SECTION 1. Introduction.

Let $E$ be an oriented orthogonal $q$-sphere bundle over a connected finite CW-complex $B$. A fibre-preserving map $f : E \to E$ is said to have degree $m$ when its restriction to some fibre is a map of degree $m$ in the familiar sense; because $B$ is path-connected it makes no difference which fibre we choose. Given $E$ and an integer $m$ is there a fibre-preserving map $f : E \to E$ of degree $m$? This question was put to me in 1971 by I. M. James, and in [2] there are some answers in fairly general situations. In the present paper I consider in more detail the special case where $B$ is a sphere $S^{r+1}$. We first make some simple observations.

The identity map has degree 1, and when $q$ is even $E$ always admits a fibre-preserving map of degree $-1$; this is because the antipodal map $a : S^q \to S^q$ commutes with the action of the group $SO(q+1)$ of rotations in $\mathbb{R}^{q+1}$ and therefore extends to a fibre-preserving map: it would be interesting to know what happens when $E$ is a general oriented $q$-spherical fibration with $q$ even. If $E$ admits fibre-preserving maps of degrees $m, n$ then their composite is a fibre-preserving map of degree $mn$. Apart from this, nothing is very obvious.

Let $\pi : E \to B$ be the projection. Then when $E$ has a cross-section $s$ the composite $s\pi : E \to E$ is a fibre-preserving map of degree 0. In [2] the converse is proved, namely that if $E$ admits a fibre-preserving map of degree 0 then $E$ has a cross-section. (It is not the case that every fibre-preserving map $f : E \to E$ of degree 0 is homotopic through fibre-preserving maps to one of the form $s\pi$ for some cross-section $s$, but if $B$ is covered by $k$ contractible open subsets then $f^k$ is homotopic through fibre-preserving maps to $s\pi$ for some cross-section $s$.) Some of the main results of [2] describe the structure of the set $A(E)$ of integers $m$ such that $E$ admits a fibre-preserving map of degree $m$. In the present paper we prove some results that allow us to estimate $A(E)$ when $B = S^{r+1}$.

If $E^*$ is a fibre bundle over $S^{r+1}$ with fibre $F^*$ let $o(E^*)$ be the obstruction to a cross-section of $E^*$, as defined in §2 below. From now on let $B = S^{r+1}$. In §2 we show that a necessary condition for there to be a fibre-preserving map $E \to E$ of degree $m$ is that $\phi_m o(E) = o(E)$. Here $\phi_m : \pi_* S^q \to \pi_* S^q$ is induced by a map of degree $m$ on $S^q$. 

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THEOREM 1. Let $q$ be odd. If there is a fibre-preserving map $E \to E$ of degree $m$ then

(i) $\phi_{-m} \circ \iota_o(E) = 0$ and
(ii) $(m(m-1)/2)[\tau_q, \tau_{q+1}] \ast \Sigma^r \circ \iota_o(E) = 0.$

Here $[,]$ is the J.H.C. Whitehead product, and $\Sigma^r$ is the suspension homomorphism. Theorem 1 is proved as (4.1), (4.3) in §4.

Our methods can also be used to give conditions sufficient for the existence of a fibre-preserving map $E \to E$ of degree $m$. In [1] Part II, §5 some calculations are carried out and, for example, necessary and sufficient conditions are given when $q$ is odd and $r \leq q+2$.

In §5 of the present paper we consider the special cases where $q=1, 3, 7$, and prove that the necessary conditions given in Theorem 1 are then sufficient. When $q=1$ condition (ii) of Theorem 1 is satisfied trivially. When $q=3$ we obtain as (5.3), (5.5)

COROLLARY 1. Let $q=3$. Then there is a fibre-preserving map $E \to E$ of degree $m$ both

(i) $(m-1) \circ \iota_o(E) = 0$ and
(ii) $(m(m-1)/2)a \ast \Sigma^r \circ \iota_o(E) = 0$

where $a \in \pi_r S^4$ is described explicitly in §5. In an interesting paper [5] Seiya Sasao considers a related problem, and in §6 of the present paper we compare Corollary 1 with Sasao's results.

When $q$ is even it may be the case that $E$ has a cross-section and yet there exist no fibre-preserving maps $E \to E$ of some degrees $m$: this cannot happen when $q$ is odd. When $E$ has a cross-section we can write $E$ as the fibre suspension of an oriented orthogonal $(q-1)$-sphere bundle $E'$, and in §7 we prove

THEOREM 2. Let $q$ be even and suppose that $E$ has a cross-section. Then there is a fibre-preserving map $E \to E$ of degree $m$:

$$(m(m-1)/2)[\tau_q, \tau_{q+1}] \ast \Sigma^r \circ \iota_o(E') \in [m \tau_q, \tau_{q+1} S^r].$$

By [9] 3.59 we have $[\tau_q, \tau_{q+1}] \ast \Sigma^r \circ \iota_o(E') = [\tau_q, \Sigma^r \circ \iota_o(E')]$ and so we have

COROLLARY 2. Let $q$ be even and suppose that $E$ has a cross-section. Then there are fibre-preserving maps $E \to E$ of all odd degrees, and of all degrees $m \equiv 0 \mod 4$.

§2. The Obstruction to a Fibre-Preserving Map—Generalities.

It is known that a fibre-preserving map of fibre bundles corresponds naturally to a cross-section of a bundle whose fibre is a function space. This point of view is traceable to I.M. James and was taken in [2] to prove results about
the structure of $A(E)$; it also turns out to be helpful when doing calculations.

Let $G^n$ be the function space of maps $f: S^q \to S^q$ of degree $m$, with the compact-open topology. We define a left action $*$ of the group $SO(q+1)$ of rotations on $G^n$ by $(A*f)(x) = A*(f(A^{-1} \cdot x))$. Here $\cdot$ is the standard action of $SO(q+1)$ on $S^q$. Let $E$ be an oriented orthogonal $q$-sphere bundle over $B$, and let $P$ be its associated principal $SO(q+1)$-bundle. Let $E_m$ be the bundle $PG^n$ associated with $P$ and with fibre $G^n$.

There is a natural one to one correspondence between fibre-preserving maps $E \to E$ of degree $m$ and cross-sections of $E_m$. When $B$ is a sphere $S^{r+1}$ there is only one obstruction which can be defined in a familiar way, but its calculation in particular cases is not so easy.

Indeed, let $E^*$ be any fibre bundle over $S^{r+1}$ with fibre $F^*$. Then the homotopy exact sequence of the fibering takes the form

$$\cdots \to \pi_{t+1}S^{r+1} \to \pi_t F^* \to \pi_t E^* \to \pi_t S^{r+1} \to \pi_{t-1} F^* \to \cdots.$$ 

Let $\iota_{r+1}$ be the generator of $\pi_{r+1}S^{r+1}$ represented by the identity map, and let $o(E^*)$ be the image in $\pi_r F^*$ of $\iota_{r+1}$. Then $E^*$ has a cross-section if and only if $o(E^*)$ is the trivial element of $\pi_r F^*$, and so $o(E^*)$ may be regarded as the obstruction to a cross-section of $E^*$. So when $B=S^{r+1}$ the obstruction to a fibre-preserving map $E \to E$ of degree $m$ is $o(E_m) \in \pi_r G^n$. We want to calculate this obstruction in terms of standard invariants of $E$, for example $o(E)$. We first point out

\begin{enumerate}
\item[(2.1)] A necessary condition for there to be a fibre-preserving map $E \to E$ of degree $m$ is that $\phi_m o(E) = o(E)$.
\end{enumerate}

Here $\phi_m: \pi_r S^q \to \pi_r S^q$ is induced by composition with a map $S^q \to S^q$ of degree $m$. We note that $\phi_m$ is not in general multiplication by $m$, although this is the case if $r<2q-1$. (For clarification of this point see [9] Theorem 5.15.)

To prove (2.1) observe that a fibre-preserving map $f: E \to E$ of degree $m$ produces the following commuting diagram, where the rows are the homotopy exact sequence of $E$.

$$\begin{array}{cccccc}
\cdots & \to & \pi_{t+1}S^{r+1} & \to & \pi_t S^q & \to & \pi_t E & \to \cdots \\
& & \downarrow 1 & & \downarrow \phi_m & & \downarrow f_* \\
& & \cdots & \to & \pi_{t+1}S^{r+1} & \to & \pi_t S^q & \to & \pi_t E & \to \cdots.
\end{array}$$

§ 3. Odd Values of $q$.

Let $k_m: S^q \to G^n$ be the map defined in [2] §2 where $n=m$, 1, 0 according as $q$ is odd, $q$ is even and $m$ is odd, or $q$ and $m$ are both even. (Given $x$, $y \in S^q$ let $\theta$ be the distance along some geodesic from $x$ to $y$. On this geodesic and
at distance $m\theta$ we have $k_m(x)(y)$. Then $k_m$ is equivariant with respect to the standard left action of $SO(q+1)$ on $S^q$ and the left action $\ast$ on $G^q_\theta$.

As in §2, $P$ is the principal $SO(q+1)$-bundle over $B$ associated with $E$, and because it is equivariant $k_m$ extends from fibres to a fibre-preserving map $P(k_m): E \to E_n$; for the rest of this section we take $q$ to be odd, so that $n=m$. Then if $E$ has a cross-section $s$ the composite $P(k_m)s$ is a cross-section of $E_m$, and therefore there is a fibre-preserving map $E \to E$ of degree $m$.

Now let $B=S^{r+1}$. When $E$ does not have a cross-section the condition that $\delta(E_m)$ should be zero translates into a condition on $\delta(E)$ as follows. Consider the commuting diagram of group homomorphisms

\[
\begin{array}{cccccccc}
\cdots & \to & \pi_{t+1}S^{r+1} & \to & \pi_tE & \to & \pi_{t+1}S^{r+1} & \to & \cdots \\
\downarrow & & \downarrow k_m & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \to & \pi_{t+1}S^{r+1} & \to & \pi_tE & \to & \pi_{t+1}S^{r+1} & \to & \cdots
\end{array}
\]

where the unlabelled vertical arrow denotes the homomorphism induced by $P(k_m)$. Taking $t=r+1$ we obtain $\delta(E_m)=k_m \delta(E)$. In practice we usually know $\delta(E)$; but the homomorphism $k_m$ is a less accessible object, in part because the computation of $\pi_rG^q_\theta$ is complicated by the appearance of Whitehead products [8].

### § 4. Homotopy Groups of Function Spaces.

Let $(1, 0, 0, \cdots, 0) \in S^q$ be chosen as the basepoint and let $F^q_{\theta}$ be the subspace of $G^q_{\theta}$ consisting of the basepoint-preserving maps of degree $m$. The evaluation map $e: G^q_{\theta} \to S^q$, given by $e(f)=f(1, 0, 0, \cdots, 0)$, is a fibration with fibre $F^q_{\theta}$. Let $I_m: \pi_tF^q_{\theta} \to \pi_{t+q}S^q$ be the Hurewicz isomorphism [8]. Then the homotopy exact sequence of $e$ gives us an exact sequence

\[
\cdots \to \pi_{t+1}S^q \xrightarrow{P_{t+1}} \pi_{t+q}S^q \xrightarrow{\delta_q} \pi_tE \xrightarrow{\pi_{t+1}S^q} \pi_tE \xrightarrow{\pi_{t+1}S^q} \cdots
\]

of homomorphisms of abelian groups. The unlabelled arrow denotes the homomorphism $i_qI_{m}^{-1}$ where $i$ is the inclusion of $F^q_{\theta}$ in $G^q_\theta$. According to [8], [10], 

\[P_t(\theta) = \pm [m_q, \theta]\]

where $[\ , \ ]$ is the Whitehead product.

For the rest of §4 let $q$ be odd, and let $E$ be an oriented orthogonal $q$-sphere bundle over $S^{r+1}$. The composite $ek_m: S^q \to S^q$ is a map of degree $1-m$, and composition with this defines the homomorphism $\phi_{1-m}: \pi_rS^q \to \pi_{r-1}S^q$. Consequently

\[(4.1)\text{ A necessary condition for } E \text{ to admit a fibre-preserving map } E \to E \text{ of degree } m \text{ is that } \phi_{1-m}(\delta(E))=0.\]

(When $r<2q-1$ this is equivalent to (2.1), and (2.1) has been proved whether $q$
is odd or even.)

Let \( j : G^q_{m} \to F^q_{m} \) be the inclusion defined by suspending maps of degree \( m \).

**Lemma (4.2).** The homomorphism

\[
I_m j_* k_m* : \pi_r S^q \to \pi_r G^q_{m} \to \pi_r F^q_{m} \to \pi_{r+q+1} S^{q+1}
\]

is given by

\[
\beta \mapsto \pm (m(m-1)/2)[\tau_{q+1}, \tau_{q+1}] \cdot \sum^{q+1}_* \beta.
\]

Here \( \sum_* : \pi_r S^q \to \pi_{r+1} S^{q+1} \) is the suspension homomorphism.

To prove (4.2) we note that \( I_m j_* k_m* \beta \) is the Hopf construction of the adjoint \( k'_m : S^q \times S^q \to S^q \) of \( k_m \), preceded by \( \sum^{q+1}_* \beta \). It therefore suffices to show that

\[
I_m j_* k_m* \tau_q = \pm (m(m-1)/2)[\tau_{q+1}, \tau_{q+1}]
\]

but when \( m = -1 \) [6] §23.5 tells us that

\[
I_{-1} j_* k_{-1}* \tau_q = \sum_* \phi = j_\tau \phi = 0
\]

where \( j \) denotes the Whitehead homomorphisms. By [4] Theorem 7.7 \( I_{-1} j_* k_{-1}* \tau_q = n[\tau_{q+1}, \tau_{q+1}] \) for some integer \( n \).

But \( k'_1 \) has type \((-1, 1)\) and so, by [9] 3.70, \( I_{-1} j_* k_{-1}* \tau_q \) has Hopf invariant \( \pm 2 \). But, by [9] Theorem 5.31, \( [\tau_{q+1}, \tau_{q+1}] \) has Hopf invariant \( \pm 2 \). So \( n = \pm 1 \).

This proves (4.2) in the special case where \( m = -1 \).

From the definition of \( k_m \) we find that \( k'_m(x, y) = k'_{m-1}(y, x) = k_{m-1}^*(y, k'_1(y, x)) \) and so \( I_m j_* k_m* \tau_q \) is the Hopf construction of the composite

\[
S^q \times S^q \xrightarrow{\text{switch}} S^q \times S^q \xrightarrow{1 \times k'_1} S^q \times S^q \xrightarrow{k_{m-1}^*} S^q
\]

which, according to [3] Theorem 2.19, differs by a multiple of a Whitehead product \([\tau_{q+1}, \tau_{q+1}]\) from the negative of the Hopf construction of just

\[
S^q \times S^q \xrightarrow{1 \times k'_1} S^q \times S^q \xrightarrow{k_{m-1}^*} S^q
\]

namely \( -I_{-1} j_* k_{-1}* \tau_q = \sum_* \phi = j_\tau \phi = 0 \). We can now argue as in the case where \( m = -1 \), noting that \( k'_m \) is a map of type \((-1, m)\). This proves (4.2).

Recall again that \( \phi(E_m) = k_{m}* \phi(E) \). Then in addition to (4.1) we have

(4.3) A necessary condition for \( E \) to admit a fibre-preserving map \( E \to E \)

of degree \( m \) is that \( (m(m-1)/2)[\tau_{q+1}, \tau_{q+1}] \cdot \sum^{q+1}_* \phi(E) = 0 \).

We emphasise that here \( q \) is odd. In the particular cases where \( q = 1, 3, 7 \) it is possible to say even more.
§ 5. Necessary and Sufficient Conditions When \( q = 1, 3, 7 \).

When \( q = 1 \) questions about fibre-preserving maps are easy to answer, because \( \pi_{r+1} S^1 \) is trivial for \( i > 0 \). Consequently \( e : G^1 \to S^1 \) is a weak homotopy equivalence. Let \( E \) be an oriented 1-sphere bundle over a connected finite CW-complex \( B \), and let \( \chi(E) \equiv H^2(B; \mathbb{Z}) \) be the Euler characteristic of \( E \), namely the obstruction to a cross-section of \( E \). Then because \( ek_m \) is a map of degree \( 1 - m \) it follows that there is a single obstruction to a cross-section of \( E_m \), namely \( (1 - m)\chi(E) \). We have

\[(5.1) \text{ When } q = 1 \text{ } E \text{ admits a fibre-preserving map } E \to E \text{ of degree } m \]

if and only if \( (1 - m)\chi(E) = 0 \).

The situation when \( q \neq 1 \) is nontrivial however, and for the remainder of § 5 we suppose that \( B = S^{r+1} \) where \( r \leq 3q - 1 \), and that \( q = 3 \) or \( 7 \). We prove

\[(5.2) \text{ The necessary conditions (4.1), (4.3) are also sufficient for } E \text{ to admit a fibre-preserving map } E \to E \text{ of degree } m.\]

**Proof of (5.2):** If (4.1) holds then we know that \( k_m \circ o(E) = i_* I_m^1 \gamma \) for some \( \gamma \in \pi_{r+q} S^q \). If (4.3) also holds then \( I_m j_* i_* I_m^1 \gamma = 0 \). But \( I_m j_* i_* I_m^1 \) is \( \Sigma^*_k \) and so we know that \( \Sigma^*_k \gamma = 0 \).

However, according to [4] Theorem 7.7 the kernel of \( \Sigma^*_k : \pi_r S^q \to \pi_{r+1} S^{r+1} \) is the image of the homomorphism

\[ \pi_{r+q+1} S^q \to \pi_r S^q \]

\[ \delta a \mapsto [\delta, \langle a \rangle]. \]

Since \( q = 3 \) or \( 7 \), \( S^q \) is an \( H \)-space and therefore \( \Sigma^*_k \) is injective. So \( \gamma = 0 \), and therefore \( o(E_m) = 0 \). This proves (5.2).

The conditions (4.1), (4.3) can be simplified somewhat in the special cases considered here, namely when \( q = 3, 7 \). Firstly, because \( S^q \) is an \( H \)-space, (4.1) is equivalent to

\[(5.3) \quad (m-1) o(E) = 0. \]

We next analyse (4.3) but restrict ourselves to the case where \( q = 3 \). We recall from [7] Lemma 4.3 that

\[(5.4) \quad [\iota_4, \iota_4] = 2\nu_4 - a_4. \]

Here \( \nu_i \in \pi_i S^1 \) is the Hopf class, namely the Hopf construction of quaternionic multiplication restricted to \( S^3 \times S^3 \), and \( a_i = \Sigma a_i \) where \( a_i \in \pi_i S^3 \) is the Hopf construction of

\[ g : S^3 \times S^3 \to S^3 \]
given by

$$(x, y) \mapsto xyx^{-1}.$$  

(We are thinking of $S^2$ as the space of unit quaternions with vanishing real part, and multiplication on $S^3$ is again quaternionic.) According to [7] Theorem 7.2, $a_5$ has order 12 and generates $\pi_6S^3$. Also $\pi_7S^4 \cong \mathbb{Z} \oplus \mathbb{Z}/12\mathbb{Z}$ where the summands are generated by $v_4$ and $a_4$.

In view of (5.4), (5.3) a necessary and sufficient condition for $E$ to admit a fibre-preserving map $E \to E$ of degree $m$ is (5.3) together with

$$\sum a_r \cdot o(E) = 0.$$  

A similar analysis can be carried out in the case where $q=7$.

§ 6. Comparison With a Result of Seiya Sasao.

It is interesting to compare (5.5) with Example 1 of [5]. In Example 1 Sasao takes $q=3$ and $E$ is a principal $S^3$-bundle over $S^{r+1}$, whereas (5.5) is applicable whether $E$ is a principal $S^3$-bundle or not. On the other hand, (5.5) applies when $r \leq 5$, whereas Sasao makes no such requirement. Sasao proves that a map of degree $m$ on fibres $S^3 \to S^3$ extends to a map $E \to E$ (not necessarily fibre-preserving) if and only if

$$\sum a_r \cdot o(E) = 0.$$  

When $r \leq 5$, (6.1) is then equivalent to

$$\sum a_r \cdot o(E) = 0.$$  

So Sasao's necessary and sufficient condition is less restrictive than (5.5) alone, at least when $r \leq 5$. (5.5) and (5.3) must together imply Sasao's condition, because a fibre-preserving map $E \to E$ of degree $m$ extends a map $S^3 \to S^3$ of degree $m$ on fibres.)

§ 7. When $q$ is Even.

Let $q$ be even. We shall see that $E$ may have a cross-section and yet fail to admit fibre-preserving maps $E \to E$ of all degrees. It follows that there is no $SO(q+1)$-equivariant map from $S^q$ to $G_k^q$, whereas when $q$ was odd we had the map $k_m$.

By (2.1), if there is a fibre-preserving map $E \to E$ of even degree $m$ then $\phi_m o(E) = o(E)$, and so if $r < 2q - 1$ we have $(m - 1) o(E) = 0$. But $q$ is even and so there is a fibre-preserving map $E \to E$ of degree $-1$. Therefore when there is a fibre-preserving map $E \to E$ of even degree, and $r < 2q - 1$, we have $o(E) = 0$, namely $E$ has a cross-section. From now on we consider only bundles $E$ which have cross-sections, namely bundles $E$ which are unreduced fibre suspensions of
orthogonal $q-1$-sphere bundles $E'$; we do not require $r<2q-1$.

Let $P'$ be the principal $SO(q)$-bundle associated with $E'$ and let $SO(q)$ act on $S^q$ by the suspension of the action on $S^{q-1}$. Then $k_m: S^{q-1} \rightarrow G_m^{q-1}, j: G_m^{q-1} \rightarrow F_m^q, i: F_m^q \rightarrow G_m^q$ are $SO(q)$-equivariant and therefore $jk_m$ extends from fibres to a fibre-preserving map $P'(jk_m): E' \rightarrow E_m$. It follows that $i\ast j\ast k_m\ast o(E')$ is the obstruction $o(E_m) \subseteq \pi_r G_m^q$ to a fibre-preserving map of degree $m$ from $E$ to itself.

(7.1) There is a fibre-preserving map $E \rightarrow E$ of degree $m \Rightarrow$

$$(m(m-1)/2)[\zeta_q, \zeta_q] \ast \sum\beta \ast o(E') \subseteq [m\eta_q, \pi_{r+q}S^q].$$

To prove (7.1) note that $i\ast j\ast k_m\ast o(E') = i\ast I_m^q(I_m j_k k_m) o(E')$. Now (4.2) says that

$$I_m j_k k_m: \pi_r G_m^{q-1} \rightarrow \pi_r G_m^{q-1} \rightarrow \pi_r F_m^{q-1} \rightarrow \pi_r \pi_r q S^q$$

is given by

$$\beta \mapsto \pm (m(m-1)/2)[\zeta_q, \zeta_q] \ast \sum\beta.$$

On the other hand, exactness of the sequence of homomorphisms in § 4 tells us that the kernel of $i\ast I_m^q$ is $[m\eta_q, \pi_{r+q}S^q]$, and this proves (7.1).

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