) \int_1^r \frac{h(t)}{t} \, dt \right\} \right) \quad (r \to \infty).$$

with some $\varepsilon > 0$. To see such a $h(r) \in S_2$ exists, we may note that any slowly varying function can be written as

$$L(r) = c(r) \exp \left( \int_1^r \varepsilon(t)t^{-1} \, dt \right),$$

where $\lim_{r \to \infty} c(r) = c > 0$ and $\lim_{t \to \infty} \varepsilon(t) = 0$. (See [8, p 45].) Then the estimate (3.14) holds for a sequence of $r \to \infty$. (See [10, Theorem 1].)

REFERENCES

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