IWASAWA THEORY AND $p$-ADIC HODGE THEORY

BY KAZUYA KATO

The aim of this paper is to formulate the Iwasawa main conjecture for varieties (or motives) over arbitrary number fields. See (4.9) for the statement of the conjecture, and (4.15) for "philosophical comments" on it. To formulate our conjecture, we need the $p$-adic Hodge theory developed by Tate [Ta], Fontaine and Messing [FM], and Faltings [Fa].

The classical Iwasawa theory relates special values of partial Riemann zeta functions to the Galois module structures of the ideal class groups of cyclotomic extensions of $\mathbb{Q}$. In our conjecture, we replace $\mathbb{Q}$ by an arbitrary number field $K$, a cyclotomic extension of $\mathbb{Q}$ by a finite abelian extension $L$ of $K$, and partial Riemann zeta functions by partial $L$-functions $L_S(M, \sigma\text{-part}, s)$ of a motive $M$ over $K$ for $\sigma \in \text{Gal}(L/K)$. (Here $S$ is a finite set of finite places of $K$ including "bad places", and $L_S$ means the $L$-function without the $S$-part. For the meaning of the $\sigma$-part, see (4.6).) Our conjecture relates special values of $L(M, \sigma\text{-part}, s)$ to the Galois module structures of the etale cohomology of $\text{Spec}(O_{L,S})$ with coefficients in an etale sheaf coming from $M$, where $O_{L,S}$ is the ring of elements of $L$ which are integral outside $S$.

In [BK], Bloch and the author formulated a conjecture on Tamagawa numbers of motives which generalizes the Birch Swinnerton-Dyer conjecture to general Hasse-Weil $L$-functions. The Iwasawa main conjecture in this paper is a natural generalization of the conjecture of [BK] (which is regarded as the case $L=K$ in the above description).

In our conjecture, we assume the variety is smooth proper over $K$ but we put no other assumption on our variety. We do not assume the variety is of ordinary reduction at the prime number $p$ in problem. We do not assume our motive is critical in the sense of Deligne [De]. However this does not mean that we can define $p$-adic $L$-functions without these assumptions. Our conjecture treats directly the special values of complex $L$-functions, but do not treat $p$-adic interpolations of special values.

In §1-§3, we review known results and conjectures on $p$-adic Hodge theory, $K$-theory, and the duality in Galois cohomology of number fields. We formulate the Iwasawa main conjecture in §4. In §5 we consider the case of Tate motives over $\mathbb{Q}$, and in §6 we show that in case of the motive $Q(r)$ over $\mathbb{Q}$ with $r$ even and positive, and $L$ real abelian extensions of $\mathbb{Q}$, our Iwasawa main conjecture
coincides with the classical Iwasawa main conjecture. In §6 we use a relation (5.12) between $p$-adic Hodge theory and values of partial Riemann zeta functions proved in a forthcoming paper $[Ka_1]$. In §7 we show that our conjecture is a generalization of the conjecture in $[BK]$ on Tamagawa number of motives.

After I completed this paper, I learned that Fontaine and Perrin-Riou found a similar approach to the arithmetic of values of L-functions. ([FP] I, II, III). They did not consider partial L-functions, but they found how to treat mixed motives (especially the height pairing of mixed motives) though my study is limited to pure motives. The motivation of my study was the hope to extend Kolyvagin's Euler systems (which are related to partial L-functions) to motives, and so partial L-functions were essential to me.

I was inspired much by the nice atmosphere in the number theory seminar at Komaba organized by Professors K. Iwasawa, G. Fujisaki, and S. Nakajima. I found the general conjecture during I was preparing a lecture in this seminar on related subjects. I am very thankful to participants of this seminar. I thank Prof. T. Saito for his advice on the proof of (4.17), and Prof. S. Bloch who introduced me to this fascinating field.

Notations: For a field $k$, $\overline{k}$ denotes an algebraic closure of $k$, and $\text{char}(k)$ denotes the characteristic of $k$. As usual, $Q$ (resp. $Q_p$, $R$, $C$) denotes the field of rational (resp. $p$-adic, real, complex) numbers.

§ 1. $p$-adic Hodge theory.

(1.1) We review some results concerning $p$-adic Hodge theory. In this section, let $K$ be a complete discrete valuation field with perfect residue field $k$ such that $\text{char}(K)=0$ and $\text{char}(k)=p>0$.

Fontaine defined a big ring $B_{dR}$ over $K$ endowed with an action of $\text{Gal}(\overline{K}/K)$. For the definition of $B_{dR}$, see $[F0]$. $B_{dR}$ is the field of fractions of a complete discrete valuation ring $B^\circ_{dR}$, $K$ is contained in $B^\circ_{dR}$, the residue field of $B^\circ_{dR}$ is isomorphic over $K$ to the $p$-adic completion of $K$, and $K$ coincides with the $\text{Gal}(\overline{K}/K)$-invariant part of $B_{dR}$.

(1.2) We review the de Rham conjecture of Fontaine [F0] proved by Fontaine and Messing [FM] under certain assumptions and by Faltings $[Fa_1]$ in general.

Let $X$ be a smooth proper scheme over $K$. Then, on one hand, we have the $p$-adic etale cohomology $H^n_{\text{et}}(X, Q_p)(X=\overline{X}\otimes_KK)$ with an action of $\text{Gal}(\overline{K}/K)$. On the other hand we have the de Rham cohomology group $H^n_{\text{dR}}(X/K)$ with the Hodge filtration. The de Rham conjecture (1.3) relates these two different $p$-adic cohomologies.

THEOREM (1.3) ($[Fa_1]\S$). For any $\alpha\in\Omega$, there exists a canonical isomorphism $H^n_{\text{et}}(\overline{X}, Q_p)\otimes_{Q_p}B_{dR}\cong H^n_{\text{dR}}(X/K)\otimes_KB_{dR}$ preserving the actions of $B_{dR}$ and $\text{Gal}(\overline{K}/K)$, and the filtrations. (Here the action
of $\sigma \in \text{Gal}(\overline{K}/K)$ on the left (resp. right) hand side is $\sigma \otimes \sigma$ (resp. $\sigma \otimes \text{id.}$). For $n \in \mathbb{Z}$, $\text{fil}^n$ of the left (resp. right) hand side is

$$H^*_d(X, Q_p) \otimes Q_p \text{fil}^n B_{dR} \text{ resp. } 2 \otimes H^*_d(X/K) \otimes_K \text{fil}^n B_{dR}.$$ 

Here the filtration on $B_{dR}$ is defined by the discrete valuation of $B_{dR}$.

(1.4) We review de Rham representations in the sense of Fontaine. Let $V$ be a finite dimensional vector space over $Q_p$ endowed with a continuous action of $\text{Gal}(\overline{K}/K)$. Let

$$D_{dR}(V) = H^*(K, V \otimes Q_p B_{dR})$$

($H^*(K, )$ means the fixed part under $\text{Gal}(\overline{K}/K)$, which is endowed with the filtration $\{D_{dR}(V)\}$ coming from the filtration $V \otimes Q_p \text{fil}^n B_{dR}$. Then

$$\dim_K(D_{dR}(V)) \leq \dim_{Q_p}(V)$$

holds always, and the following two conditions (i) (ii) are equivalent.

(i) $\dim_K(D_{dR}(V)) = \dim_{Q_p}(V)$.
(ii) The canonical map

$$D_{dR}(V) \otimes_K B_{dR} \longrightarrow V \otimes Q_p B_{dR}$$

is bijective.

We say $V$ is a de Rham representation of $\text{Gal}(\overline{K}/K)$ if these equivalent conditions (i) (ii) are satisfied. If $V$ is a de Rham representation, the bijection in (ii) gives an isomorphism of filtrations.

The theorem (1.3) says that for a smooth proper scheme $X$ over $K$, $H^*_d(X, Q_p)$ is a de Rham representation of $\text{Gal}(\overline{K}/K)$ and $D_{dR}(H^*_d(XQ_p))$ is canonically isomorphic to $H^*_d(X/K)$ as a filtered $K$-vector space.

(1.5) We review the exponential map of a de Rham representation defined in $[BK]$. Let $V$ be a de Rham representation of $\text{Gal}(\overline{K}/K)$ and let $H^*(K, V)$ be the continuous Galois cohomology. Then we have a canonical homomorphism

$$\exp : D_{dR}(V)/D^0_{dR}(V) \longrightarrow H^*(K, V)$$

defined as follows. Recall that Fontaine defined a subring $B_{crys}$ of $B_{dR}$ containing the field of fractions of the $p$-Witt ring $W(k)$, endowed with the Frobenius operator $\beta : B_{crys} \longrightarrow B_{crys}$ (cf. $[F_0]$ $[FM]$), and defined the functor $D_{crys}$ by

$$D_{crys}(V) = H^*(K, V \otimes Q_p B_{crys}) \subset D_{dR}(V).$$

The sequence

$$0 \longrightarrow Q_p \longrightarrow B_{crys} \otimes B_{dR} \beta \longrightarrow B_{crys} \otimes B_{dR} \longrightarrow 0$$

is exact where $\alpha(x) = (x, x)$, $\beta(x, y) = ((1-f)(x), x - y)$. By tensoring with $V$ and by taking Galois cohomologies, we have an exact sequence.
(1.5.2) \[ 0 \longrightarrow H^n(K, V) \longrightarrow D_{\text{crys}}(V) \oplus D_{\text{dR}}(V) \]
\[ r \delta \longrightarrow D_{\text{crys}}(V) \oplus D_{\text{dR}}(V) \longrightarrow H^1(K, V), \]
where \( \gamma(x, y) = (1-f)(x), x-y \) The exponential map (1.5.1) is defined by the second component of \( \delta \).

(1.6) Let \( l \) be a prime number and let \( V \) be a finite dimensional \( \mathbb{Q}_l \)-vector space endowed with a continuous action of \( \text{Gal}(\bar{K}/K) \). In the case \( l=p \), assume that \( V \) is a de Rham representation of \( \text{Gal}(\bar{K}/K) \). Then, the "finite part" \( H^i_f(K, V) \) of \( H^i(K, V) \) is defined as follows ([BK] §3). If \( l \neq p \),
\[ H^i_f(K, V) = \ker(H^i(K, V) \longrightarrow H^i(K_{nr}, V)) \]

where \( K_{nr} \) denotes a maximal unramified extension of \( K \). If \( l=p \), \( H^i_f(K, V) \) is the image of the map \( \delta \) in (1.5.2). If \( l=p \), we have an exact sequence ([BK] (3.8.4))

(1.6.1) \[ 0 \longrightarrow H^0(K, 7) \longrightarrow \ker(1-f: D_{\text{crys}}(V)) \]
\[ \exp \longrightarrow D_{\text{dR}}(V)/D_{\text{dR}}^+(V) \longrightarrow H^1_f(K, V) \]
\[ \longrightarrow \text{coker}(1-f: D_{\text{crys}}(V)) \longrightarrow 0. \]

Assume the residue field of \( K \) is finite. Then, for any \( i \), the cup product
\[ H^i(K, V) \times H^{*+i}(K, V^*(1)) \longrightarrow H^i(K, Q_l(1)) \]
gives a perfect duality of finite dimensional \( \mathbb{Q}_l \)-vector spaces (Tate duality). Here \( V^* = \text{Hom}(V, Q_l) \) on which \( \sigma \in \text{Gal}(\bar{K}/K) \) acts by \( h \rightarrow h^\sigma \), and (1) means the Tate twist. In this pairing, if \( l \neq p \) or if \( l=p \) and 7 is de Rham, \( H^i_f(K, 7) \) and \( H^i_f(K, V^*(1)) \) are the annihilators of each other ([BK] (3.8)).

\[ \S \, 2. \, K\text{-theory}. \]

In this section, \( K \) is a number field, \( p \) is a fixed prime number, and \( M \) is a pure motive (in \( \mathbb{Q} \)-coefficients) over \( K \) of weight \( \text{wt}(M) \). We review standard conjectures concerning the "\( K\)-theory of \( M \)."

(2.1) We do not ask seriously what motives are, but it is better to fix a definition. A pure motive over \( K \) of weight \( w \in \mathbb{Z} \) is a finite family of 4-ple \( \{X_i, m_i, r_i, e_i\}_i \), where \( X_i \) are smooth proper schemes of pure dimension over \( K \), \( m_i, r_i \in \mathbb{Z} \) with \( w=m_i-2r_i \), and \( e_i \) is an idempotent in the ring of algebraic cycles on \( X_{i, K} \) with \( \mathbb{Q} \)-coefficients modulo rational equivalence which are regarded as algebraic correspondences from \( X_i \) to \( X_i \). We denote the single family \( (X, m, r, \Delta_X) \) by \( H^w(X)(r) \Delta_X \) denotes the diagonal, which is regarded as the identity correspondence. We interpret \( \{(X_i, m_i, r_i, e_i)_i\} \), as the direct
sum of the direct summands of \( H^n(X_t)(r_t) \) corresponding to \( \varepsilon_t \). (This is just a very non-smart modification of the original definition of the motive of Grothen- dieck.) We do not discuss morphisms of motives. For a motive \( M = \{(X_t, m_t, r_t, \varepsilon_t)\} \), the Tate twist \( M(r) \) for \( r \in \mathbb{Z} \) (resp. the dual \( M^* \)) is defined as \( \{(X_t, m_t, ni+r, \varepsilon_t)\} \) (resp. \( \{(X_t, 2n_t-m_t, n_t-r, \varepsilon_t^*)\} \)) where \( n_t = \dim(X_t) \) and \( \varepsilon_t^* \) is the transpose of \( \varepsilon_t \).

(2.2) We fix notations for various realizations of \( M \).

Let \( V_p(M) \) be the \( p \)-adic etale realization of \( M \) which is a \( \mathbb{Q}_p \)-sheaf on \( \text{Spec}(K)_{et} \). Once we fix an algebraic closure \( K \) of \( K \), \( V_p(M) \) is identified with a finite dimensional \( \mathbb{Q}_p \)-vector space endowed with a continuous action of \( \text{Gal}(\overline{K}/K) \). Let \( M_h \) be the \( \mathbb{Q} \)-structure in the Hodge structure of \( M \) which we regard as a sheaf of \( \mathbb{Q} \)-vector spaces on \( \text{Spec}(\mathbb{Q}^\mathbb{R})_{et} \). Finally let \( D(M) \) be the de Rham realization of \( M \), which is a \( \mathbb{Q} \)-vector space endowed with the Hodge filtration \( \{D^i(M)\} \) \( i \in \mathbb{Z} \). These realizations are defined as follows. Assume \( M=H^m(X)(r) \) with \( X, m, r \) as in (2.1). Then, \( V_p(M) = H^m_p(X_{\mathbb{Q}_p})(r) \) as a Gal(\( \overline{K}/K \))-module, where \( X=X \otimes K \mathbb{A} \). For \( \alpha \in \text{Spec}(K \otimes \mathbb{Q}) \), the stalk \( M_h(\{\alpha\}) \) of \( M_h \) at the algebraic closure \( \overline{\alpha} \) of \( \alpha \) is \( H^m_{\mathbb{C}l}(X \otimes \overline{\alpha} \mathbb{Q} (2\pi i)^r) \) where \( H^m_{\mathbb{C}l} \) is the classical cohomology, and \( M_h(\{\alpha\}) \) is the Gal(\( \overline{\alpha}/\alpha \))-invariant part of \( M_h(\{\alpha\}) \) where Gal(\( \overline{\alpha}/\alpha \)) acts simultaneously on \( \overline{\alpha} \) and on \( \mathbb{Q} (2\pi i)^r \). Finally \( D(M) = H^m_{\mathbb{C}l}(X/K) \) with the filtration \( D^i(M) = \text{fil}^i H^m_{\mathbb{C}l}(X/K) \) where \( \text{fil}^i \) is the Hodge filtration.

In general if \( M = \{(X_t, m_t, r_t, \varepsilon_t)\} \), a realization of \( M \) is defined as the direct sum of the direct summands of the realizations of \( H^m(X_t)(r_t) \) corresponding to \( \varepsilon_t \).

(2.3) We have a canonical map

\[
H^*(K \otimes \mathbb{Q} R, M_h) \otimes \mathbb{Q} R \longrightarrow \langle D(M)/D^*(M) \rangle \otimes \mathbb{Q} R,
\]

which is injective if \( wt(M) \leq -1 \). This map (2.3.1) is induced from the isomorphism

\[
H^*(K \otimes \mathbb{Q} C, M_h) \otimes \mathbb{Q} C \cong D(M) \otimes \mathbb{Q} C
\]

which is compatible with the action of \( \text{Gal}(\mathbb{C}/\mathbb{R}) \). (\( \sigma \in \text{Gal}(\mathbb{C}/\mathbb{R}) \) acts by \( \sigma \otimes \sigma \) on the left hand side and by \( 1 \otimes \sigma \) on the right hand side. If \( M \) is the motive \( H^m(X)(r) \) for \( X, m, r \) as in (2.1), this isomorphism is the classical isomorphism

\[
H^m_{\mathbb{C}l}(X \otimes \mathbb{Q} C, C) \cong H^m_{\mathbb{C}l}(X \otimes \mathbb{Q} C)/\mathbb{C}.
\]

(2.4) Here we fix some notations concerning etale cohomology. For a scheme \( Y \) of finite type over \( O_K \) and for a smooth \( \mathbb{Z}_p \)-sheaf \( F \) on \( Y_{et} \), let \( H^i(Y, F) = \lim\limits_{\rightarrow p} H^i_{\mathbb{Z}_p}(Y_{et}, F/p^n F) \). Then if \( p \) is invertible on \( Y \), each \( H^i_{\mathbb{Z}_p}(Y_{et}, F/p^n F) \) is a finite group and \( H^i(Y, F) \) is a finitely generated \( \mathbb{Z}_p \)-module. For such \( Y \) and for a smooth \( \mathbb{Q}_p \)-sheaf \( F \) on \( Y_{et} \), let \( H^i(Y, F) = H^i(Y, F') \otimes \mathbb{Z}_p Q_p \) where \( F' \) is a smooth \( \mathbb{Z}_p \)-sheaf such that \( F = F' \otimes \mathbb{Z}_p Q_p \), which is independent of the choice of \( F' \). For a scheme \( Y \) of finite type over \( K \) and for a smooth \( \mathbb{Z}_p \) (resp. \( \mathbb{Q}_p \))-sheaf \( F \) on \( Y_{et} \), which comes from some smooth \( \mathbb{Z}_p \) (resp. \( \mathbb{Q}_p \))-sheaf on a scheme \( Y' \) of finite type over \( O_K \) such that \( Y = Y' \otimes_{O_K} K \), let
where $U$ ranges over all non-empty open subsets of Spec$(O_K)$. Then $H_{\text{lim}}(Y, F)$ depends only on $Y$ and $F$, and is independent of the choices of $Y'$ and $F'$.

(2.5) We shall need $K$-theoretic $Q$-vector spaces denoted by $H^i(K, M)$ ($i \in \mathbb{Z}$) and a subspace $H^i(K, M)$ of $H^i(K, M)$. For $i \geq 2$ or for $i < 0$, define $H^i(K, M) = 0$. We define $H^i(K, M)$ as follows.

First assume $M = H^m(X)(r)$ with $X, m, r$ as in (2.1). Let $K^i(X)$ be Quillen’s $K$-group. Then

$$H_{\text{lim}}(Y, F) = \lim_{\rightarrow} (H^i(Y' \times_{O_K} U, F))$$

where $U$ ranges over all non-empty open subsets of Spec$(O_K)$. Then $H_{\text{lim}}(Y, F)$ depends only on $Y$ and $F$, and is independent of the choices of $Y'$ and $F'$.

(2.5) We shall need $K$-theoretic $Q$-vector spaces denoted by $H^i(K, M)$ ($i \in \mathbb{Z}$) and a subspace $H^i(K, M)$ of $H^i(K, M)$. For $i \geq 2$ or for $i < 0$, define $H^i(K, M) = 0$. We define $H^i(K, M)$ as follows.

First assume $M = H^m(X)(r)$ with $X, m, r$ as in (2.1). Let $K^i(X)$ be Quillen’s $K$-group. Then

$$K^i(X) \otimes Q = \bigoplus_{n \in \mathbb{Z}} (K^i(X) \otimes Q)^{(n)}$$

where $(K^i(X) \otimes Q)^{(n)}$ is the part of $K^i(X) \otimes Q$ on which the Adams operators $\phi^k$ act by $k^n$ for any $k \in \mathbb{Z}$. If $\text{wt}(M) = \ell - 1$ (i.e. if $m = 2r - 1$), define

$$H^i(K, M) = (K_{2r - m - 1}(X) \otimes Q)^{(i)}.$$ 

If $\text{wt}(M) = -1$ (i.e. if $m = 2r - 1$), define $H^i(K, M)$ to be the part of $(K^i(X) \otimes Q)^{(i)} = CH^r(X) \otimes Q$ ($CH^r$ denotes the Chow group of algebraic cycles of codimension $r$) consisting of elements which are homologically equivalent to zero.

In general $H^i(K, M)$ is defined from the case $M = H^m(X)(r)$ by taking the direct summands and the direct sum.

Next we define $H^i_j(K, M)$. Consider the Chern class mapping

$$(2.5.1) \quad H^i(K, M) \longrightarrow H^i_{\text{lim}}(K, V_p(M))$$

which is induced when $M = H^m(X)(r)$ from the Chern class map

$$K_{2r - m - 1}(X) \longrightarrow H^{m + i}_1(X, Q_p(r))$$

(Soule [S II]) and from the map

$$\text{Ker} \{ H^i_{\text{lim}}(X, Q_p(r)) \longrightarrow H^{m + i}_1(X, Q_p(r)) \} \longrightarrow H^i_{\text{lim}}(K, H^i_{\text{et}}(X, Q_p(r)))$$

coming from Leray’s spectral sequence

$$E^2 = H^i_{\text{lim}}(K, H^i_{\text{et}}(X, Q_p(r))) \Longrightarrow H^i_{\text{et}}(X, Q_p(r)).$$

Let $H^i_j(K, V_p(M))$ be the subspace of $H^i(K, V_p(M))$ consisting of elements whose images in $H^i(K_v, V_p(M))$ belong to $H^i_j(K_v, V_p(M))$ for all finite places $v$ of $K$ (cf. (1.6)). Let $H^i_j(K, M) \subset H^i(K, M)$ be the inverse image of $H^i_j(K, V_p(M)) \subset H^i(K, V_p(M))$.

Finally we define $H^i(K, M)$ as follows. It is enough to consider the case $M = H^m(X)(r)$. If $m \neq 2r$ (i.e. if $\text{wt}(M) \neq 0$), define $H^i(K, M) = 0$. If $m = 2r$, let

$$H^i(K, M) = (CH^r(X) \otimes Q) / (\text{hom.} \sim 0)$$

where $(\text{hom.} \sim 0)$ means the part homologically equivalent to zero.
CONJECTURE (2.6). (1) $H^j(K, M)$ and $H^0(K, M)$ are finite dimensional $Q$-vector spaces.
(2) $H^j(K, M) \otimes Q_p \cong H^j(K, V_p(M))$ (cf. [BK] (5.3))
(3) (Tate conjecture.) $H^0(K, M) \otimes Q_p \cong H^0(K, V_p(M))$.
(4) // $M \in H^n(X)(r)$ and $X$ is a proper flat regular scheme over $O_K$ such that $X = \mathcal{X} \otimes_{O_K} K$, $H^j(K, M) \subseteq K_{r, m-1}(X) \otimes Q$ coincides with the intersection of $H^1(K, M)$ and the image of $K_{r, m-1}(X) \otimes Q$.

The following is an old conjecture of Beilinson [Be] (H^1(K, M) was defined in the way of (2.6) (4) by him).

CONJECTURE (2.7). Assume $wt(M) \neq -1$. Then we have an exact sequence

$$0 \to H^j(K, M) \otimes Q_p \to P \to Hom_Q(H^0(K, M^*(1)), R) \to 0.$$ where $P = \{(D(M)/D^n(M)) \otimes \mathcal{R}/\text{Image}(H^0(K \otimes \mathcal{R}, M_n) \otimes Q)\}$

Here $a$ is the regulator map of Beilinson, “Image” is taken with respect to the map (2.2.1), and $\beta$ is defined as follows: If $M = H^m(X)(r)$ with $m - 2r = -2$ and with $X$ purely of dimension $n$, each element of $H^0(K, M^*(1)) = (CH^{n-r+1}(X) \otimes Q)/(\text{hom.} \sim 0)$ defines a cycle class in $\mathbf{H}^{n-r+1}H_{dR}^{2n-2r+3}(X/K)$ which induces

$$D(M)/D^n(M) = H^m_{dR}(X/K)/\mathbf{H}^{n-r+1}H_{dR}^{2n-2r+3}(X/K) \to H^m_{dR}(X/K)/\mathbf{H}^{n-r+1}H_{dR}^{2n-2r+3}(X/K).$$

CONJECTURE (2.8). Assume $wt(M) = -1$. Then:
(1) $H^j(K, M) = H^j(K, M)$.
(2) The height pairing

$$(H^j(K, M) \otimes Q) \times (H^j(K, M^*(1)) \otimes Q) \to R$$

defined by Beilinson [Be$_+$] and Bloch [Bl$_+$] under a certain assumption is defined in general, and it is a perfect pairing of finite dimensional vector spaces over $R$.

§ 3. Global $p$-adic duality.

In this section, $K$ denotes a number field and $p$ is a fixed prime number. Let $S$ be a finite set of finite places of $K$ containing all places lying over $p$. Let $O_K$ be the ring of integers of $K$, and let $O_{K,S}$ be the ring of $S$-integers, that is, $O_{K,S}$ is the ring of elements of $K$ which belongs to the local ring of $O_K$ at $v$ for all places $v$ outside $S$. 

(3.1) Let $V$ be a smooth $\mathbb{Q}_p$-sheaf on $\text{Spec}(O_K, S)_{et}$, or in other words, a finite dimensional $\mathbb{Q}_p$-vector space with a continuous action of $\text{Gal}(K/K)$ which is unramified outside $S$. Then, by the duality of Poitou-Tate and Artin-Verdier ([Ser] II (6.3), [Po], [Ta], [AV], [Ma]), we have an exact sequence of finite dimensional $\mathbb{Q}_p$-vector spaces

$$0 \to H^i(O_K, s, V) \to \bigoplus_{v \in S} H^i(K_v, V) \to H^i(O_K, s, V^*(1))^*$$

and

$$0 \to H^i(O_K, s, V) \to \bigoplus_{v \in S} H^i(K_v, V) \to H^i(O_K, s, V^*(1))^*$$

Here the cohomology groups are etale cohomology groups or Galois cohomology groups, $(\_)^*=\text{Hom}_{\mathbb{Q}_p}(\_ , \mathbb{Q}_p)$, and the maps $\bigoplus_{v \in S} H^i(K_v, V) \to H^{i-1}(O_K, s, V^*(1))^*$ are induced by the canonical map

$$H^{i-1}(O_K, s, V^*(1)) \to \bigoplus_{v \in S} H^{i-1}(K_v, V^*(1))$$

and the local Tate duality

$$H^i(K_v, V) \times H^{i-1}(K_v, V^*(1)) \to H^0(K_v, \mathbb{Q}_p)^* \cong \mathbb{Q}_p.$$

COROLLARY (3.2). Assume for any place $v$ of $K$ which divides $p$, $V$ is a de Rham representation of $\text{Gal}(K_v/K_v)$. We have an exact sequence of finite dimensional $\mathbb{Q}_p$-vector spaces

$$0 \to H^i(K, V) \to \bigoplus_{v \in S} H^i(K_v, V) \to H^i(O_K, s, V^*(1))^*$$

and

$$0 \to H^i(K, V) \to \bigoplus_{v \in S} H^i(K_v, V) \to H^i(O_K, s, V^*(1))^*$$

(3.3) In the rest of this section, let $M$ be a pure motive over $K$ of weight $\text{wt}(M)$, which is of good reduction outside $S$. Note that for any place $v$ of $K$ which divides $p$, $V_p(M)$ is a de Rham representation of $\text{Gal}(K_v/K_v)$ by (1.3).

The following conjecture (3.4) is a real theorem of Deligne ([De2] (resp. of Fontaine, Messing [FM] and Faltings [Fa]) if $v$ does not divide $p$ (resp. $v$ divides $p$) and $M$ is of good reduction at $v$. For general $v$ which does not divide $p$ (resp. divides $p$), a geometric analogue of (3.4) was proved in [De2] II (resp. [Fa]).

CONJECTURE (3.4). Let $v$ be a finite place of $K$. If $v$ does not divide $p$, let $\varphi_v$ be the Frobenius automorphism in $\text{Gal}(K_{v, nR}/K_v)$ acting on $H^0(K_{nR}, \_)$, and let $P_v(t) = \det_{Q_p}(1 - \varphi_v^{-1} \cdot H^0(K_{nR}, V_p(M)))\subseteq Q_p[t]$. If $v$ divides $p$, let $f_v : D_{cryst}(V_p(M)) \to D_{cryst}(V_p(M))$ be the Frobenius operator, $\kappa(v)$ the residue field of $v$, $K_{v, 0}$ the field of fractions of the $p$-Witt ring $W(\kappa(v))$, $d(v) = [\kappa(v) : F_p]$, and let
IWASAWA THEORY AND $p$-ADIC HODGE THEORY

$P_p(t) = \text{det}_F\left(1 - f_p^{(v)}t; D_{cr}(V_p(M)) \right) \in K_{v, 0}[t]$. 

Then, in any case, $p_p(t)$ is with Q-coefficients, and has the form

$$\prod_i (1 - \alpha_i t) \quad (\alpha_i \in \mathbb{C}, |\alpha_i| \leq N(v)^{w(M)/2})$$

($N(v)$ is the norm of $v$) in $\mathbb{C}[t]$.

PROPOSITION (3.5). Assume $\text{wt}(M) \leq -1$ and assume Conj. (3.4) is true for $M$.

Let $v$ be a finite place of $K$.

1. $v$ divides $p$, the exponential map of §1 induces an isomorphism

$$D(M)/D^q(M) \otimes_K K \overset{\approx}{\longrightarrow} H^1(K_v, V_p(M)).$$

2. $v$ does not divide $p$, then $H^1(K_v, V_p(M)) = 0$.

3. $H^q(K_v, V_p(M)) = 0$.

Proof. The case $v|p$ follows from (1.6.1). The case $v$ does not divide $p$ follows from the fact that $H^1(K_v, )$ (resp. $H^o(K_v, )$) is isomorphic to the cokernel (resp. kernel) of $1 - \varphi_v^{-1}$ on $H^0(K_v, )$.

From (3.2) and (3.5), we can deduce easily:

PROPOSITION (3.6). Assume $\text{wt}(M) \leq -1$ and assume that the conjectures (2.6) and (3.4) are true for $M$. Then:

1. For any $i$, we have

$$H^i(O_{K,s}, V_p(M)^*(1)) \overset{\approx}{\longrightarrow} H^i_{\text{lim}}(K, V_p(M)^*(1)).$$

2. We have an exact sequence of finite dimensional $Q_p$-vector spaces

$$0 \longrightarrow H^i_{\text{lim}}(K, V_p(M)^*(1)) \longrightarrow H^i(K, M)^{\otimes q} \longrightarrow \cdots$$

Proof. The sequence in (2) plays the role of the $p$-adic version of the sequence in (2.7) of vector spaces over $R$.

Remark (3.7). In (3.6), assume $\text{wt}(M) \leq -2$. Then $H^1(K, M^*(1)) = 0$. Indeed, since $M^*(1)$ is of weight $\geq 0$, it is a direct sum of subspaces of $\left(K_{r-m-1}(X) \otimes Q\right)^{(r)}$ with $m - 2r \geq 0$, i.e. with $2r - m - 1 < 0$. Furthermore it is probable that $H^i_{\text{lim}}(K, V_p(M)^*(1))$ vanishes (this is conjectured by Jannsen), or equivalently (if we assume (2.6) and (3.4)), the $p$-adic regulator map $H^1(K, M)^{\otimes q} \otimes Q_p \rightarrow D(M)/D^q(M)^{\otimes q} Q_p$ in (3.6) (2) is injective. If this is the case, the sequence in (3.6) (2) has the simple form.
§ 4. Iwasawa main conjecture.

(4.1) In this section, let $K$ be a number field and let $M$ be a pure motive over $K$ of weight $\leq -1$. Let $L$ be a finite abelian extension of $K$ with Galois group $G$, and let $p$ be a fixed prime number.

(4.2) We define $Q[G]$-modules $H_h, H_d, H_k, H'_k$ and $Q_p[G]$-modules $H^i$ $(i \in \mathbb{Z})$ as follows. With the notations in § 2, let

$$H_h = H^i (L \otimes_R R, M_h), \quad H_d = (D(M)/D^p(M)) \otimes_K L, \quad H_k = H^i (L, M),$$

$$H'_k = H^i (L, M^*(1)), \quad H''_k = H^i (L, M^*(1)),$$

and let

$$H^i = H_{l,m}(L, V_p(M)^*(1)) \quad (2.9).$$

Then $H'_k = 0$ unless $wt(M) = -2$, $H''_k = 0$ unless $wt(M) = -1$, $H^i_p = 0$ for $i \geq 3$, and $H^2_p = 0$ unless $wt(M) = -2$. If $X = H^n(X)(r)$, then

$$H_h = H^n (X \otimes_K L) \otimes Q, \quad Q (2\pi i)^r +$$

where $+$ means the fixed part by $Gal(C/R)$ which acts simultaneously on $C$ and on $Q (2\pi i)^r$, $H_d = H^n (X \otimes_K L) / fil^r$, and $H_k$ is a certain subspace of $K_{s - 1} (X \otimes_K L) \otimes Q$.

(4.3) In (4.3) and (4.4) we give purely module theoretic preliminaries.

Here we give a preliminary concerning determinant modules. Let $R$ be a commutative ring. Recall that for a finitely generated projective $R$-module $L$, the determinant $R$-module $det_R(L)$ is defined to be the exterior power $\wedge L$ where $r$ is the rank of $L$ which is a locally constant function on $Spec (R)$ (so $\wedge L$ is defined locally on $Spec (R)$ first and glued globally on $Spec (R)$). This definition of the determinant module is generalized to perfect complexes as follows.

Let $C$ be the derived category of the category of $R$-modules. An object $C$ of $C$ is called a perfect complex if there is a bounded complex of $R$-modules consisting of finitely generated projective $R$-modules which represents $C$. For a perfect complex $C$ in $C$, the determinant module $det_R(C)$ is the invertible $R$-module defined as follows. Take a representative of $C$

$$\cdots \longrightarrow L_i \longrightarrow L_{i-1} \longrightarrow \cdots$$
which is bounded and which consists of finitely generated projective $R$-modules. Then
\[
\det_R(C) = \bigotimes_{i \in \mathcal{G}} \{\det_R(L_i)\}^{\otimes(-1)^i}.
\]

It is known that $\det_R(C)$ is independent, modulo canonical isomorphisms, of the choice of a representative as above.

(4.4) For a ring $R$, which will be $Q$, $Q_p$, $Z_p$ or $Z_{\mathbb{Q}} = Z_p \cap Q$ below, and for an $R[G]$-module $F$, let $F^*$ be the $R[G]$-module whose underlying $R$-module is $F$ but on which $\sigma \in G$ acts by the original action of $\sigma^{-1}$ on $F$. Let $F^* = \text{Hom}_R(F, R)$ on which $\sigma \in G$ acts by $h \mapsto h \sigma$. Then $F^{**} = F^*$ is identified with the dual module $\text{Hom}_{R[G]}(F, R[G])$ on which $\sigma \in R[G]$ acts by $h \mapsto h \sigma a - a + h$, via the isomorphism
\[
F^{**} \rightarrow \text{Hom}_{R[G]}(F, R[G]) h \mapsto (x \mapsto \sum_{\sigma \in G} h(\sigma^{-1}x)\sigma).
\]

(4.5) In the rest of this section except in (4.15)-(4.17), we assume that the conjectures (2.6) (2.7) (2.8) (3.4) are true for the pull back of $M$ over $L$.

We define a free $Q[G]$-module $\Phi_{mot}$ of rank 1 with an isomorphism of $R[G]$-modules
\[
(4.5.1) \quad \Phi_{mot} \otimes_q R \cong R[G],
\]
and a free $Q_p[G]$-module $\Phi_{Fr}^p$ of rank 1 with an isomorphism of $Q_p[G]$-modules
\[
(4.5.2) \quad \Phi_{mot} \otimes_q Q_p \cong \Phi_{Fr}^p.
\]

Let
\[
\Phi_{mot} = \det_{Q[G]}(H_n) \otimes_{Q[G]}(H_{d}) \otimes_{Q[G]}(H_{h}) * \otimes_{Q[G]}(H_{s})^*.
\]

Note that the fourth (resp. the last) $\det(\ )$ is $\det_{Q[G]}(\{0\}) = Q[G]$ and can be cancelled if $\omega(M) \neq 0$. Let
\[
\Phi_{Fr}^p = \det_{Q[G]}(H_n) \otimes_{Q[G]}(H_{d}) \otimes_{Q[G]}(H_{h}) R^\Gamma_{1+1}(K, V_p(M)*(1))^*.
\]

We define the isomorphisms (4.5.1) (4.5.2) as follows. First we consider the Archimedean side (4.5.1). If $\omega(M) \leq -2$, by Conj. (2.7) (with $K$ replaced by $L$) we have an exact sequence
\[
0 \rightarrow H_s \otimes_q R \rightarrow (H_d \otimes_q R)/(H_n \otimes_q R) \rightarrow (H_h) \otimes_q R \rightarrow 0
\]
which gives (4.5.1). If $\omega(M) = -1$, the height pairing
\[
H_s \otimes_q R \cong (H_h)^* \otimes_q R \quad (2.8)
\]
and the isomorphism
Next we consider the $p$-adic side (4.5.2). By (3.6) (2), we have an exact sequence

$$0 \longrightarrow (H^*_p)^* \longrightarrow H^*_p \otimes \mathbb{Q}_p \longrightarrow \mathbb{H} \longrightarrow 0.$$ 

This sequence and

$$H^*_p \otimes \mathbb{Q}_p \longrightarrow \mathbb{H}$$

give (4.5.2).

(4.6) We consider the partial $L$-functions of $M$ relative to the abelian extension $L/K$. Let $S$ be a finite set of finite places of $K$ containing all finite places at which $M$ has bad reduction and all finite places which ramify in $L/K$.

Let $L_{S}(M, s) = \sum_{\mathcal{Q} \in S} a(\mathcal{Q})N(\mathcal{Q})^{-s}$

be the Hasse-Weil $L$-function of $M$ without Euler factors for places in $S$. Here $\mathcal{Q}$ ranges over all non-zero ideals of $O_K$, $a(\mathcal{Q}) \in \mathcal{Q}$, and $N(\mathcal{Q})$ denotes the norm of $\mathcal{Q}$.

For $\sigma \in G$, we define the partial $L$-function

$$L_{S}(M, \sigma, s)$$

to be $\sum_{\mathcal{Q} \in S} a(\mathcal{Q})N(\mathcal{Q})^{-s}$ where $\mathcal{Q}$ ranges over all non-zero ideals of $O_K$ which are prime to $S$ such that the Artin symbols $(L/K, \mathcal{Q}) \in G$ coincide with $\sigma$.

This function $L_{S}(M, \sigma, s)$ converges absolutely when $\Re(s) > wt(M)/2 + 1$.

We will consider the values of $L_{S}(M, \sigma, s)$ at $s = 0$. If $M = H^*(X)(r)$, these are the values of the partial Hasse-Weil $L$-functions $L_{S}(H^*(X), \sigma, s)$ at $s = r = (m + 1)/2$ (note $L_{S}(N(r), \sigma, s) = L_{S}(N, \sigma, s + r)$ for any motive $N$), where $(m + 1)/2$ is the central point of the conjectural functional equations of $L_{S}(H^*(X), \sigma, s)$ under the substitution $s \leftrightarrow m + 1 - s$.

Define "the analytic zeta element"

$$\zeta_{L_{S}, K, \sigma}(M) \in \mathbb{R}[G]$$

assuming no conjecture if $wt(M) \leq -3$, and assuming some conjectures if $wt(M) = -2$ or $-1$ as follows. If $wt(M) \leq -3$, define

$$\zeta_{L_{S}, K, \sigma}(M) = \sum_{\sigma \in G} L_{S}(M, \sigma, 0) \sigma.$$ 

Assume $wt(M) = -2$ (resp. $wt(M) = -1$). We proceed making conjectures. We conjecture that the functions $L_{S}(M, \sigma, s)$ are extended to the whole complex plane as meromorphic functions. Let $f'_i$ (resp. $H'_i$) be the coherent sheaf on $\text{Spec}(\mathbb{Q}[G])$ associated to the $\mathbb{Q}[G]$-module $H'_i$ (resp. $H'_i$). Take an open
set $U$ of $\text{Spec}(\mathbb{Q}[G])$ on which $\tilde{H}_k^O$ (resp. $\tilde{f}_g$) has constant rank $r(U)$. Then we conjecture that the image $f_\sigma(s) \in \mathcal{O}(U \otimes \mathbb{Q}_R)$ of $s^\tau(s) \sum_{\sigma \in \mathcal{O}} L_\sigma(M, \sigma\text{-part}, s) \sigma \in \mathcal{R}[G]$ with $\varepsilon=1$ (resp. $-1$) is holomorphic at $s=0$ as a vector valued function in $s$. We define $\zeta_{L/K, s}(M)$ to be the unique element of $\mathcal{O}[G]$ such that for any $U$ as above, the image of $\zeta_{L/K, s}(M)$ in $\mathcal{O}(U \otimes \mathbb{Q}_R)$ coincides with $f_\sigma(0)$.

If $S'$ is a finite set of finite places of $K$ containing $S$, we have

$$(4.6.1) \quad \zeta_{L/K, s}(M) = \left( \prod_{v \in S'-S} P_v(\varphi_v) \right) \zeta_{L/K, s}(M)$$

where $P_v(t) \in \mathbb{Q}[t]$ is the polynomial such that $P_v(N(v)^{-\tau})$ is the Euler factor of $L_\sigma(M, s)$ at $v$, $\varphi_v \in G$ is the Frobenius at $v$, and we assumed in the case $\text{wt}(M) = -1, -2$ the conjectures needed in the definition of $\zeta_{L/K, s}(M)$ (which are equivalent to the conjectures needed for $\zeta_{L/K, s}(M)$) are true.

The following Conj. (4.7) is a famous Beilinson conjecture when $L=K$. The generalization to abelian extensions $L/K$ is proposed by several peoples (Stark, Gross, Beilinson, •••). In the critical case, (4.7) was conjectured be Deligne $[D_1, \text{•••}]$.

**CONJECTURE (4.7).** The image of $\zeta_{L/K, s}(M)$ under the isomorphism $(4.5.1)$ is contained in $\Phi_{\text{mot}} \subset \Phi_{\text{mot}} \otimes \mathbb{Q}_R$.

Assuming this conjecture we denote by $\zeta_{L/K, s}(M)$ the element of $\Phi_{\text{mot}}$ corresponding to $\zeta_{L/K, s}(M)$ via $(4.5.1)$, and call it "the motivic zeta element". We denote by $\zeta_{L/K, s}(M)_p$ the image of $\zeta_{L/K, s}(M)$ in $\Phi_p$ and call it "the $(p\text{-adic})$ arithmetic zeta element".

The relation $(4.6.1)$ concerning the change of the zeta element when we enlarge $S$ is extended to motivic zeta elements and arithmetic zeta elements, if the conjectures needed for the definitions of them are true.

$$(4.8) \quad \text{Fix a } \mathbb{Z}_p\text{-sheaf } T \text{ in } V_p(M) \text{ such that } T \otimes \mathbb{Z}_p \mathcal{F}=V_p(M), \text{ in other words, a Gal}(K/K)\text{-stable } \mathbb{Z}_p\text{-lattice } T \text{ of } V_p(M). \text{ Let } H_{h, T} \subset H_{h} \text{ be the inverse image of } H^0(L \otimes \mathbb{Q}_C, T) \subset H^0(L \otimes \mathbb{Q}_C, V_p(M)) \text{ under the composite map }$$

$$H_{h} \rightarrow H^0(L \otimes \mathbb{Q}_C, M_h) \otimes \mathbb{Q}_p \approx H^0(L \otimes \mathbb{Q}_C, V_p(M)).$$

For example, if $M=H^m(X)(r)$, we can take

$$T=H^m(\mathcal{O}, \mathcal{Z}_p)(r)/\text{(torsion)} \subset V_p(M) = H^m(X \otimes \mathcal{O}_p)(r)$$

(then $H_{h, T} = \{ H^m(\mathcal{O}, \mathcal{Z}_p)(r)/\text{(torsion)} \}^+ \otimes \mathbb{Z}_p)$. We have

$$H_{h, T} \otimes \mathbb{Z}_p = H^0(L \otimes \mathbb{Q}_R, T).$$

Assume $p\neq 2$. Then $H^0(L \otimes \mathbb{Q}_R, T)$ is a finitely generated projective $\mathbb{Z}_p[G]$-module as is easily seen. It follows that $H_{h, T}$ and hence $H^*_h, T=\text{Hom}_{\mathbb{Z}_p}(H^*_h, T, \mathcal{Z}(G))$ are finitely generated projective $\mathbb{Z}_p[G]$-modules. Let $T^*_p=\text{Hom}_{\mathbb{Z}_p}(T, \mathcal{Z}_p)$.

Let $O_L, s$ be the ring of elements of $L$ which belong to the local ring of $O_L$
at $v$ for any finite place $v$ of $L$ not lying over $S$. Assume $S$ contains all places of $K$ lying over $p$. We see in (4.17) below that if $p \neq 2$, $R\Gamma(O_{L,s}, T^*(1))$ is a perfect complex in the derived category of the category of $Z_p[G]$-modules. Let

$$\Phi^a_{p,s,T} = (\det_{Z_p[G]}(H_{a,T})) \otimes_{Z_p[G]} \det_{Z_p[G]} R\Gamma(O_{L,s}, T^*(1))^*$$

where $(\cdot)^* = \text{Hom}(\cdot, Z_p)$. Since

$$H^i(O_{L,s}, T^*(1)) \otimes_{Z_p} Q_p \cong H_{i+1}(\text{Hull}_p(M)^*(1)) \ (3.6) \ (1)$$

we have $\Phi^a_{p,s,T} \otimes_{Z_p} Q_p = \Phi^a_{p,s,T}$.

Iwasawa main conjecture (4.9). Assume $p \neq 2$, and assume $S$ contains all places of $K$ lying over $p$, all finite places of $K$ at which $M$ has bad reduction, and all finite places of $K$ which ramify in $L/K$. Then, the arithmetic zeta element

$$\xi_{L/K,s}^p(M) \in \Phi^a_{p,s,T} = \Phi^a_{p,s,T} \otimes_{Z_p} Q_p$$

is a $Z_p[G]$-basis of $\Phi^a_{p,s,T}$.

Note that $Z_{(p)}[G]$ and $Z_p[G]$ are semi-local rings and hence any invertible modules over them are free modules.

Remark (4.10). Conj. (4.9) is compatible with isogeny. That is, if $T$ and $T'$ are two $Z_p$-sheaves in $V_p(M)$ such that $T \otimes_{Z_p} Q_p = V_p(M) = T' \otimes_{Z_p} Q_p$, $\Phi^a_{p,s,T} = \Phi^a_{p,s,T'}$ holds in $\Phi^a_{p,s,T}$ by (4.17) (3) below, and hence Conj. (4.9) for the pair $(M, T)$ is equivalent to the Conj. (4.9) for the pair $(M, T')$.

Remark (4.11). Conj. (4.9) is compatible with localization, i.e. with enlarging $S$. Let $S'$ be a finite set of finite places of $K$ containing $S$. Then we have a distinguished triangle

$$R\Gamma(O_{L,S}, T^*(1)) \rightarrow \rightarrow R\Gamma(O_{L,S'}, T^*(1)) \rightarrow \rightarrow \bigoplus_{v \in S'} R\Gamma(v, T^*)[-1] \rightarrow \rightarrow .$$

From this we have easily

$$\Phi^a_{p,s,T'}(\text{veve}) = \Phi^a_{p,s,T'} \text{ in } \Phi^a_{p,s,T}$$

By comparing this with $\xi_{L/K,S'}^p(M) P_\text{ve}(p) \xi_{L/K,S}^p(M) p$, we see that Conj. (4.9) for the pair $(M, S)$ is equivalent to Conj. (4.9) for $(M, S')$.

Remark (4.12). I refer to the case $p=2$ which was excluded in Conj. (4.9). Let $p=2$ and assume that all Archimedean places of $K$ split in $L$. Then $H_{a,T}$ is a free $Z_{(2)}[G]$-module of finite rank. Under this assumption, I conjecture that the truncation $\tau_{s,t} R\Gamma(O_{L,s} T^*(1))$ is a perfect complex in the derived category of the category of $Z_2[G]$-modules, and the statement of Conj. (4.9)
holds if we replace \( R\Gamma(O_L, s, T^*(1)) \) by \( \tau_{\infty s} R\Gamma(O_L, s, T^*(1)) \).

Remark (4.13). To state Conj. (4.9) we needed many conjectures concerning K-theory which are difficult to verify. If \( wt(M) \leq -3 \) and \( M \) is critical in the sense of Deligne (i.e., \( H_0 \otimes \mathbb{Q} \cong H_d \otimes \mathbb{Q} R \)), there is a way to get rid of conjectures on K-theory. In this case, define \( H_0 = 0 \). Then, once we have expressions of values of partial \( L \)-functions of \( M \) at \( s = 0 \) in terms of period integrals, Conj. (4.9) becomes purely a problem on \( p \)-adic Hodge theory and Galois cohomology.

Remark (4.14) (on Euler systems). If \( L'/K \) is a subextension of \( L/K \) with Galois group \( G' \), the norm maps induce isomorphisms \( H_0 \otimes \mathbb{Q}\mathbb{G}_m \cong H_0 \otimes \mathbb{Z}/p \mathbb{G}_m \), \( H_0 \otimes \mathbb{Q}\mathbb{G}_m \cong H_0 \otimes \mathbb{Z}/p \mathbb{G}_m \), etc., where \( H_n \) etc. mean the \( H_n \) etc. defined for the extension \( L'/K \), and hence isomorphisms

\[
(4.14.1) \quad \Phi_{L'/K} \otimes \mathbb{Q}\mathbb{G}_m \cong \Phi_{L'/K} \mathbb{G}_m
\]

\[
(4.14.2) \quad \Phi_{L'/K} \otimes \mathbb{Q}\mathbb{G}_m \cong \Phi_{L'/K} \mathbb{G}_m
\]

Furthermore, if the conjectures needed for the definitions of \( \zeta_{L/K, s}(M) \) (resp. \( \zeta_{L/K, s}(M) \)) are true, the isomorphism (4.14.1) (resp. (4.14.2)) sends \( \zeta_{L/K, s}(M) \) to \( \zeta_{L'/K, s}(M) \). This fact and the change of zeta elements with enlarging 5 described after (4.7) suggest that when \( L \) and 5 vary, the systems \( \{ \zeta_{L/K, s}(M) \}_L \) and \( \{ \zeta_{L/K, s}(M) \}_5 \) should be called "Euler systems of Kolyvagin" ([K3]) for the motive \( M \). Is it possible to apply theories of Kolyvagin on his Euler systems to these general systems?

(4.15) In the next section, we will see that if \( K = \mathbb{Q} \), \( M = \mathbb{Q}(1) \) (resp. \( M = \mathbb{Q}(r) \) with \( r \) a positive even integer) and \( L \) is the real part of \( \mathbb{Q}(1) \) with a a root of 1, the algebraic (resp. arithmetic) zeta element is essentially the fundamental cyclotomic unit \( (1-\alpha)(1-\alpha^{-1}) \) (resp. the \( p \)-adic cyclotomic element of Soulé and Deligne in the Galois cohomology of \( \mathbb{Z}_p(1) \)).

Zeta functions live in some world. They come to \( \mathbb{R}[G] \) and become \( \zeta_{L/K, s}(M) \) to be called special values of zeta functions. When we find they come to \( \Phi_{L/K} \) and become \( \zeta_{L/K, s}(M) \), we call them expressions of special values of zeta functions in terms of period integrals, in terms of regulators of K-theory, \( \cdots \). We have realized they come to \( \Phi_{L/K} \) and become \( \zeta_{L/K, s}(M) \) only in very special cases. In such cases, we have called them expressions of special values of zeta functions in terms of explicit reciprocity laws, describing explicitly and map \( H_0 \otimes \mathbb{Q} \mathbb{G}_m \to (H_0)^* \) (cf. [dS] for the case \( M = \mathbb{H}^1(E)(1) \) with \( E \) elliptic curves with complex multiplication. This point will be discussed in more detail in [Kan].)

There should be many beautiful extensions of the theory of cyclotomic units and of the known theory of explicit reciprocity laws, to motives.

(4.16) To describe how arithmetic zeta elements are important for the
study of arithmetic of varieties, I introduce a result of a forthcoming paper [Ka2]. (We assume no conjecture in this (4.16).) Let $E$ be an elliptic curve over $\mathbb{Q}$ dominated by a modular curve, let $M=H^1(E)(1)$ and let $S$ be a finite set of primes containing all primes at which $E$ has bad reduction. In [Ka2], we give a new proof of the theorem of Kolyvagin

$$L(H^1(E),1)\neq 0 \implies E(\mathbb{Q})$$

is a finite group in the following way without using Heegner points. In [Ka2], we construct an element $c$ of $H_h \otimes_{\mathbb{Q}} H_p^* \subset H_h \otimes_{\mathbb{Q}} H_p^* \otimes_{\mathbb{Q}} Q_p$ such that the $Q_p$-dual $H_p^* \rightarrow H_p^* \otimes_{\mathbb{Q}} Q_p$ of $H_d \otimes_{\mathbb{Q}} Q_p \rightarrow (H_p^*)^*$ sends $c$ to

$$(4.16.1) \quad \gamma \otimes \left\{ L_S(H^1(E),1) \left( \left( \int_{\gamma} \omega \right)^{-1} \right) \right\} \in H_h \otimes_{\mathbb{Q}} H_p^* \subset H_h \otimes_{\mathbb{Q}} H_p^* \otimes_{\mathbb{Q}} Q_p$$

where $\gamma \in H_h - \{0\}$, $\omega \in \Gamma(E, \Omega_{E/\mathbb{Q}})-\{0\}$, we identify $H_h$ with $H_1(E(\mathbb{R}), \mathbb{Q})$, $H_1$ with $\Gamma(E, \Omega_{E/\mathbb{Q}})$, and we denote by $\int_{\gamma} \omega$ the integration of $\omega$ against $\gamma$. Note $H_h$ and $H_d$ are one dimensional over $\overline{\mathbb{Q}}$. $L_S(H^1(E),1)(\left( \int_{\gamma} \omega \right)^{-1}) \in \mathbb{Q}$, and the element (4.16.1) is independent of the choices of $\gamma$ and $\omega$.

Assume $L_S(H^1(E),1)\neq 0$. We obtain $E(\mathbb{Q})$ finite as follows. The property of $c$ implies that the map $H_d \otimes_{\mathbb{Q}} Q_p \rightarrow (H_p^*)^*$ is injective. By the exact sequence (3.2), we have that $H_l^1(Q, V_p(M)) \rightarrow H_d \otimes_{\mathbb{Q}} Q_p$ is the zero map. On the other hand, the map $E(\mathbb{Q}) \rightarrow H_l^1(Q, V_p(M))$ induced by the exact sequences

$$0 \rightarrow \rho^n E \rightarrow E \rightarrow \frac{E}{\rho^n E} \rightarrow 0 \quad (\rho^n E = \text{Ker}(\rho^n: E \rightarrow E))$$

lands in $H_l^1(Q, V_p(M))$ ([BK] (3.11)), and the composite map $E(\mathbb{Q}) \rightarrow H_l^1(Q, V_p(M)) \rightarrow H_d \otimes_{\mathbb{Q}} Q_p$ coincides with

$$E(\mathbb{Q}) \rightarrow E(Q) \otimes_{\mathbb{Q}} Q \rightarrow \text{Lie}(E) \otimes_{\mathbb{Q}} Q_p = H_d \otimes_{\mathbb{Q}} Q_p.$$

From these facts, it follows that $E(\mathbb{Q}) \rightarrow E(Q) \otimes_{\mathbb{Q}} Q$ is the zero map. This shows that $E(\mathbb{Q})$ is finite. (This kind of method was used by Coates-Wiles [CW], Bloch [Bl1] §2, and by K. Rubin [de Shalit [dS] IV §2] for elliptic curves with complex multiplication).

The element $c$ is the zeta element $\zeta_5 \|_{\mathbb{Q}, S(\mathbb{M})}$ if $L(H^1(E),1)\neq 0$ (strictly speaking, we know $c$ is the arithmetic zeta element only after we know $H_{l,\mathbb{Q}}(Q, V_p(M))$ is one dimensional over $Q_p$ and $H_{l,\mathbb{Q}}(Q, V_p(M))=0$ as consequences of Kolyvagin's theorem on the finiteness of the Tate-Shafarevich group; I do not have a new proof of this theorem). The element $c$ above is defined in [Ka2] by using "modular units in $K_3$ of modular curves" of Beilinson [Be1], just as the $p$-adic cyclotomic elements of Deligne and Soulé are defined (cf. (5.11)) by using classical cyclotomic units in $K_1=\mathbb{C}_m$. The fact $c$ is sent to the element (4.16.1) is shown by applying the explicit reciprocity law for two dimensional local fields by Vostokov and Kirillov ([VK]) to the completed
modular function field
\[
\lim_{\rightarrow} (Z/p^nZ[[q]][q^{-1}])[[1/p]]
\]
where \( q \) is the \( q \)-invariant.

Finally we prove the following result used in (4.8) and (4.10).

**Proposition (4.17).** (Here we assume no conjecture.) Assume \( p \neq 2 \), and let \( S \) be a finite set of finite places of \( K \) containing all prime divisors of \( p \) in \( K \). Let \( F \) be a smooth \( Z_p\)-sheaf on \( \text{Spec}(O_K, s) \). Then:

1. \( R\Gamma(O_{L, s}, F) \) is a perfect complex in the derived category of the category of \( Z_p[G] \)-modules.
2. If \( L/K \) is a subextension of \( L/K \) with Galois group \( G' \), we have a canonical isomorphism
   \[
   R\Gamma(O_{L, s}, F) \otimes_{Z_p[G]} Z_p[G'] \cong R\Gamma(O_{L', s}, F). 
   \]
3. If \( p \) is 0 for some \( n \geq 0 \), we have
   \[
   \det_{Z_p[G]}(R\Gamma(O_{L, s}, F) \otimes_{Z_p[G]} \det_{Z_p[G]}(H^q(L \otimes qR, F^*(1)))^* = Z_p[G] \quad \text{in } Q_p[G],
   \]
   where \((\ )^* = \text{Hom}(, Q_p/Z_p)\) and the left hand side is regarded as embedded in its \( \otimes_{Z_p[G]} Q_p \) which is identified with
   \[
   \det_{Q_p[G]}(0) \otimes_{Q_p[G]} \det_{Q_p[G]}(0) = \mathbb{Q}_p[G].
   \]

**Proof.** (1) and (2) are proved by the methods of Deligne (SGA4, Ch. 17) as follows. It is enough to prove that the morphism
\[
h_N : R\Gamma(O_{L, s}, F) \otimes_{Z_p[G]} N \longrightarrow R\Gamma(O_{L, s}, F \otimes_{Z_p[G]} N)
\]
is an isomorphism for any finitely generated \( Z_p[G] \)-module \( N \). To prove that the map \( H^q(h_N) \) induced on the \( q \)-th cohomology groups of these complexes is an isomorphism, take an exact sequence of \( Z_p[G] \)-modules of the form
\[
0 \rightarrow N' \rightarrow L_r \rightarrow \cdots \rightarrow L_0 \rightarrow L \rightarrow N \rightarrow 0
\]
with \( L_1 \) free of finite type and with \( r > 2 - q \). Since \( h_{L_1} \) are clearly isomorphisms, the bijectivity of \( H^q(h_N) \) is reduced to the bijectivity of \( H^{q+r}(h_N) \), but the cohomology groups are zero in degree \( > 2 \).

We prove (3). If \( G = \{1\} \), the statement of (3) is equivalent to the formula of Tate ([Ta_2, Thm. (2.2)])
\[
\prod_{0 \leq i \leq q}(H^i(O_{L, s}, F))^{(-1)^i} = (H^q(L \otimes qR, F^*(1))^{-1}.
\]
\((\#(\ ) \text{ denotes the order of the set). Our proof of (3) is essentially the same with the argument of Tate in his proof (suggested by Serre) of this formula.\)
We are reduced to the case where there is a cyclic extension $K'$ of $K$ of degree prime to $p$ which is unramified outside $S$, and $F$ corresponds to a finite $\text{Gal}(K'/K)$-module killed by $p$. By replacing $L$ with the composite field $K'L(\zeta_p)$ where $\zeta_p$ is a primitive $p$-th root of 1, we may assume $F=\mathbb{Z}/p\mathbb{Z}(l)$. Let $K_0(\mathbb{Z}/p\mathbb{Z}[G])$ (resp. $K_0'(\mathbb{Z}/p\mathbb{Z}[G])$) be the Grothendieck group of the category of finitely generated projective (resp. finitely generated) $\mathbb{Z}/p\mathbb{Z}[G]$-modules. Then for a perfect complex $C$ in the derived category of the category of $\mathbb{Z}/p\mathbb{Z}[G]$-modules, the $\mathbb{Z}_p[G]$-submodule $\det\mathbb{Z}_p[G](C)$ of $\mathbb{Z}_p[G]$ is determined by the class of $C$ in $K_0(\mathbb{Z}/p\mathbb{Z}[G])$. Since $K_0(\mathbb{Z}/p\mathbb{Z}[G])\to K_0'(\mathbb{Z}/p\mathbb{Z}[G])$ is injective, it is sufficient to prove that the sum of the class of $R\Gamma(O_{L,\mathfrak{s}}, \mathbb{Z}/p\mathbb{Z}(1))$ and the class of $H^0(L\otimes_{\mathbb{Q}} R, \mathbb{Z}/p\mathbb{Z})$ in $K_0'(\mathbb{Z}/p\mathbb{Z}[G])$ is zero. This fact is proved, just as Tate says in [T3], by considering the cohomology sequence of $\mathbb{Q}-\mathbb{Z}/p\mathbb{Z}(l)\to \mathbb{G}_m\to 0$, together with the knowledge of the cohomology of $G_m$ furnished by class field theory.

§ 5. Zeta elements of $\mathbb{Q}(r)$ ($r\geq 1$) for cyclotomic extensions.

In this section, let $K=\mathbb{Q}$, $M=\mathbb{Q}(r)$ with $r\geq 1$, let $N\geq 1$, and let $L$ be the extension of $\mathbb{Q}$ generated by a primitive $N$-th root of 1. We give an explicit description of the motivic zeta element ((5.6), which is a rewriting of known results) and a "half description" of the arithmetic zeta element ((5.14), for which we need a result (5.12) proved in [Ka]).

(5.1) For $e\in(\mathbb{Z}/N\mathbb{Z})^*$ let $\sigma_e$ be the element of $G=\text{Gal}(L/K)$ characterized by the property $\sigma_e(a)=ae$ for $N$-th roots of 1 of $a$. Define the rings

$$A=\mathbb{Q}[G]/(\sigma_{-1}=(-1)^r), \quad B=\mathbb{Q}[G]/(\sigma_{-1}+(-1)^r).$$

Then

$$\mathbb{Q}[G] \cong A \times B.$$

In the case $r=1$, let $B'=\mathbb{Q}[G]/(\sigma_{-1}-1, \sum_{\sigma \in G} \sigma)$. Then, $B \cong \mathbb{Q} \times B'$ where the part $B \to \mathbb{Q}$ sends elements of $G$ to 1.

The following lemma is easy seen.

**Lemma (5.2).**

1. $H_0=L$, and it is a free $\mathbb{Q}[G]$-module of rank 1.
2. $H_0$ is identified with the space of systems $\{a(\iota)\}_1$ which associate to each embedding $\iota: L \to \mathbb{C}$ an element $a(\iota)$ of $\mathbb{Q}(2\pi i)^r$ satisfying $a(\iota)=\bar{a}(\iota)$. (Here $\bar{}$ denote the complex conjugation.) The canonical injection

$$H_0 \otimes_{\mathbb{Q}} R \to H_0 \otimes_{\mathbb{Q}} R = L \otimes_{\mathbb{Q}} R$$

associates to $a=\{a(\iota)\}_1\in H_0$, the unique element of $L \otimes_{\mathbb{Q}} R$ whose image in $C$ for any embedding $\iota: L \to \mathbb{C}$ coincides with $a(\iota)$. The action of $\mathbb{Q}[G]$ on $H_0$ is given by $\sigma(a(\iota))=a(\iota \cdot \sigma)$.
3. The action of $\mathbb{Q}[G]$ on $H_0$ factors through $A$, and $H_0$ is a free $A$-
module of rank one.

(5.3) Assume \( r=1 \). Then \( H_k = O_T \otimes Q \), and \( H'_k = Q \) with the trivial action of \( G \). The classical theory of regulator shows that Conj. (2.7) is true for \( Q(1) \) over any number field, the action of \( Q[G] \) on \( H_k \) factors through \( B' \), and \( H_k \) is a free \( B' \)-module of rank 1.

For an \( N \)-th root \( a \) of 1 in \( L \) such that \( a \neq 1 \), we define the cyclotomic element \( c_i(a) \in H_k \) as follows. If the order of \( a \) is not a power of a prime number, then \( 1 - a \) is a unit, and we define \( c_i(a) \) to be the image of \( (1 - a)^{-1} \in O_T \) in \( H_k \). If the order of \( a \) is a power of a prime number \( l \), \( 1 - a \) does not belong to \( O_T \), but we have

\[
O_T \otimes Q \longrightarrow O_T \otimes_{Z[G]} B \longrightarrow \left( O_L \left[ \frac{1}{l} \right] \right)^* \otimes_{Z[G]} B.
\]

We define \( c_i(a) \in H_k \) in this case to be the image of \( (1 - a)^{-1} \in (O_L[1/l])^* \). It is easily seen that if \( N \geq 2 \) and \( a \) is a primitive \( N \)-th root of 1, \( c_i(a) \) is a basis of the \( B' \)-module \( H_k \).

(5.4) Assume \( r \geq 2 \). Borel proved Conj. (2.7) is true for \( Q(r) \) over any number field \([B_0] \). Borel used his regulator map but the coincidence of it with the regulator map of Beilinson is checked in \([Ra]\). It follows that the action of \( Q[G] \) on \( H_k \) factors through \( B' \), and \( H_k \) is a free \( B' \)-module of rank one.

For an \( N \)-th root \( a \) of 1 in \( L \), Beilinson defined an element \( c_r(a) \in H_k \) whose image in \((H_d \otimes qR)/(H_h \otimes qR)\) is the class of

\[
\sum_{n \geq 1} a^n (t^n - 1) \in L \otimes qR = H_d \otimes qR.
\]

If \( a \) is a primitive \( N \)-th root of 1, \( c_r(a) \) is a basis of the \( B \)-module \( H_k \).

(5.5) We fix some notations. Let \( a \) be an \( N \)-th root of 1 in \( L \).

(5.5.1) We define \( w_r(a) \in L \) as follows. For \( n \geq 1 \), let

\[
\nu_n : Q[t]/(t^n - 1) \longrightarrow Q[t]/(t^n - 1)
\]

be the ring homomorphism \( t \mapsto t^n \). We define \( w_r(a) \) to be the image of

\[
( \prod_{i=1}^{N-1} (1 - t^{-r} \nu_i(t)) \text{ under } Q[t]/(t^n - 1) \rightarrow L ; t \mapsto a. \]

Here \( 1 - t^{-r} \nu_i : Q[t]/(t^n - 1) \rightarrow Q[t]/(t^n - 1) \) is bijective, since the eigenvalues of \( \nu_i \) are 0 or a root of 1 (indeed, there are \( i > j \geq 0 \) such that \( \nu_i = \nu_j \)). If \( N \geq 1 \) and \( a \) is a primitive \( N \)-th root of 1, \( w_r(a) \) is the sum of \( a \) and a linear combination over \( Q \) of powers of \( a \) which are not primitive \( N \)-th roots of 1, and it is a basis of the \( Q[G] \)-module \( L \).

(5.5.2) For a primitive \( N \)-th root \( a \) of 1 and for \( x \in Q(2\pi i)^{\nu} \), let \( x \langle a \rangle \) be the following element of \( H_k \). To an embedding \( \iota : L \rightarrow \mathbb{C} \), \( x \langle a \rangle \) associates \( x \in Q(2\pi i)^{\nu} \) if \( \iota(a) = \exp(2\pi i/N) \), \( \bar{x} \) if \( \iota(a) = \exp(-2\pi i/N) \) and associates 0 \( \in Q(2\pi i)^{\nu} \) otherwise. (Cf. (5.2) (2).)

(5.5.3) We define \( d_r(a) \in L = H_k \) as follows. Let \( g(t) \) be the rational func-
tion in $t$ defined by
\[ g(t) = \left( \frac{d}{t^{-1}dt} \right)^{r-1} \left( 1 - \frac{t}{1-t} \right) = \sum_{n \geq 1} n^{-r} t^n. \]

If $\alpha \neq 1$, we define $d_\alpha(\alpha) = (-1)^r (r-1)^{-1} g(\alpha)$. For $\alpha = 1$, take any integer $c$ which is different from 0, 1, $-1$, and let $d_r(1)$ be $(1-c^r)^{-1}$ times the value at $t=1$ of the rational function $(-1)^r (r-1)^{-1} (g(t)-g(t^r))$. Then $rf(l)$ is independent of the choice of $c$.

We have $d_\alpha(\alpha^{-1}) = (-1)^r d_r(\alpha)$ if $r \geq 2$, and $d_1(\alpha^{-1}) = -d_1(\alpha)-1$.

Let $\mathcal{M}_\alpha(\alpha) \in \Phi_{\text{mot}}(\mathbb{Q})$ be the $Q$-linear map
\[ H_d \to L; \quad x \mapsto Tr_{L/Q}(x d_r(\alpha)). \]

The motivic zeta element is described as follows.

**Proposition (5.6).** Let the notations be as above, and let $S$ be the set of places of $Q$ consisting of $\infty$ and all prime divisors of $N$.

1. The image of $\zeta_{\mathbb{E}/K, s(M)} \in \Phi_{\text{mot}}(\mathbb{Q})$ belongs to $\Phi_{\text{mot}}$.
2. The image of $\zeta_{\mathbb{E}/K, s(M)} \in \Phi_{\text{mot}}(\mathbb{Q})$ coincides with
\[ \left( \frac{2\pi i}{N} \right)^r \langle \alpha \rangle \otimes d_r^*(\alpha) \]
for any primitive $N$-th root $\alpha$ of 1 in $L$.

3. If $r \geq 2$ (resp. $r=1$ and $N \geq 2$), the image of $\zeta_{\mathbb{E}/K, s(M)}$ in
\[ \Phi_{\text{mot}}(\mathbb{Q}) \otimes \mathbb{Q}[G] \]
coincides with
\[ c_r(\alpha) \otimes w_r(\alpha)^{-1} \]
for any primitive $N$-th root $\alpha$ of 1 in $L$. (Here $H_{\Phi}^{-1}$ means the inverse of the invertible $\mathbb{Q}[G]$-module $H_{\Phi}$.) If $r=1$, the image of $\zeta_{\mathbb{E}/K, s(M)}$ in
\[ \Phi_{\text{mot}}(\mathbb{Q}) \otimes \mathbb{Q}[G] \]
coincides with the trace map $L \to Q$.

**Proof.** All things follow from the results introduced in (5.2)-(5.5) by direct computation, except that; for (2), we have to recall the following fact which is a consequence of the functional equations of partial Riemann zeta functions. The value at $s=r$ of
\[ \left( \sum_{n \equiv c \pmod{N}} n^{-s} \right) + (-1)^r \left( \sum_{n \equiv -c \pmod{N}} n^{-s} \right) \]
for $c \in \mathbb{Z}$ coincides with
Now we fix a prime number $p$, and consider the $p$-adic side. We first recall a result of Soulé.

**Theorem ([So, §1]).** Let $F$ be a number field, and let $r \geq 2$. Then the chern class map induces an isomorphism

$$K_{2r-1}(O_F[\frac{1}{p}]) \otimes Q_p \cong H^{r}(O_F[\frac{1}{p}], Q_p(r))$$

for $i=1, 2$.

Both groups are zero if $i \neq 1$.

**Corollary (5.8).** Let $F$ and $r$ be as in (5.7). Then, for any $i \in \mathbb{Z}$, the chern class map induces an isomorphism

$$(5.8.1)\ H^i(F, Q(r)) \otimes Q_p \cong H^i_{l\text{m}}(F, Q_p(r)).$$

Both groups are zero if $i \neq 1$.

**Corollary (5.9).** For any number field $F$ and for any $r \in \mathbb{Z}$, the chern class map induces an isomorphism

$$(5.9.1)\ H^i_{f}(F, Q(r)) \otimes Q_p \cong H^i_{l\text{m}}(F, Q_p(r)).$$

Both groups are zero if $r \leq 0$.

**Proof.** As is easily seen, $H^i_{f}(F, Q_p(r))$ is isomorphic to the kernel of $H^i_{l}(O_F[1/p], Q_p(r)) \to \bigoplus_{v \mid p} H^i_{l}(F_v, Q_p(r))$ for a place $v$ of $F$ lying over $p$. For a place $v$ of $F$ lying over $p$, $H^i_{f}(F_v, Q_p(r))$ coincides with $H^i_{l}(F_v, Q_p(r))$ if $r \geq 2$, coincides with the image of $O_F \otimes Q$ in $H^i_{l}(F_v, Q_p(1))$ if $r=1$, coincides with the “unramified part” of $H^i_{l}(F_v, Q_p)$ if $r=0$, and is zero if $r < 0$ ([BK] §3).

We have from these facts

$$H^i_{f}(F, Q_p(r)) = H^i_{l\text{m}}(Q_p(r)) \quad \text{if } r \geq 2,$$

$$H^i_{f}(F, Q_p) = H^i_{l}(O_F, Q_p) = 0.$$
(5.10) Now we return to the cyclotomic case. By (3.2) and (5.9), we have an exact sequence (without any conjecture)

$$0 \rightarrow (H^F_p)^* \rightarrow H_k \otimes \mathbb{Q}_p \rightarrow \Gamma_p \otimes (H^F_p)^* \rightarrow 0.$$ 

One conjectures $H^F_p = 0$, but this is not yet proved. However, since $H_k \otimes \mathbb{Q}_p \mathfrak{A}$ = 0, we have $H^F_p \otimes \mathbb{Q}_p \mathfrak{A} = 0$. From this and (5.2) (1), we see that $H^F_p \otimes \mathbb{Q}_p \mathfrak{A}$ is a free $A \otimes \mathbb{Q}_p \mathfrak{A}$-module of rank 1. We have

$$\Phi^F_p \otimes \mathbb{Q}_p \mathfrak{A} \cong H_k \otimes \mathbb{Q}_p (H^F_p)^*$$

canonically.

We will give an explicit description of the image of the arithmetic zeta element in $\Phi^F_p \otimes \mathbb{Q}_p \mathfrak{A}$, by using $p$-adic cyclotomic elements of Deligne and Soule, which we recall here.

In the rest of this section, let $S$ be the set of places of $\mathbb{Q}$ consisting all prime divisors of $N$, and let $S'$ be the union of $S$ and $\{p\}$. Of course one has $S = S'$ if $p \mid N$.

(5.11) Let $a$ be a primitive $N$-th root of 1 in $L$ and let $m \in \mathbb{Z}$. Then, the $p$-adic cyclotomic elements

$$c_m(\alpha), c'_m(\alpha) \in H^i(O_{L,s'}, \mathbb{Z}_p(m))$$

of Deligne and Soule are defined unless $(\alpha, \sigma) = (1, 1)$ ($[De_1], [So_1]$).

In the case $m = 1$, $c_i(\alpha)$ will be the image $1 - \alpha$ of $1 - \alpha$ under the canonical map $O_{L,s} \rightarrow H^i(O_{L,s'}, \mathbb{Z}_p(1))$ defined by Kummer theory. The elements $c_m(\alpha)$ and $c'_m(\alpha)$ will be related to each other by

$$c'_m(\alpha) = (1 - p^{m-1} \phi_p^{-1}) c_m(\alpha) \quad \text{in} \quad H^i(L, \mathbb{Q}_p(m))$$

where $\phi_p$ is the Frobenius of $p$.

Let $n \geq 1$. Take a $p^n$-th root $\beta$ of $\alpha$ of order $p^n N$. Then, we obtain an element

$$\{1 - \beta\} \otimes \beta^{n-1} \in H^i(Q(\beta), \mathbb{Z}/p^n \mathbb{Z}(1)) \otimes \mathbb{Z}/p^n \mathbb{Z}(m-1)$$

$$\cong H^i(Q(\beta), \mathbb{Z}/p^n \mathbb{Z}(m)).$$

Here $\{1 - \beta\}$ denotes the image of $1 - \beta$ by Kummer theory, and $[\beta^n]$ is just $\beta^n$ but one puts $[\ ]$ to avoid a confusion for we consider $\mathbb{Z}/p^n \mathbb{Z}(1)$ as an additive group. One sees easily that the elements

$$c'_m(\alpha)_n \overset{\text{def}}{=} N_{Q(\beta)/L}(1 - \beta) \otimes [\beta^{n-1}] \in H^i(O_{L,s'}, \mathbb{Z}/p^n \mathbb{Z}(m))$$

$(N_{Q(\beta)/L}$ denotes the norm map) is independent of the choice of $\beta$, and $c'_m(\alpha)_n$ forms a projective system when $n$ varies. Let

$$c'_m(\alpha) = \lim_n c'_m(\alpha)_n \in H^i(O_{L,s'}, \mathbb{Z}_p(m)).$$
Now we define $c_m(\alpha)$. In the case $p \nmid N$, we define

$$c_m(\alpha) = c'_m(\alpha).$$

In the case $m=1$, let $c_m(\alpha)$ be the image of $1 - \alpha \in \mathcal{O}_K, s$ in $H^4(O_L, s', Z_p(1))$. Then these two definitions agree when $p \mid N$ and $m=1$. If $(N, p)=1$ and $m \geq 2$ (resp. if $(N, p)=1$ and $m \leq 0$), we define

$$c_m(\alpha) = \sum_{i \geq 0} (p^{m-i})^i c'_m(\alpha^{p^{m-1}})$$

where $\alpha^{p^{m-1}}$ is the unique $p^i$-th root of $\alpha$ of order $N$

(resp. $c_m(\alpha) = -\sum_{i \geq 1} (p^{1-i})^i c'_m(\alpha^{p^i})$).

The following theorem will be proved in [Ka, §2], by using an "explicit reciprocity law" for the motive $Q(r)$.

**THEOREM (5.12).** Let $a$ be a primitive $N$-th root of $1$ in $L$. Then, the image of $c_{1-r}(\alpha)$ ($r \geq 1$) in $H^*_{\text{dec}}$ under the dual map $H^*_b \to H^*_{\text{dec}}$ of $H^*_{\text{dec}} \to (H^*_b)^*$ coincides with $-N^rd_1(a)$.

In the case $N$ is prime to $p$, this result follows easily from [BK]§2 (2.1).

**Remark (5.13).** Deligne and Soulé, and also Gros and Kurihara ([Gr]) consider these cyclotomic elements in $H^1$ of $Q_p(m)$ mainly for positive $m$, and relate them to special values of $p$-adic zeta functions, though we consider here these elements with $m \leq 0$ which are related to special values of complex zeta functions.

By (5.6) (2) and (5.13), we have

**THEOREM (5.14).** The image of $\zeta_{L/K, s}(M)_p$ (resp. $\zeta_{L/K, s'}(M)_p \in \Phi^{\text{et}}_{\text{dec}}$) in $\Phi^{\text{et}}_{\text{dec}} \otimes Q_{(1)} A$ coincides with

$$-(2\pi i)^{r}(\alpha) \otimes c_{1-r}(\alpha)$$

(resp. $-(2\pi i)^{r}(\alpha) \otimes c_{1-r}'(\alpha)$).

for any primitive $N$-th root $\alpha$ of $1$.

§ 6. Relation with classical Iwasawa theory.

In this section, we show that when we consider the situation where $K=Q$, $M=Q(r)$ with $r$ a positive even integer and $L$ is the maximal real subfield of $Q(\alpha)$ with $\alpha$ a root of $1$ of order a power of $p$, our Iwasawa main conjecture coincides with the classical Iwasawa conjecture.

The classical Iwasawa theory uses characteristic polynomials of torsion modules. We first relate this concept to determinant modules.
PROPOSITION (6.1). Let $R$ be a Noetherian normal ring and let $F$ be the total quotient ring of $R$ (that is, $F$ is obtained from $R$ by inverting all non-zero-divisors in $R$). Let $Y$ be a finitely generated $R$-module of finite tor-dimension such that $Y \otimes_R F = 0$. Then, the image of

\[
\det_R(Y) \rightarrow \det_R(Y) \otimes_R F = \det_F(Y \otimes_R F) = \det_F(0) = F
\]

coincides with $\text{Char}(Y)^{-1}$, where $\text{Char}(Y)$ is the unique invertible ideal of $R$ such that for any prime ideal $\mathfrak{p}$ of $R$ of height one, the stalk $\text{Char}(Y)_\mathfrak{p}$ coincides with $(\mathfrak{p}R)^{n(\mathfrak{p})}$ where

\[
n(\mathfrak{p}) = \text{length}_{R_\mathfrak{p}}(Y_\mathfrak{p}).
\]

**Proof.** Since $R$ is normal, an invertible $R$-module in $F$ (for example $\det_R(Y)$) is characterized by its stalks in codimension one. Hence we are reduced to the case where $R$ is a discrete valuation ring. In this case, $Y \cong \bigoplus_i R/a_i R$ for a finite family $(a_i)_i$ of non-zero elements of $R$. By using the resolution $0 \rightarrow \bigoplus_i R^{\alpha_i} \rightarrow Y \rightarrow 0$ with $\alpha = (a_i)_i$, we obtain $\det_R(Y) = (\prod \alpha_i)^{-1} R \otimes F$.

(6.2) We recall the classical Iwasawa main conjecture proved by Mazur and Wiles. (Cf. [1w], [Wa] Ch. 13).

Let $K = \mathbb{Q}$, $M = \mathbb{Q}(\zeta)$, and let $r$ be an even positive integer. Let $S = \{\mathfrak{p}\}$ with $\mathfrak{p}$ an odd prime. For $n \geq 1$, take a primitive $p^n$-th root $\alpha_n$ of 1 in $\overline{\mathbb{Q}}$. Let $L_n$ be the maximal real subfield of $\mathbb{Q}(\alpha_n)$, and let $L_{\infty} = \bigcap_n L_n$, $O_{L_{\infty}, s} = \bigcup_n O_{L_n, s}$. Unless the contrary is explicitly stated, $(\cdot)^*$ means $\text{Hom}(\cdot, Q_p/Z_p)$ in what follows.

Let

\[
\mathfrak{X} = H^1_{\text{et}}(O_{L_{\infty}, s}, Q_p/Z_p)^*.
\]

Then $\mathfrak{X}$ is the Galois group of the maximal abelian extension of $L_{\infty}$ which is unramified outside $p$. Let

\[
q_j = H^1(L_{\infty} \otimes Q_p, Q_p/Z_p)^*
\]

\[= \lim_{m \rightarrow 0} (L_{\infty} \otimes Q_p)^*/((L_{\infty} \otimes Q_p)^*)^{p^m}
\]

(by class field theory)

where $\lim_{m \rightarrow 0}$ is taken with respect to norms. Let

\[
c = (\{1 - \alpha_n(1 - \alpha_n^{-1})\}_{n \geq 1}) \in q_j.
\]

Let $\Gamma = \text{Gal}(L_{\infty}/\mathbb{Q})$. Then the completed group ring $Z_p[[\Gamma]]$ is a regular ring and so any finitely generated $Z_p[[\Gamma]]$-module is a perfect complex over $Z_p[[\Gamma]]$ when regarded as an object of the derived category.

In Iwasawa theory, it is well known that $\mathfrak{X}$ and $q_j/Z_p[[\Gamma]]c$ are finitely generated torsion $Z_p[[\Gamma]]$-modules (Iwasawa [1w]). The classical Iwasawa main conjecture, in one formulation, is stated as

\[
\text{Char}_{Z_p[[\Gamma]]c}(\mathfrak{X}) = \text{Char}_{Z_p[[\Gamma]]c}(q_j/Z_p[[\Gamma]]c).
\]
We relate our conjecture (4.9) to (6.2.1). For $n \geq 1$, let $\Gamma_n = \text{Gal}(L_n/L_n)$, $G_n = \text{Gal}(L_n/Q)$.

**Lemma (6.3).**

1. $\mathfrak{X}(-r)_{r_n}$ if $i = 0$
2. $0$ otherwise.

3. There is a distinguished triangle

$$ R\Gamma(O_{L_n,s}, Z_p(1-r))[1] \longrightarrow q(-r)_{\gamma_n} \longrightarrow \mathfrak{X}(-r)_{r_n} \longrightarrow . $$

Here $(\gamma_n)$ denote coinvariants by $\Gamma_n$.

**Proof.** Consider the spectral sequences

$$ E_{1,2}^{i,j} = H^i(\Gamma_n, H^j(O_{L_n,s}, (Q_p/Z_p)(r))) \implies H^{i+j}(O_{L_n,s}, (Q_p/Z_p)(r)) $$

$$ E_{1,0}^{i,j} = H^i(\Gamma_n, H^j(L_n \otimes Q_p, (Q_p/Z_p)(r))) \implies H^{i+j}(L_n \otimes Q_p, (Q_p/Z_p)(r)). $$

Then we have:

1. $E_{1,1}^{i,j} = 0$ and $E_{1,0}^{i,j} = 0$ except the cases $(i, j) = (0, 0), (1, 0), (0, 1)$.

2. $E_{1,1}^{i,j} \approx E_{1,0}^{i,j}$ if $j = 0$.

Indeed, (6.4.1) follows from the facts that the cohomological $p$-dimension of $\Gamma_n$ is 1, the cohomological dimension of $L_n \otimes Q_p$ (resp. $O_{L_n,s}$) is $\leq 2$, and

$$ H^i(L_n \otimes Q_p, (Q_p/Z_p)(r)) = H^i(L_n \otimes Q_p, Z_p(1-r)^s) = 0, $$

$$ H^i(O_{L_n,s}, (Q_p/Z_p)(r)) = 0, \quad H^i(O_{L_n,s}, (Q_p/Z_p)(r)) = 0 $$

([S0])

The proof of (6.4.2) is easy.

Finally (6.3) (3) follows from the above facts and the distinguished triangle

$$ R\Gamma(O_{L_n,s}, Z_p(1-r)) \longrightarrow R\Gamma(L_n \otimes Q_p, (Q_p/Z_p)(r))^s[-2] $$

$$ \longrightarrow R\Gamma(O_{L_n,s}, (Q_p/Z_p)(r))^s[-2] \longrightarrow . $$

which comes from Artin-Verdier duality.

We relate Conj. (4.9) to (6.2.1). By (5.14), Conj. (4.9) for $K = Q$, $L = L_n$ with $n \geq 1$ and $M = Q(r)$ is equivalent to the statement that $c_{1-r}(\alpha_n) \in H^i(O_{L_n,s}, Z_p(1-r))$ is a $Z_p[G_n]$-base of $\det_{Y_{\gamma_n}} R\Gamma(O_{L_n,s}, Z_p(1-r))[1]$. Let

$$ \gamma = \{((1-\alpha_n)(1-\alpha_n^s) \otimes \alpha_n^{\gamma(-r)}) \}_{n \geq 1} \subseteq q(-r), $$

and let $\gamma_n \in q(-r)_{r_n}$ be the image of $\gamma$. Then the image of $c_{1-r}(\alpha_n)$ in $H^i(L_n \otimes Q_p, Z_p(1-r))$ coincides with the image of $2^{-i} \gamma_n$. By (6.3) (3), (4.9) is
equivalent to the statement that $\gamma_n$ is a $\mathbb{Z}_p[\mathbb{G}_n]$-basis of $\det_{\mathbb{Z}_p[\mathbb{G}_n]}(\eta(r)r_n \to \mathcal{X}(-r)r_n)$. This holds for any $n \geq 1$ if and only if $\gamma$ is a $\mathbb{Z}_p[[T]]$-basis of $\det_{\mathbb{Z}_p[[T]]}(\eta(r) \to \mathcal{X}(-r))$, that is, if and only if (6.2.1) holds.

§ 7. Relation with Tamagawa numbers of motives.

In this section, we see that the conjecture on Tamagawa numbers of motives in $[BK]$ is regarded as the case of trivial abelian extension of our Iwasawa main conjecture. For simplicity, we treat motives of weight $\leq -3$ and we consider numbers (Tamagawa numbers, values of $L$-functions, ...) modulo multiplication by powers of 2.

In this section we assume the conjectures (2.6) (2.7) (3.4) (these conjectures were assumed also in $[BK]$).

(7.1) We fix notations. In this section, let $M$ be a pure motive over $\mathbb{Q}$ of weight $\leq -3$. Fix an odd prime number $p$. Take a $\mathbb{Z}_p$-sheaf $T$ in $\mathbb{V}_{p}(M)$ such that $T \otimes_{\mathbb{Z}_p} \mathbb{Q} = \mathbb{V}_p(M)$.

Let

$$H^i(Q, T) \subseteq H^i(Q, T) \quad \text{(resp. } H^i(Q_p, T) \subseteq H^i(Q_p, T))$$

be the inverse image of

$$H^i(Q, \mathbb{V}_p(M)) \subseteq H^i(Q, \mathbb{V}_p(M)) \quad \text{(resp. } H^i(Q_p, \mathbb{V}_p(M)) \subseteq H^i(Q_p, \mathbb{V}_p(M))).$$

Let $H_{k, r} \subseteq H^i(Q, T)$ be the inverse image of $H_{k} \subseteq H^i(Q, \mathbb{V}_p(M))$. Then, $H_{k, r}$ is a finitely generated $\mathbb{Z}(p)$-module such that

$$H_{k, r} \otimes_{\mathbb{Z}(p)} Q \cong H_{k}, \quad H_{k, r} \otimes_{\mathbb{Z}(p)} \mathbb{Z}_p \cong H^i(Q, T).$$

(7.2) We review the definition of the Tamagawa number of the pair $(M, T)$, which is an element of $\mathbb{R}^*/\mathbb{Z}(p)^*$. (In $[BK]$, the Tamagawa number is defined as a number without modulo $\mathbb{Z}(p)$, by using $/\$-adic realizations of $M$ for all prime numbers $l$. We work here only with the $p$-adic realization, so we have a number modulo $\mathbb{Z}(p)^*$. By varying $p$, we can recover the Tamagawa number in $[BK]$.)

Take a $\mathbb{Z}(p)$-lattice $\Delta$ of $H_{k}$. By Conj. (2.7) which we assumed to hold, we have an isomorphism

$$H_{k, r} \otimes_{\mathbb{Z}(p)} \mathbb{R} \cong (\Delta \otimes_{\mathbb{Z}(p)} \mathbb{R})/(H_{k, r} \otimes_{\mathbb{Z}(p)} \mathbb{R}).$$

Let

$$\Phi_{\Delta}^{\otimes_{\mathbb{Z}(p)} \mathbb{R}} = \det_{\mathbb{Z}(p)}(H_{k, r} \otimes_{\mathbb{Z}(p)} \mathbb{R}) \otimes_{\mathbb{Z}(p)} \det_{\mathbb{Z}(p)}(H_{k, r} \otimes_{\mathbb{Z}(p)} \mathbb{Z}_p, \det_{\mathbb{Z}(p)}(\Delta)^*).$$

$((^* = \text{Hom}_{\mathbb{Z}(p)}(\ , \mathbb{Z}(p)))$. Then (7.2.1) induces

$$\Phi_{\Delta}^{\otimes_{\mathbb{Z}(p)} \mathbb{R}} \cong \quad \,.$$
Let $\alpha \in R^*/Z_{(p)}^*$ be the image of a $Z_{(p)}$-basis of $\Phi_{T_{(p)}}^\sv$ under (7.2.2). (With the notations in [BK], $a$ is the volume of $A(R)/A(Q)$ modulo $Z_{(p)}$. The choice of $\Delta$ here corresponds to the choice of $\det_{Q}(H_{\Delta}) \otimes_{Z_{(p)}} Q$ in [BK].) On the other hand, let $\beta \in Q^*/Z_{(p)}^*$ be the element such that the image of a $Z_{(p)}$-basis of $\det_{Z_{(p)}}(\Delta)$ under the isomorphism

$\det_{Z_{(p)}}(\Delta) \otimes_{Z_{(p)}} Q_p \cong \det_{Z_{(p)}}(H_{(Q_p, T)}) \otimes_{Z_p} Q_p$

induced by $\exp: \Delta \otimes_{Z_{(p)}} Q_p \cong H_{(Q_p, T)} \otimes_{Z_p} Q_p$ is (a representative in $Q^*$ of) $\beta$ times a $Z_p$-basis of $\det_{Z_{(p)}}(H_{(Q_p, T)})$. (With the notation of [BK], $\beta$ is the volume of $A(Q_p) \mod Z_{(p)}$.)

Let $S$ be a finite set of places of $Q$ containing $\infty$, $p$, and all finite places at which $M$ has bad reduction. Define

$\mu_{S,f} \in Q^*/Z_{(p)}^*$, $\mu_S \in R^*/Z_{(p)}^*$ by

$\mu_{S,f} = \beta \prod_{v \in S, p} \#(H^0(Q_v, T \otimes (Q_p/Z_p)))$, $\mu_S = \alpha \mu_{S,f}$.

Then $\mu_S$ is independent of the choice of $\Delta$. Define

$\text{Tam}(M, T) = \mu_S \cdot L_S(M, 0)^{-1} \cdot R^*/Z_{(p)}^*$.

This element, called the Tamagawa number of the pair $(M, T)$, is independent of the choice of $S$.

The following (7.3) is clear.

**LEMMA (7.3).** The image of $L_S(M, 0) \in R$ under the isomorphism (7.2.2) is equal to (a representative in $Q^*$ of) $\mu_{S,f} \cdot \text{Tam}(M, T)^{-1} \cdot Z_{(p)}$-basis of $\Phi_{T_{(p)}}^\sv$.

(7.4) By (7.3), we see that Conj. (4.7) is true in this situation (with $K = L = Q$) if and only if $\text{Tam}(M, T) \in Q$. Assume $\text{Tam}(M, T) \in Q$. Then, the motivic zeta element $\zeta_{Q, S}(M)$ is $\mu_{S,f} \cdot \text{Tam}(M, T)^{-1}$ times a $Z_{(p)}$-basis of $\Phi_{T_{(p)}}^\sv$.

(7.5) To state the conjecture in [BK] on Tamagawa numbers, we have to consider the Tate-Shafarevich group of a motive.

Consider the map $\iota: P \to Q$ where

$P = H^1(Q, T \otimes (Q_p/Z_p)) / (H_k \otimes (Q_p/Z_p))$

$Q = H^1(Q_p, T \otimes (Q_p/Z_p)) / (H^1(Q_p, T \otimes (Q_p/Z_p)) \oplus (\bigoplus_{v \neq p} H^1(Q_v, T \otimes (Q_p/Z_p))))$

The kernel of $\iota$ is a generalization of the Tate-Shafarevich group of an abelian variety.

**PROPOSITION (7.6).** $\ker(\iota)$ and $\coker(\iota)$ are finite groups.

The proof of (7.6) is given below. (Recall we assumed the conjectures (2.6) (2.7) (3.4). Otherwise such finiteness becomes very difficult).
The \( p \)-primary part of the conjecture on Tamagawa numbers of motives in \([BK]\) is stated as follows.

**Conjecture (7.7).** \( \text{Tam}(M, T) = \#(\text{Coker}(\iota)) \cdot \#(\text{Ker}(\iota))^{-1} \text{ in } Q^*/Z_p^* \).

The following (7.8) shows the equivalence between (7.7) and our Iwasawa main conjecture in this situation.

**Proposition (7.8).** Under the isomorphism \( \Phi^\text{mot} \otimes Q_p = \Phi^\text{sr} \), the image of the zeta element \( \zeta^\text{mot}_{Q, s}(M) \) under \( \Phi \) is

\[
\#(\text{Coker}(\iota)) \cdot (\text{Ker}(\iota))^{-1} \text{Tam}(M, T)^{-1}
\]

times a \( Z_p \)-basis of

\[
\Phi^\text{sr}_{p, \tau} = \text{det}_{Z_p}(H_{k, \tau} \otimes Z_p) \{ \text{det}_{Z_p}(R \Gamma(Z_S, T^*(1))) \}^*.
\]

Proofs of (7.6) and (7.8).

By Artin-Verdier duality and the localization theory for étale cohomology, we have an acyclic complex

\[
C : 0 \rightarrow H^0(Q, T \otimes (Q_p/Z_p)) \rightarrow 0 \rightarrow 0
\]

\[
\rightarrow H^1(Z_S, T^*(1))^* \rightarrow H^1(Q, T \otimes (Q_p/Z_p))
\]

\[
\rightarrow \bigoplus_{v \in S} H^1(Q_v, T \otimes (Q_p/Z_p)) \oplus (\bigoplus_{v \in S} H^1(Q, T \otimes (Q_p/Z_p)(-1)))
\]

\[
\rightarrow H^1(Z_p, T^*(1))^* \rightarrow H^1(Q, T \otimes (Q_p/Z_p))
\]

\[
\gamma \rightarrow \bigoplus_{v \in S} H^1(Q_v, T \otimes (Q_p/Z_p)) \rightarrow 0.
\]

Here the * outside the notation of cohomology \( H^i(\cdot) \) are \( \text{Hom}(\cdot, Q_p/Z_p) \), whereas the * inside \( H^i(\cdot) \) are \( \text{Hom}(\cdot, Z_p) \).

Jannsen proved that the map \( \gamma \) is an isomorphism. ([Ja] §4 Thm. 3d).

We define a subcomplex \( C' \) of \( C \). Since the image of

\[
\text{Hom}_{Q_p}(H^1(Z_S, V_p(M)^*(1))^* \rightarrow H^1(Z_S, V_p(M))
\]

belongs to \( H^1(Z_S, V_p(M)) = H_{k, \tau} \otimes_{Z_p} Q_p \), we see that there is a \( Z_p \)-submodule \( D \) of \( H^1(Z_S, T^*(1))^* \) of finite index whose image in \( H^1(Z_S, T \otimes (Q_p/Z_p)) \) is contained in \( H_{k, \tau} \otimes (Q_p/Z_p) \). Let \( C'_B \) be the complex

\[
0 \rightarrow \bigoplus (\text{deg. } 2) D \rightarrow H_{k, \tau} \otimes (Q_p/Z_p) \rightarrow H^1(Q_p, T \otimes (Q_p/Z_p)(-1))
\]

\[
\rightarrow H^1(Z_S, T^*(1))^* \rightarrow 0.
\]

We have an exact sequence of complexes

\[
0 \rightarrow C'_B \rightarrow C \rightarrow C' \rightarrow 0
\]
where \( C'_p \) is the complex

\[
0 \longrightarrow H(Q, T \otimes (Q_p/Z_p)) \longrightarrow \bigoplus_{n \geq 0} H^n(Q_v, T \otimes (Q_p/Z_p)) \rightarrow^t \longrightarrow H^*(Z_S, T^* (1)) \rightarrow P \rightarrow Q \rightarrow 0.
\]

Here \( P \) and \( Q \) are as in (7.5). Since \( C \) is acyclic and the cohomology groups of the complex \( C'_p \) are finite, it follows that the cohomology groups of the complex \( C'_p \) are finite, that is, \( \text{Ker}(\iota) \) and \( \text{Coker}(\iota) \) are finite.

For a bounded complex \( E \) of abelian groups whose cohomology groups are finite, let \( \chi(E) = \prod_i \#(H^i(E))^{-1} \). We have

\[(7.9.1) \quad \chi(C'_p) = \chi(C'_p)^{-1}.
\]

It is easily seen that the isomorphism \( \Phi^{\text{mot}} \otimes \mathbb{Q}_p \cong \Phi^{\text{mot}} \otimes \mathbb{Q}_p \otimes \mathbb{A}_p \) sends a \( \mathbb{Z}_p \)-basis of \( \Phi^{\text{mot}} \) to

\[(7.9.2) \quad \beta^{-1} \chi(C'_p)^{-1} \#(H^i(Q, T)_{tor})^{-1} \#(H^*(Z_S^* (1)) \rightarrow D) \cdot \#(H^i(Q_p, T)_{tor})
\]

times a generator of \( \Phi^{\text{mot}} \otimes \mathbb{Q}_p \). Here \( \text{tor} \) denote the torsion part. Since

\[(H^i(Q, T)_{tor}) = H^i(Q, T_{tor}) \cong H^i(Q, T \otimes (Q_p/Z_p))
\]

and

\[H^i(Q_p, T)_{tor} \cong H^i(Q_p, T \otimes (Q_p/Z_p)),
\]

we have by (7.9.1) that the element (7.9.2) is equal to

\[\#(\text{Ker}(\iota)) \cdot (\text{Ker}(\iota))^{-1} \cdot \mu^{-1}_{S,1}.
\]

Hence by (7.3), the map \( \Phi^{\text{mot}} \otimes \mathbb{Q}_p \cong \Phi^{\text{mot}} \otimes \mathbb{Q}_p \otimes \mathbb{A}_p \) sends \( \zeta^{\text{mot}}_{Q_p}, s(M) \) to

\[\#(\text{Coker}(\iota)) \cdot (\text{Ker}(\iota))^{-1} \cdot \text{Tam}(M, T)^{-1}\]

times a \( \mathbb{Z}_p \)-basis of \( \Phi^{\text{mot}} \otimes \mathbb{Q}_p \).

\[\text{REFERENCES}\]


Tate, J., $p$-divisible groups, in Proc. of a Conf. on local fields, 1966 Springer (1967) 153-183.


